Instructors' Solutions

for

Mathematical Methods for Physics and Engineering (*third edition*)

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Contents

Introduction

The second edition of *Mathematical Methods for Physics and Engineering* carried more than twice as many exercises, based on its various chapters, as did the first. In the Preface we discussed the general question of how such exercises should be treated but, in the end, decided to provide hints and outline answers to all problems, as in the first edition. This decision was an uneasy one as, on the one hand, it did not allow the exercises to be set as totally unaided homework that could be used for assessment purposes but, on the other, it did not give a full explanation of how to tackle a problem when a student needed explicit guidance or a model answer.

In order to allow both of these educationally desirable goals to be achieved we have, in the third edition, completely changed the way this matter is handled. All of the exercises from the second edition, plus a number of additional ones testing the newly-added material, have been included in penultimate subsections of the appropriate, sometimes reorganised, chapters. Hints and outline answers are given, as previously, in the final subsections, but *only to the odd-numbered exercises*. This leaves all even-numbered exercises free to be set as unaided homework, as described below.

For the four hundred plus **odd-numbered** exercises, complete solutions are available, to both students and their teachers, in the form of a separate manual, K. F. Riley and M. P. Hobson, *Student Solutions Manual for Mathematical Methods for Physics and Engineering, 3rd edn.* (Cambridge: CUP, 2006). These full solutions are additional to the hints and outline answers given in the main text. For each exercise, the original question is reproduced and then followed by a fully-worked solution. For those exercises that make internal reference to the main text or to other (even-numbered) exercises not included in the manual, the questions have been reworded, usually by including additional information, so that the questions can stand alone.

The remaining four hundred or so **even-numbered** exercises have no hints or answers, outlined or detailed, available for general access. They can therefore be used by instructors as a basis for setting unaided homework. Full solutions to these exercises, in the same general format as those appearing in the manual (though they may contain cross-references to the main text or to other exercises), form the body of the material on this website.

In many cases, in the manual as well as here, the solution given is even fuller than one that might be expected of a good student who has understood the material. This is because we have aimed to make the solutions instructional as well as utilitarian. To this end, we have included comments that are intended to show how the plan for the solution is fomulated and have given the justifications for particular intermediate steps (something not always done, even by the best of students). We have also tried to write each individual substituted formula in the form that best indicates how it was obtained, before simplifying it at the next or a subsequent stage. Where several lines of algebraic manipulation or calculus are needed to obtain a final result they are normally included in full; this should enable the instructor to determine whether a student's incorrect answer is due to a misunderstanding of principles or to a technical error.

In all new publications, on paper or on a website, errors and typographical mistakes are virtually unavoidable and we would be grateful to any instructor who brings instances to our attention.

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Preliminary algebra

Polynomial equations

> ***1.2*** *Determine how the number of real roots of the equation*
>
> $$g(x) = 4x^3 - 17x^2 + 10x + k = 0$$
>
> *depends upon k. Are there any cases for which the equation has exactly two distinct real roots?*

We first determine the positions of the turning points (if any) of $g(x)$ by equating its derivative $g'(x) = 12x^2 - 34x + 10$ to zero. The roots of $g'(x) = 0$ are given, either by factorising $g'(x)$, or by the standard formula,

$$\alpha_{1,2} = \frac{34 \pm \sqrt{1156 - 480}}{24},$$

as $\frac{5}{2}$ and $\frac{1}{3}$.

We now determine the values of $g(x)$ at these turning points; they are $g(\frac{5}{2}) = -\frac{75}{4} + k$ and $g(\frac{1}{3}) = \frac{43}{27} + k$. These will remain of opposite signs, as is required for three real roots, provided k remains in the range $-\frac{43}{27} < k < \frac{75}{4}$. If k is equal to one of these two extreme values, a graph of $g(x)$ just touches the x-axis and two of the roots become coincident, resulting in only two *distinct* real roots.

> ***1.4*** *Given that $x = 2$ is one root of*
>
> $$g(x) = 2x^4 + 4x^3 - 9x^2 - 11x - 6 = 0,$$
>
> *use factorisation to determine how many real roots it has.*

Given that $x = 2$ is one root of $g(x) = 0$, we write $g(x) = (x - 2)h(x)$ or, more explicitly,

$$2x^4 + 4x^3 - 9x^2 - 11x - 6 = (x - 2)(b_3x^3 + b_2x^2 + b_1x + b_0).$$

Equating the coefficients of successive (decreasing) powers of x, we obtain

$$b_3 = 2, \quad b_2 - 2b_3 = 4, \quad b_1 - 2b_2 = -9, \quad b_0 - 2b_1 = -11, \quad -2b_0 = -6.$$

These five equations have the consistent solution for the four unknowns b_i of $b_3 = 2$, $b_2 = 8$, $b_1 = 7$ and $b_0 = 3$. Thus $h(x) = 2x^3 + 8x^2 + 7x + 3$.

Clearly, since all of its coefficients are positive, $h(x)$ can have no zeros for positive values of x. A few tests with negative integer values of x (with the initial intention of making a rough sketch) reveal that $h(-3) = 0$, implying that $(x + 3)$ is a factor of $h(x)$. We therefore write

$$2x^3 + 8x^2 + 7x + 3 = (x + 3)(c_2x^2 + c_1x + c_0),$$

and, proceeding as previously, obtain $c_2 = 2$, $c_1 + 3c_2 = 8$, $c_0 + 3c_1 = 7$ and $3c_0 = 3$, with corresponding solution $c_2 = 2$, $c_1 = 2$ and $c_0 = 1$.

We now have that $g(x) = (x - 2)(x + 3)(2x^2 + 2x + 1)$. If we now try to determine the zeros of the quadratic term using the standard form (1.4) we find that, since $2^2 - (4 \times 2 \times 1)$, i.e. -4, is negative, its zeros are complex. In summary, the only real roots of $g(x) = 0$ are $x = 2$ and $x = -3$.

1.6 *Use the results of* (i) *equation (1.13),* (ii) *equation (1.12) and* (iii) *equation (1.14) to prove that if the roots of* $3x^3 - x^2 - 10x + 8 = 0$ *are* α_1, α_2 *and* α_3 *then*

(a) $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} = 5/4$,
(b) $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 61/9$,
(c) $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = -125/27$.
(d) *Convince yourself that eliminating (say)* α_2 *and* α_3 *from* (i), (ii) *and* (iii) *does* not *give a simple explicit way of finding* α_1.

If the roots of $3x^3 - x^2 - 10x + 8 = 0$ are α_1, α_2 and α_3, then:

(i) from equation (1.13), $\alpha_1 + \alpha_2 + \alpha_3 = -\frac{-1}{3} = \frac{1}{3}$;

(ii) from equation (1.12), $\alpha_1\alpha_2\alpha_3 = (-1)^3\frac{8}{3} = -\frac{8}{3}$;

(iii) from equation (1.14), $\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \frac{-10}{3} = -\frac{10}{3}$.

We now use these results in various combinations to obtain expressions for the given quantities:

(a) $\dfrac{1}{\alpha_1} + \dfrac{1}{\alpha_2} + \dfrac{1}{\alpha_3} = \dfrac{\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_2\alpha_1}{\alpha_1\alpha_2\alpha_3} = \dfrac{-(10/3)}{-(8/3)} = \dfrac{5}{4}$;

(b) $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (\alpha_1 + \alpha_2 + \alpha_3)^2 - 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)$

$= \left(\frac{1}{3}\right)^2 - 2\left(-\frac{10}{3}\right) = \frac{61}{9}$;

(c) $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 =$

$(\alpha_1 + \alpha_2 + \alpha_3)^3 - 3(\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) + 3\alpha_1\alpha_2\alpha_3$

$= (\frac{1}{3})^3 - 3(\frac{1}{3})(-\frac{10}{3}) + 3(-\frac{8}{3}) = -\frac{125}{27}$.

(d) No answer is given as it cannot be done. All manipulation is complicated and, at best, leads back to the original equation. Unfortunately, the ‘convincing’ will have to come from frustration, rather than from a proof by contradiction!

Trigonometric identities

1.8 *The following exercises are based on the half-angle formulae.*

(a) *Use the fact that* $\sin(\pi/6) = 1/2$ *to prove that* $\tan(\pi/12) = 2 - \sqrt{3}$.

(b) *Use the result of* (a) *to show further that* $\tan(\pi/24) = q(2 - q)$, *where* $q^2 = 2 + \sqrt{3}$.

(a) Writing $\tan(\pi/12)$ as t and using (1.32), we have

$$\frac{1}{2} = \sin\frac{\pi}{6} = \frac{2t}{1+t^2},$$

from which it follows that $t^2 - 4t + 1 = 0$.

The quadratic solution (1.6) then shows that $t = 2 \pm \sqrt{2^2 - 1} = 2 \pm \sqrt{3}$; there are two solutions because $\sin(5\pi/6)$ is also equal to 1/2. To resolve the ambiguity, we note that, since $\pi/12 < \pi/4$ and $\tan(\pi/4) = 1$, we must have $t < 1$; hence, the negative sign is the appropriate choice.

(b) Writing $\tan(\pi/24)$ as u and using (1.34) and the result of part (a), we have

$$2 - \sqrt{3} = \frac{2u}{1-u^2}.$$

Multiplying both sides by $q^2 = 2 + \sqrt{3}$, and then using $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$, gives

$$1 - u^2 = 2q^2u.$$

This quadratic equation has the (positive) solution

$$\begin{aligned} u &= -q^2 + \sqrt{q^4 + 1} \\ &= -q^2 + \sqrt{4 + 4\sqrt{3} + 3 + 1} \\ &= -q^2 + 2\sqrt{2 + \sqrt{3}} \\ &= -q^2 + 2q = q(2 - q), \end{aligned}$$

as stated in the question.

> ***1.10*** *If* $s = \sin(\pi/8)$, *prove that*
>
> $$8s^4 - 8s^2 + 1 = 0,$$
>
> *and hence show that* $s = [(2 - \sqrt{2})/4]^{1/2}$.

With $s = \sin(\pi/8)$, using (1.29) gives

$$\sin \frac{\pi}{4} = 2s(1 - s^2)^{1/2}.$$

Squaring both sides, and then using $\sin(\pi/4) = 1/\sqrt{2}$, leads to

$$\frac{1}{2} = 4s^2(1 - s^2),$$

i.e. $8s^4 - 8s^2 + 1 = 0$. This is a quadratic equation in $u = s^2$, with solutions

$$s^2 = u = \frac{8 \pm \sqrt{64 - 32}}{16} = \frac{2 \pm \sqrt{2}}{4}.$$

Since $\pi/8 < \pi/4$ and $\sin(\pi/4) = 1/\sqrt{2} = \sqrt{2/4}$, it is clear that the minus sign is the appropriate one. Taking the square root of both sides then yields the stated answer.

Coordinate geometry

> ***1.12*** *Obtain in the form (1.38), the equations that describe the following:*
>
> (a) *a circle of radius* 5 *with its centre at* $(1, -1)$;
> (b) *the line* $2x + 3y + 4 = 0$ *and the line orthogonal to it which passes through* $(1, 1)$;
> (c) *an ellipse of eccentricity* 0.6 *with centre* $(1, 1)$ *and its major axis of length* 10 *parallel to the* y*-axis.*

(a) Using (1.42) gives $(x-1)^2 + (y+1)^2 = 5^2$, i.e. $x^2 + y^2 - 2x + 2y - 23 = 0$.

(b) From (1.24), a line orthogonal to $2x + 3y + 4 = 0$ must have the form $3x - 2y + c = 0$, and, if it is to pass through $(1, 1)$, then $c = -1$. Expressed in the form (1.38), the pair of lines takes the form

$$0 = (2x + 3y + 4)(3x - 2y - 1) = 6x^2 - 6y^2 + 5xy + 10x - 11y - 4.$$

(c) As the major semi-axis has length 5 and the eccentricity is 0.6, the minor semi-axis has length $5[1 - (0.6)^2]^{1/2} = 4$. The equation of the ellipse is therefore

$$\frac{(x-1)^2}{4^2} + \frac{(y-1)^2}{5^2} = 1,$$

which can be written as $25x^2 + 16y^2 - 50x - 32y - 359 = 0$.

1.14 *For the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with eccentricity e, the two points $(-ae, 0)$ and $(ae, 0)$ are known as its foci. Show that the sum of the distances from any *point on the ellipse to the foci is $2a$.*

[*The constancy of the sum of the distances from two fixed points can be used as an alternative defining property of an ellipse.*]

Let the sum of the distances be s. Then, for a point (x, y) on the ellipse,

$$s = [\,(x + ae)^2 + y^2\,]^{1/2} + [\,(x - ae)^2 + y^2\,]^{1/2},$$

where the positive square roots are to be taken.

Now, $y^2 = b^2[1 - (x/a)^2]$, with $b^2 = a^2(1 - e^2)$. Thus, $y^2 = (1 - e^2)(a^2 - x^2)$ and

$$\begin{aligned} s &= (x^2 + 2aex + a^2e^2 + a^2 - a^2e^2 - x^2 + e^2x^2)^{1/2} \\ &\qquad + (x^2 - 2aex + a^2e^2 + a^2 - a^2e^2 - x^2 + e^2x^2)^{1/2} \\ &= (a + ex) + (a - ex) = 2a. \end{aligned}$$

This result is independent of x and hence holds for *any* point on the ellipse.

Partial fractions

1.16 *Express the following in partial fraction form:*

$$\text{(a)}\ \frac{2x^3 - 5x + 1}{x^2 - 2x - 8}, \qquad\qquad \text{(b)}\ \frac{x^2 + x - 1}{x^2 + x - 2}.$$

(a) For

$$f(x) = \frac{2x^3 - 5x + 1}{x^2 - 2x - 8},$$

we note that the degree of the numerator is higher than that of the denominator, and so we must first divide through by the latter. Write

$$2x^3 - 5x + 1 = (2x + s_0)(x^2 - 2x - 8) + (r_1 x + r_0).$$

Equating the coefficients of the powers of x: $0 = s_0 - 4$, $-5 = -16 - 2s_0 + r_1$, and $1 = -8s_0 + r_0$, giving $s_0 = 4$, $r_1 = 19$, and $r_0 = 33$. Thus,

$$f(x) = 2x + 4 + \frac{19x + 33}{x^2 - 2x - 8}.$$

The denominator in the final term factorises as $(x - 4)(x + 2)$, and so we write the term as

$$\frac{A}{x - 4} + \frac{B}{x + 2}.$$

Using the third method given in section 1.4:

$$A = \frac{19(4) + 33}{4 + 2} \quad \text{and} \quad B = \frac{19(-2) + 33}{-2 - 4}.$$

Thus,

$$f(x) = 2x + 4 + \frac{109}{6(x - 4)} + \frac{5}{6(x + 2)}.$$

(b) Since the highest powers of x in the denominator and numerator are equal, the partial–fraction expansion takes the form

$$f(x) = \frac{x^2 + x - 1}{x^2 + x - 2} = 1 + \frac{1}{x^2 + x - 2} = 1 + \frac{A}{x + 2} + \frac{B}{x - 1}.$$

Using the same method as above, we have

$$A = \frac{1}{-2 - 1}; \quad B = \frac{1}{1 + 2}.$$

Thus,

$$f(x) = 1 - \frac{1}{3(x + 2)} + \frac{1}{3(x - 1)}.$$

1.18 *Resolve the following into partial fractions in such a way that x does not appear in any numerator:*

$$\text{(a)}\ \frac{2x^2 + x + 1}{(x - 1)^2(x + 3)}, \qquad \text{(b)}\ \frac{x^2 - 2}{x^3 + 8x^2 + 16x}, \qquad \text{(c)}\ \frac{x^3 - x - 1}{(x + 3)^3(x + 1)}.$$

Since no factor x may appear in a numerator, all repeated factors appearing in the denominator give rise to as many terms in the partial fraction expansion as the power to which that factor is raised in the denominator.

(a) The denominator is already factorised but contains the repeated factor $(x-1)^2$. Thus the expansion will contain a term of the form $(x-1)^{-1}$, as well as one of the form $(x-1)^{-2}$. So,

$$\frac{2x^2+x+1}{(x-1)^2(x+3)} = \frac{A}{x+3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}.$$

We can evaluate A and B using the third method given in section 1.4:

$$A = \frac{2(-3)^2-3+1}{(-3-1)^2} = 1 \quad \text{and} \quad B = \frac{2(1)^2+1+1}{1+3} = 1.$$

We now evaluate C by setting $x = 0$ (say):

$$\frac{1}{(-1)^2 3} = \frac{1}{3} + \frac{1}{(-1)^2} + \frac{C}{-1},$$

giving $C = 1$ and the full expansion as

$$\frac{2x^2+x+1}{(x-1)^2(x+3)} = \frac{1}{x+3} + \frac{1}{(x-1)^2} + \frac{1}{x-1}.$$

(b) Here the denominator needs factorising, but this is elementary,

$$\frac{x^2-2}{x^3+8x^2+16x} = \frac{x^2-2}{x(x+4)^2} = \frac{A}{x} + \frac{B}{(x+4)^2} + \frac{C}{x+4}.$$

Now, using the same method as in part (a):

$$A = \frac{0-2}{(0+4)^2} = -\frac{1}{8} \quad \text{and} \quad B = \frac{(-4)^2-2}{-4} = -\frac{7}{2}.$$

Setting $x = 1$ (say) determines C through

$$\frac{-1}{25} = -\frac{1}{8(1)} - \frac{7}{2(5)^2} + \frac{C}{5}.$$

Thus $C = 9/8$, and the full expression is

$$\frac{x^2-2}{x^3+8x^2+16x} = -\frac{1}{8x} - \frac{7}{2(x+4)^2} + \frac{9}{8(x+4)}.$$

(c)

$$\frac{x^3-x-1}{(x+3)^3(x+1)} = \frac{A}{x+1} + \frac{B}{(x+3)^3} + \frac{C}{(x+3)^2} + \frac{D}{x+3}.$$

As in parts (a) and (b), the third method in section 1.4 gives A and B as

$$A = \frac{(-1)^3-(-1)-1}{(-1+3)^3} = -\frac{1}{8} \quad \text{and} \quad B = \frac{(-3)^3-(-3)-1}{-3+1} = \frac{25}{2}.$$

Setting $x = 0$ requires that

$$\frac{-1}{27} = -\frac{1}{8} + \frac{25}{54} + \frac{C}{9} + \frac{D}{3} \qquad \text{i.e. } C + 3D = -\frac{27}{8}.$$

Setting $x = 1$ gives the additional requirement that

$$\frac{-1}{128} = -\frac{1}{16} + \frac{25}{128} + \frac{C}{16} + \frac{D}{4} \qquad \text{i.e. } C + 4D = -\frac{18}{8}.$$

Solving these two equations for C and D now yields $D = 9/8$ and $C = -54/8$. Thus,

$$\frac{x^3 - x - 1}{(x+3)^3(x+1)} = -\frac{1}{8(x+1)} + \frac{1}{8}\left[\frac{100}{(x+3)^3} - \frac{54}{(x+3)^2} + \frac{9}{x+3}\right].$$

If necessary, that the expansion is valid for all x (and not just for 0 and 1) can be checked by writing all of its terms so as to have the common denominator $(x+3)^3(x+1)$.

Binomial expansion

> ***1.20*** *Use a binomial expansion to evaluate* $1/\sqrt{4.2}$ *to five places of decimals, and compare it with the accurate answer obtained using a calculator.*

To use the binomial expansion, we need to express the inverse square root in the form $(1 + a)^{-1/2}$ with $|a| < 1$. We do this as follows.

$$\begin{aligned}\frac{1}{\sqrt{4.2}} &= \frac{1}{(4 + 0.2)^{1/2}} = \frac{1}{2(1 + 0.05)^{1/2}} \\ &= \frac{1}{2}\left[1 - \frac{1}{2}(0.05) + \frac{3}{8}(0.05)^2 - \frac{15}{48}(0.05)^3 + \cdots\right] \\ &= 0.487949218.\end{aligned}$$

This four-term sum and the accurate value differ by about 8×10^{-7}.

Proof by induction and contradiction

> ***1.22*** *Prove by induction that*
>
> $$1 + r + r^2 + \cdots + r^k + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

To prove that

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r},$$

assume that the result is valid for $n = N$, and consider the corresponding sum for $n = N + 1$, which is the original sum plus one additional term:

$$\begin{aligned}
\sum_{k=0}^{N+1} r^k &= \sum_{k=0}^{N} r^k + r^{N+1} \\
&= \frac{1 - r^{N+1}}{1 - r} + r^{N+1}, \quad \text{using the assumption,} \\
&= \frac{1 - r^{N+1} + r^{N+1} - r^{N+2}}{1 - r} \\
&= \frac{1 - r^{N+2}}{1 - r}.
\end{aligned}$$

This is the same form as in the assumption, except that N has been replaced by $N + 1$, and shows that the result is valid for $n = N + 1$ if it is valid for $n = N$.

But, since $(1 - r)/(1 - r) = 1$, the result is trivially valid for $n = 0$. It therefore follows that it is valid for all n.

1.24 *If a sequence of terms u_n satisfies the recurrence relation $u_{n+1} = (1 - x)u_n + nx$, with $u_1 = 0$, then show by induction that, for $n \geq 1$,*

$$u_n = \frac{1}{x}[nx - 1 + (1 - x)^n].$$

Assume that the stated result is valid for $n = N$, and consider the expression for the next term in the sequence:

$$\begin{aligned}
u_{N+1} &= (1 - x)u_N + Nx \\
&= \frac{1 - x}{x}\left[Nx - 1 + (1 - x)^N\right] + Nx, \quad \text{using the assumption,} \\
&= \frac{1}{x}\left[Nx - Nx^2 - 1 + x + (1 - x)^{N+1} + Nx^2\right] \\
&= \frac{1}{x}\left[(N + 1)x - 1 + (1 - x)^{N+1}\right].
\end{aligned}$$

This has the same form as in the assumption, except that N has been replaced by $N + 1$, and shows that the result is valid for $n = N + 1$ if it is valid for $n = N$.

The assumed result gives u_1 as $x^{-1}(x - 1 + 1 - x) = 0$ (i.e. as stated in the question), and so is valid for $n = 1$. It now follows, from the result proved earlier, that the given expression is valid for all $n \geq 1$.

1.26 *The quantities a_i in this exercise are all positive real numbers.*

(a) *Show that*

$$a_1 a_2 \leq \left(\frac{a_1 + a_2}{2} \right)^2.$$

(b) *Hence, prove by induction on m that*

$$a_1 a_2 \cdots a_p \leq \left(\frac{a_1 + a_2 + \cdots + a_p}{p} \right)^p,$$

where $p = 2^m$ with m a positive integer. Note that each increase of m by unity doubles the number of factors in the product.

(a) Consider $(a_1 - a_2)^2$ which is always non-negative:

$$\begin{aligned}
(a_1 - a_2)^2 &\geq 0, \\
a_1^2 - 2a_1 a_2 + a_2^2 &\geq 0, \\
a_1^2 + 2a_1 a_2 + a_2^2 &\geq 4a_1 a_2, \\
(a_1 + a_2)^2 &\geq 4a_1 a_2, \\
\left(\frac{a_1 + a_2}{2} \right)^2 &\geq a_1 a_2.
\end{aligned}$$

(b) With $p = 2^m$, assume that

$$a_1 a_2 \cdots a_p \leq \left(\frac{a_1 + a_2 + \cdots + a_p}{p} \right)^p$$

is valid for some $m = M$. Write $P = 2^M$, $P' = 2P$, $b_1 = a_1 + a_2 + \cdots + a_P$ and $b_2 = a_{P+1} + a_{P+2} + \cdots + a_{P'}$. Note that both b_1 and b_2 consist of P terms.

Now consider the multiple product $u = a_1 a_2 \cdots a_P a_{P+1} a_{P+2} \cdots a_{P'}$.

$$\begin{aligned}
u &\leq \left(\frac{a_1 + a_2 + \cdots + a_P}{P} \right)^P \left(\frac{a_{P+1} + a_{P+2} + \cdots + a_{P'}}{P} \right)^P \\
&= \left(\frac{b_1 b_2}{P^2} \right)^P,
\end{aligned}$$

where the assumed result has been applied twice, once to a set consisting of the first P numbers, and then for a second time to the remaining set of P numbers, $a_{P+1}, a_{P+2}, \ldots, a_{P'}$. We have also used the fact that, for positive real numbers, if $q \leq r$ and $s \leq t$ then $qs \leq rt$.

But, from part (a),

$$b_1 b_2 \leq \left(\frac{b_1 + b_2}{2} \right)^2.$$

Thus,

$$\begin{aligned} a_1 a_2 \cdots a_P a_{P+1} a_{P+2} \cdots a_{P'} &\leq \left(\frac{1}{P^2}\right)^P \left(\frac{b_1+b_2}{2}\right)^{2P} \\ &= \frac{(b_1+b_2)^{P'}}{(2P)^{2P}} \\ &= \left(\frac{b_1+b_2}{P'}\right)^{P'}. \end{aligned}$$

This shows that the result is valid for $P' = 2^{M+1}$ if it is valid for $P = 2^M$. But for $m = M = 1$ the postulated inequality is simply result (a), which was shown directly. Thus the inequality holds for all positive integer values of m.

1.28 *An arithmetic progression of integers a_n is one in which $a_n = a_0 + nd$, where a_0 and d are integers and n takes successive values $0, 1, 2, \ldots$.*

(a) *Show that if any one term of the progression is the cube of an integer, then so are infinitely many others.*
(b) *Show that no cube of an integer can be expressed as $7n+5$ for some positive integer n.*

(a) We proceed by the method of contradiction. Suppose $d > 0$. Assume that there is a finite, but non-zero, number of natural cubes in the arithmetic progression. Then there must be a largest cube. Let it be $a_N = a_0 + Nd$, and write it as $a_N = a_0 + Nd = m^3$. Now consider $(m+d)^3$:

$$\begin{aligned} (m+d)^3 &= m^3 + 3dm^2 + 3d^2m + d^3 \\ &= a_0 + Nd + d(3m^2 + 3dm + d^2) \\ &= a_0 + dN_1, \end{aligned}$$

where $N_1 = N + 3m^2 + 3dm + d^2$ is necessarily an integer, since N, m and d all are. Further, $N_1 > N$. Thus $a_{N_1} = a_0 + N_1 d$ is also the cube of a natural number and is greater than a_N; this contradicts the assumption that it is possible to select a largest cube in the series and establishes the result that, if there is one such cube, then there are infinitely many of them. A similar argument (considering the smallest term in the series) can be carried through if $d < 0$.

We note that the result is also formally true in the case in which $d = 0$; if a_0 is a natural cube, then so is every term, since they are all equal to a_0.

(b) Again, we proceed by the method of contradiction. Suppose that $7N + 5 = m^3$

for some pair of positive integers N and m. Consider the quantity

$$\begin{aligned}(m-7)^3 &= m^3 - 21m^2 + 147m - 343 \\ &= 7N + 5 - 7(3m^2 - 21m + 49) \\ &= 7N_1 + 5,\end{aligned}$$

where $N_1 = N - 3m^2 + 21m - 49$ is an integer smaller than N. From this, it follows that if m^3 can be expressed in the form $7N + 5$ then so can $(m-7)^3$, $(m-14)^3$, etc. Further, for some finite integer p, $(m - 7p)$ must lie in the range $0 \le m - 7p \le 6$ and will have the property $(m - 7p)^3 = 7N_p + 5$.

However, explicit calculation shows that, when expressed in the form $7n + q$, the cubes of the integers 0, 1, 2, $\cdots$, 6 have respective values of q of 0, 1, 1, 6, 1, 6, 6; none of these is equal to 5. This contradicts the conclusion that followed from our initial supposition and subsequent argument. It was therefore wrong to assume that there is a natural cube that can be expressed in the form $7N + 5$.

[Note that it is not sufficient to carry out the above explicit calculations and then rely on the construct from part (a), as this does not guarantee to generate every cube.]

Necessary and sufficient conditions

> ***1.30*** *Prove that the equation* $ax^2 + bx + c = 0$*, in which* a*,* b *and* c *are real and* $a > 0$*, has two real distinct solutions IFF* $b^2 > 4ac$*.*

As is usual for IFF proofs, this answer will consist of two parts.

Firstly, assume that $b^2 > 4ac$. We can then write the equation as

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0,$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0,$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a} = \lambda^2.$$

Since $b^2 > 4ac$ and $a > 0$, λ is real, positive and non-zero. So, taking the square roots of both sides of the final equation gives

$$x = -\frac{b}{2a} \pm \frac{\lambda}{\sqrt{a}},$$

i.e. both roots are real and they are distinct; thus, the ‘if’ part of the proposition is established.

Now assume that both roots are real, α and β say, with $\alpha \neq \beta$. Then,

$$a\alpha^2 + b\alpha + c = 0,$$
$$a\beta^2 + b\beta + c = 0.$$

Subtraction of the two equations gives

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0 \quad \Rightarrow \quad b = -(\alpha + \beta)a, \text{ since } \alpha - \beta \neq 0.$$

Multiplying the first displayed equation by β and the second by α and then subtracting, gives

$$a(\alpha^2\beta - \beta^2\alpha) + c(\beta - \alpha) = 0 \quad \Rightarrow \quad c = \alpha\beta a, \text{ since } \alpha - \beta \neq 0.$$

Now, recalling that $\alpha \neq \beta$ and that $a > 0$, consider the inequality

$$\begin{aligned} 0 < (\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= \frac{b^2}{a^2} - 4\frac{c}{a} = \frac{b^2 - 4ac}{a^2}. \end{aligned}$$

This inequality shows that b^2 is necessarily greater than $4ac$, and so establishes the ‘only if’ part of the proof.

> ***1.32*** *Given that at least one of a and b, and at least one of c and d, are non-zero, show that ad = bc is both a necessary and sufficient condition for the equations*
>
> $$ax + by = 0,$$
> $$cx + dy = 0,$$
>
> *to have a solution in which at least one of x and y is non-zero.*

First, suppose that $ad = bc$ with at least one of a and b, and at least one of c and d, non-zero. Assume, for definiteness, that a and c are non-zero; if this is not the case, then the following proof is modified in an obvious way by interchanging the roles of a and b and/or of c and d, as necessary:

$$ax + by = 0 \Rightarrow x = -\frac{b}{a}y,$$
$$cx + dy = 0 \Rightarrow x = -\frac{d}{c}y.$$

Now

$$ad = bc \quad \Rightarrow \quad d = \frac{bc}{a} \quad \Rightarrow \quad \frac{d}{c} = \frac{b}{a},$$

where we have used, in turn, that $a \neq 0$ and $c \neq 0$. Thus the two solutions for x in terms of y are the same. Any non-zero value for y may be chosen, but that for x is then determined (and may be zero). This establishes that the condition is sufficient.

To show that it is a necessary condition, suppose that there is a non-trivial solution to the original equations and that, say, $x \neq 0$. Multiply the first equation by d and the second by b to obtain

$$dax + dby = 0,$$
$$bcx + bdy = 0.$$

Subtracting these equations gives $(ad - bc)x = 0$ and, since $x \neq 0$, it follows that $ad = bc$.

If $x = 0$ then $y \neq 0$, and multiplying the first of the original equations by c and the second by a leads to the same conclusion.

This completes the proof that the condition is both necessary and sufficient.

2

Preliminary calculus

> ***2.2*** *Find from first principles the first derivative of* $(x + 3)^2$ *and compare your answer with that obtained using the chain rule.*

Using the definition of a derivative, we consider the difference between $(x+\Delta x+3)^2$ and $(x + 3)^2$, and determine the following limit (if it exists):

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x + \Delta x + 3)^2 - (x + 3)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[(x + 3)^2 + 2(x + 3)\Delta x + (\Delta x)^2] - (x + 3)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(2(x + 3)\Delta x + (\Delta x)^2}{\Delta x} \\ &= 2x + 6. \end{aligned}$$

The limit does exist, and so the derivative is $2x + 6$.

Rewriting the function as $f(x) = u^2$, where $u(x) = x+3$, and using the chain rule:

$$f'(x) = 2u \times \frac{du}{dx} = 2u \times 1 = 2u = 2x + 6,$$

i.e. the same, as expected.

> ***2.4*** *Find the first derivatives of*
>
> (a) $x/(a + x)^2$, (b) $x/(1 - x)^{1/2}$, (c) $\tan x$, *as* $\sin x/\cos x$,
>
> (d) $(3x^2 + 2x + 1)/(8x^2 - 4x + 2)$.

In each case, using (2.13) for a quotient:

(a) $$f'(x) = \frac{[(a+x)^2 \times 1] - [x \times 2(a+x)]}{(a+x)^4} = \frac{a^2 - x^2}{(a+x)^4} = \frac{a-x}{(a+x)^3};$$

(b) $$f'(x) = \frac{[(1-x)^{1/2} \times 1] - [x \times -\frac{1}{2}(1-x)^{-1/2}]}{1-x} = \frac{1 - \frac{1}{2}x}{(1-x)^{3/2}};$$

(c) $$f'(x) = \frac{[\cos x \times \cos x] - [\sin x \times (-\sin x)]}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x;$$

(d) $$f'(x) = \frac{[(8x^2 - 4x + 2) \times (6x + 2)] - [(3x^2 + 2x + 1) \times (16x - 4)]}{(8x^2 - 4x + 2)^2}$$
$$= \frac{x^3(48 - 48) + x^2(16 - 24 + 12 - 32) + \cdots}{(8x^2 - 4x + 2)^2}$$
$$\frac{\cdots + x(-8 + 12 + 8 - 16) + (4 + 4)}{(8x^2 - 4x + 2)^2}$$
$$= \frac{-28x^2 - 4x + 8}{(8x^2 - 4x + 2)^2} = \frac{-7x^2 - x + 2}{(4x^2 - 2x + 1)^2}.$$

2.6 *Show that the function* $y(x) = \exp(-|x|)$ *defined as*

$$\begin{aligned} \exp x \quad & \textit{for } x < 0, \\ 1 \quad & \textit{for } x = 0, \\ \exp(-x) \quad & \textit{for } x > 0, \end{aligned}$$

is not *differentiable at* $x = 0$. *Consider the limiting process for both* $\Delta x > 0$ *and* $\Delta x < 0$.

For $x > 0$, let $\Delta x = \eta$. Then,

$$y'(x > 0) = \lim_{\eta \to 0} \frac{e^{-0-\eta} - 1}{\eta}$$
$$= \lim_{\eta \to 0} \frac{1 - \eta + \frac{1}{2!}\eta^2 \cdots - 1}{\eta} = -1.$$

For $x < 0$, let $\Delta x = -\eta$. Then,

$$y'(x > 0) = \lim_{\eta \to 0} \frac{e^{0-\eta} - 1}{-\eta}$$
$$= \lim_{\eta \to 0} \frac{1 - \eta + \frac{1}{2!}\eta^2 \cdots - 1}{-\eta} = 1.$$

The two limits are not equal, and so $y(x)$ is not differentiable at $x = 0$.

2.8 *If* $2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$, *show that* $dy/dx = 16$ *when* $x = 1$.

For this equation neither x nor y can be made the subject of the equation, i.e neither can be written explicitly as a function of the other, and so we are forced to use implicit differentiation. Starting from

$$2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$$

implicit differentiation, and the use of the chain rule when differentiating $\sin y$ with respect to x, gives

$$2\frac{dy}{dx} + \cos y \frac{dy}{dx} = 4x^3 + 12x^2.$$

When $x = 1$ the original equation reduces to $2y + \sin y = 2\pi$ with the obvious (and unique, as can be verified from a simple sketch) solution $y = \pi$. Thus, with $x = 1$ and $y = \pi$,

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{4 + 12}{2 + \cos \pi} = 16.$$

2.10 *The function* $y(x)$ *is defined by* $y(x) = (1 + x^m)^n$.

(a) *Use the chain rule to show that the first derivative of* y *is* $nmx^{m-1}(1+x^m)^{n-1}$.

(b) *The binomial expansion (see section 1.5) of* $(1 + z)^n$ *is*

$$(1 + z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!} z^r + \cdots .$$

Keeping only the terms of zeroth and first order in dx, *apply this result twice to derive result* (a) *from first principles.*

(c) *Expand* y *in a series of powers of* x *before differentiating term by term. Show that the result is the series obtained by expanding the answer given for* dy/dx *in part* (a).

(a) Writing $1 + x^m$ as u, $y(x) = u^n$, and so $dy/du = nu^{n-1}$, whilst $du/dx = mx^{m-1}$. Thus, from the chain rule,

$$\frac{dy}{dx} = nu^{n-1} \times mx^{m-1} = nmx^{m-1}(1 + x^m)^{n-1}.$$

(b) From the defining process for a derivative,

$$
\begin{aligned}
y'(x) &= \lim_{\Delta x\to 0} \frac{[1+(x+\Delta x)^m]^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{[1+x^m(1+\frac{\Delta x}{x})^m]^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{[1+x^m(1+\frac{m\Delta x}{x}+\cdots)]^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{(1+x^m+mx^{m-1}\Delta x+\cdots)^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\left[(1+x^m)\left(1+\frac{mx^{m-1}\Delta x}{1+x^m}+\cdots\right)\right]^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{(1+x^m)^n\left(1+\frac{mnx^{m-1}\Delta x}{1+x^m}+\cdots\right) - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{mn(1+x^m)^{n-1}x^{m-1}\Delta x+\cdots}{\Delta x} \\
&= nmx^{m-1}(1+x^m)^{n-1},
\end{aligned}
$$

i.e. the same as the result in part (a).

(c) Expanding in a power series before differentiating:

$$
\begin{aligned}
y(x) &= 1 + nx^m + \frac{n(n-1)}{2!}x^{2m} + \cdots \\
&\quad + \frac{n(n-1)\cdots(n-r+1)}{r!}x^{rm} + \cdots, \\
y'(x) &= m\,nx^{m-1} + \frac{2m\,n(n-1)}{2!}x^{2m-1} + \cdots \\
&\quad + \frac{rm\,n(n-1)\cdots(n-r+1)}{r!}x^{rm-1} + \cdots.
\end{aligned}
$$

Now, expanding the result given in part (a) gives

$$
\begin{aligned}
y'(x) &= nmx^{m-1}(1+x^m)^{n-1} \\
&= nmx^{m-1}\left(1+\cdots+\frac{(n-1)(n-2)\cdots(n-s)}{s!}x^{ms}+\cdots\right) \\
&= nmx^{m-1}+\cdots+\frac{mn(n-1)(n-2)\cdots(n-s)}{s!}x^{ms+m-1}+\cdots.
\end{aligned}
$$

This is the same as the previous expansion of $y'(x)$ if, in the general term, the index is moved by one, i.e. $s = r - 1$.

> ***2.12*** *Find the positions and natures of the stationary points of the following functions:*
>
> (a) $x^3 - 3x + 3$; (b) $x^3 - 3x^2 + 3x$; (c) $x^3 + 3x + 3$;
> (d) $\sin ax$ *with* $a \neq 0$; (e) $x^5 + x^3$; (f) $x^5 - x^3$.

In each case, we need to determine the first and second derivatives of the function. The zeros of the 1st derivative give the positions of the stationary points, and the values of the 2nd derivatives at those points determine their natures.

(a) $$y = x^3 - 3x + 3; \quad y' = 3x^2 - 3; \quad y'' = 6x.$$

$y' = 0$ has roots at $x = \pm 1$, where the values of y'' are ± 6. Therefore, there is a minimum at $x = 1$ and a maximum at $x = -1$.

(b) $$y = x^3 - 3x^2 + 3x; \quad y' = 3x^2 - 6x + 3; \quad y'' = 6x - 6.$$

$y' = 0$ has a double root at $x = 1$, where the value of y'' is 0. Therefore, there is a point of inflection at $x = 1$, but no other stationary points. At the point of inflection, the tangent to the curve $y = y(x)$ is horizontal.

(c) $$y = x^3 + 3x + 3; \quad y' = 3x^2 + 3; \quad y'' = 6x.$$

$y' = 0$ has no real roots, and so there are no stationary points.

(d) $$y = \sin ax; \quad y' = a \cos ax; \quad y'' = -a^2 \sin ax.$$

$y' = 0$ has roots at $x = (n + \frac{1}{2})\pi/a$ for integer n. The corresponding values of y'' are $\mp a^2$, depending on whether n is even or odd. Therefore, there is a maximum for even n and a minimum where n is odd.

(e) $$y = x^5 + x^3; \quad y' = 5x^4 + 3x^2; \quad y'' = 20x^3 + 6x.$$

$y' = 0$ has, as its only real root, a double root at $x = 0$, where the value of y'' is 0. Thus, there is a (horizontal) point of inflection at $x = 0$, but no other stationary point.

(f) $$y = x^5 - x^3; \quad y' = 5x^4 - 3x^2; \quad y'' = 20x^3 - 6x.$$

$y' = 0$ has a double root at $x = 0$ and simple roots at $x = \pm(\frac{3}{5})^{1/2}$, where the respective values of y'' are 0 and $\pm 6(\frac{3}{5})^{1/2}$. Therefore, there is a point of inflection at $x = 0$, a maximum at $x = -(\frac{3}{5})^{1/2}$ and a minimum at $x = (\frac{3}{5})^{1/2}$.

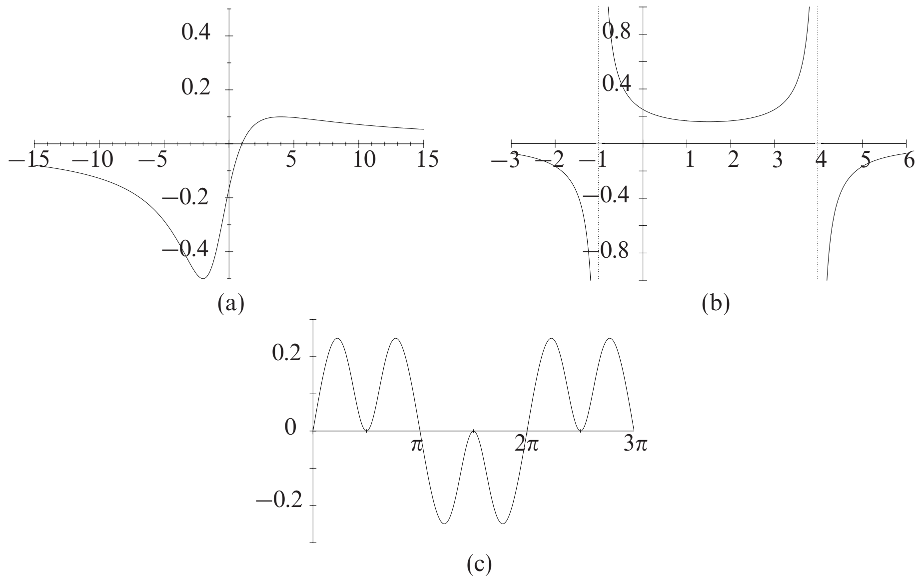

Figure 2.1 The solutions to exercise 2.14.

***2.14** By finding their stationary points and examining their general forms, determine the range of values that each of the following functions $y(x)$ can take. In each case make a sketch-graph incorporating the features you have identified.*

(a) $y(x) = (x-1)/(x^2+2x+6)$.
(b) $y(x) = 1/(4+3x-x^2)$.
(c) $y(x) = (8\sin x)/(15+8\tan^2 x)$.

See figure 2.1 (a)–(c).

(a) Some simple points to calculate for

$$y = \frac{x-1}{x^2+2x+6}$$

are $y(0) = -\frac{1}{6}$, $y(1) = 0$ and $y(\pm\infty) = 0$, and, since the denominator has no real roots ($2^2 < 4 \times 1 \times 6$), there are no infinities. Its 1st derivative is

$$y' = \frac{-x^2+2x+8}{(x^2+2x+6)^2} = \frac{-(x+2)(x-4)}{(x^2+2x+6)^2}.$$

Thus there are turning points only at $x = -2$, with $y(-2) = -\frac{1}{2}$, and at $x = 4$, with $y(4) = \frac{1}{10}$. The former must be a minimum and the latter a maximum. The range in which $y(x)$ lies is $-\frac{1}{2} \le y \le \frac{1}{10}$.

(b) Some simple points to calculate for

$$y = \frac{1}{4 + 3x - x^2}.$$

are $y(0) = \frac{1}{4}$ and $y(\pm\infty) = 0$, approached from negative values. Since the denominator can be written as $(4-x)(1+x)$, the function has infinities at $x = -1$ and $x = 4$ and is positive in the range of x between them.

The 1st derivative is

$$y' = \frac{2x - 3}{(4 + 3x - x^2)^2}.$$

Thus there is only one turning point; this is at $x = \frac{3}{2}$, with corresponding $y(\frac{3}{2}) = \frac{4}{25}$. Since $\frac{3}{2}$ lies in the range $-1 < x < 4$, at the ends of which the function $\to +\infty$, the stationary point must be a minimum. This sets a lower limit on the positive values of $y(x)$ and so the ranges in which it lies are $y < 0$ and $y \geq \frac{4}{25}$.

(c) The function

$$y = \frac{8 \sin x}{15 + 8 \tan^2 x}$$

is clearly periodic with period 2π.

Since $\sin x$ and $\tan^2 x$ are both symmetric about $x = \frac{1}{2}\pi$, so is the function. Also, since $\sin x$ is antisymmetric about $x = \pi$ whilst $\tan^2 x$ is symmetric, the function is antisymmetric about $x = \pi$.

Some simple points to calculate are $y(n\pi) = 0$ for all integers n. Further, since $\tan(n + \frac{1}{2})\pi = \infty$, $y((n + \frac{1}{2})\pi) = 0$. As the denominator has no real roots there are no infinities.

Setting the derivative of $y(x) \equiv 8u(x)/v(x)$ equal to zero, i.e. writing $vu' = uv'$, and expressing all terms as powers of $\cos x$ gives (using $\tan^2 z = \sec^2 z - 1$ and $\sin^2 z = 1 - \cos^2 z$)

$$\begin{aligned}
(15 + 8 \tan^2 x) \cos x &= 16 \sin x \tan x \sec^2 x, \\
15 + \frac{8}{\cos^2 x} - 8 &= \frac{16(1 - \cos^2 x)}{\cos^4 x}, \\
7 \cos^4 x + 24 \cos^2 x - 16 &= 0.
\end{aligned}$$

This quadratic equation for $\cos^2 x$ has roots of $\frac{4}{7}$ and -4. Only the first of these gives real values for $\cos x$ of $\pm\frac{2}{\sqrt{7}}$. The corresponding turning values of $y(x)$ are $\pm\frac{8}{7\sqrt{21}}$. The value of y always lies between these two limits.

2.16 *The curve* $4y^3 = a^2(x + 3y)$ *can be parameterised as* $x = a\cos 3\theta$, $y = a\cos\theta$.

(a) *Obtain expressions for* dy/dx (i) *by implicit differentiation and* (ii) *in parameterised form. Verify that they are equivalent.*
(b) *Show that the only point of inflection occurs at the origin. Is it a stationary point of inflection?*
(c) *Use the information gained in* (a) *and* (b) *to sketch the curve, paying particular attention to its shape near the points* $(-a, a/2)$ *and* $(a, -a/2)$ *and to its slope at the 'end points'* (a, a) *and* $(-a, -a)$.

(a) (i) Differentiating the equation of the curve implicitly:

$$12y^2\frac{dy}{dx} = a^2 + 3a^2\frac{dy}{dx}, \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a^2}{12y^2 - 3a^2}.$$

(ii) In parameterised form:

$$\frac{dy}{d\theta} = -a\sin\theta, \quad \frac{dx}{d\theta} = -3a\sin 3\theta, \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-a\sin\theta}{-3a\sin 3\theta}.$$

But, using the results from section 1.2, we have that

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &= 2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta\\ &= \sin\theta(4\cos^2\theta - 1),\end{aligned}$$

thus giving dy/dx as

$$\frac{dy}{dx} = \frac{1}{12\cos^2\theta - 3} = \frac{a^2}{12a^2\cos^2\theta - 3a^2},$$

with $a\cos\theta = y$, i.e. as in (i).

(b) At a point of inflection $y'' = 0$. For the given function,

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right) \times \frac{dy}{dx} = -\frac{a^2}{(12y^2 - 3a^2)^2} \times 24y \times \frac{a^2}{12y^2 - 3a^2}.$$

This can only equal zero at $y = 0$, when $x = 0$ also. But, when $y = 0$ it follows from (a)(i) that $dy/dx = 1/(-3) = -\frac{1}{3}$. As this is non-zero the point of inflection is not a stationary point.

(c) See figure 2.2. Note in particular that the curve has vertical tangents when $y = \pm a/2$ and that $dy/dx = \frac{1}{9}$ at $y = \pm a$, i.e. the tangents at the end points of the 'S'-shaped curve are not horizontal.

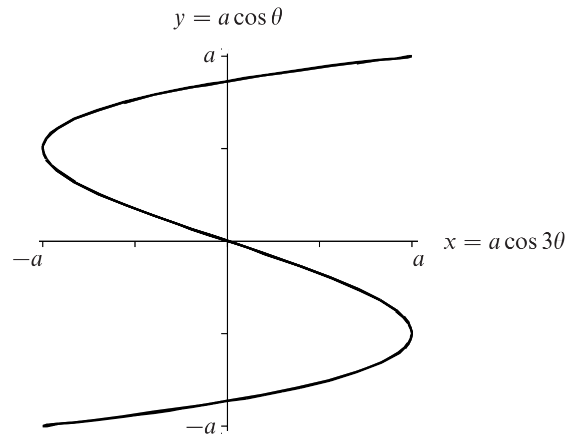

Figure 2.2 The parametric curve described in exercise 2.16.

***2.18** Show that the maximum curvature on the catenary $y(x) = a\cosh(x/a)$ is $1/a$. You will need some of the results about hyperbolic functions stated in subsection 3.7.6.*

The general expression for the curvature, ρ^{-1}, of the curve $y = y(x)$ is

$$\frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{3/2}},$$

and so we begin by calculating the first two derivatives of y. Starting from $y = a\cosh(x/a)$, we obtain

$$y' = a\frac{1}{a}\,\sinh\frac{x}{a},$$
$$y'' = \frac{1}{a}\,\cosh\frac{x}{a}.$$

Therefore the curvature of the catenary at the point (x, y) is given by

$$\frac{1}{\rho} = \frac{\dfrac{1}{a}\,\cosh\dfrac{x}{a}}{\left[1 + \sinh^2\dfrac{x}{a}\right]^{3/2}} = \frac{1}{a}\,\frac{\cosh\dfrac{x}{a}}{\cosh^3\dfrac{x}{a}} = \frac{a}{y^2}.$$

To obtain this result we have used the identity $\cosh^2 z = 1 + \sinh^2 z$. We see that the curvature is maximal when y is minimal; this occurs when $x = 0$ and $y = a$. The maximum curvature is therefore $1/a$.

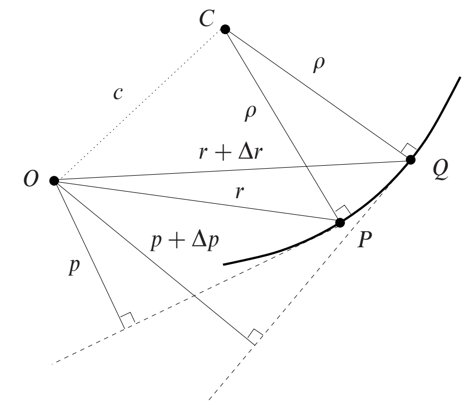

Figure 2.3 The coordinate system described in exercise 2.20.

2.20 *A two-dimensional coordinate system useful for orbit problems is the tangential-polar coordinate system (figure 2.3). In this system a curve is defined by* r*, the distance from a fixed point* O *to a general point* P *of the curve, and* p*, the perpendicular distance from* O *to the tangent to the curve at* P*. By proceeding as indicated below, show that the radius of curvature at* P *can be written in the form* $\rho = r\,dr/dp$*.*

Consider two neighbouring points P *and* Q *on the curve. The normals to the curve through those points meet at* C*, with (in the limit* $Q \to P$*)* $CP = CQ = \rho$*. Apply the cosine rule to triangles* OPC *and* OQC *to obtain two expressions for* c^2*, one in terms of* r *and* p *and the other in terms of* $r+\Delta r$ *and* $p+\Delta p$*. By equating them and letting* $Q \to P$ *deduce the stated result.*

We first note that cos OPC is equal to the sine of the angle between OP and the tangent at P, and that this in turn has the value p/r. Now, applying the cosine rule to the triangles OCP and OCQ, we have

$$\begin{aligned} c^2 &= r^2 + \rho^2 - 2r\rho\cos OPC = r^2 + \rho^2 - 2\rho p \\ c^2 &= (r+\Delta r)^2 + \rho^2 - 2(r+\Delta r)\rho\cos OQC \\ &= (r+\Delta r)^2 + \rho^2 - 2\rho(p+\Delta p). \end{aligned}$$

Subtracting and rearranging then yields

$$\rho = \frac{r\Delta r + \frac{1}{2}(\Delta r)^2}{\Delta p},$$

or, in the limit $Q \to P$, that $\rho = r(dr/dp)$.

2.22 *If* $y = \exp(-x^2)$*, show that* $dy/dx = -2xy$ *and hence, by applying Leibnitz' theorem, prove that for* $n \geq 1$

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0.$$

With $y(x) = \exp(-x^2)$,

$$\frac{dy}{dx} = -2x\exp(-x^2) = -2xy.$$

We now take the nth derivatives of both sides and use Leibnitz' theorem to find that of the product xy, noting that all derivatives of x beyond the first are zero:

$$y^{(n+1)} = -2[\,(y^{(n)})(x) + n(y^{(n-1)})(1) + 0\,].$$

i.e.

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0,$$

as stated in the question.

2.24 *Determine what can be learned from applying Rolle's theorem to the following functions* $f(x)$*:* (a)e^x*;* (b)$x^2 + 6x$*;* (c)$2x^2 + 3x + 1$*;* (d)$2x^2 + 3x + 2$*;* (e)$2x^3 - 21x^2 + 60x + k$. (f)*If* $k = -45$ *in* (e)*, show that* $x = 3$ *is one root of* $f(x) = 0$*, find the other roots, and verify that the conclusions from* (e) *are satisfied.*

(a) Since the derivative of $f(x) = e^x$ is $f'(x) = e^x$, Rolle's theorem states that between any two consecutive roots of $f(x) = e^x = 0$ there must be a root of $f'(x) = e^x = 0$, i.e. another root of the same equation. This is clearly a contradiction and it is wrong to suppose that there is more than one root of $e^x = 0$. In fact, there are no finite roots of the equation and the only zero of e^x lies formally at $x = -\infty$.

(b) Since $f(x) = x(x+6)$, it has zeros at $x = -6$ and $x = 0$. Therefore the (only) root of $f'(x) = 2x + 6 = 0$ must lie between these values; it clearly does, as $-6 < -3 < 0$.

(c) With $f(x) = 2x^2 + 3x + 1$ and hence $f'(x) = 4x + 3$, any roots of $f(x) = 0$ (actually -1 and $-\frac{1}{2}$) must lie on either side of the root of $f'(x) = 0$, i.e. $x = -\frac{3}{4}$. They clearly do.

(d) This is as in (c), but there are no real roots. However, it can be more generally stated that if there are two values of x that give $2x^2 + 3x + k$ equal values then they lie one on each side of $x = -\frac{3}{4}$.

(e) With $f(x) = 2x^3 - 21x^2 + 60x + k$,

$$f'(x) = 6x^2 - 42x + 60 = 6(x-5)(x-2)$$

and $f'(x) = 0$ has roots 2 and 5. Therefore, if $f(x) = 0$ has three real roots α_i with $\alpha_1 < \alpha_2 < \alpha_3$, then $\alpha_1 < 2 < \alpha_2 < 5 < \alpha_3$.

(f) When $k = -45$, $f(3) = 54 - 189 + 180 - 45 = 0$ and so $x = 3$ is a root of $f(x) = 0$ and $(x-3)$ is a factor of $f(x)$. Writing $f(x) = 2x^3 - 21x^2 + 60x - 45$ as $(x-3)(a_2x^2 + a_1x + a_0)$ and equating coefficients gives $a_2 = 2$, $a_1 = -15$ and $a_0 = 15$. The other two roots are therefore

$$\frac{15 \pm \sqrt{225 - 120}}{4} = \frac{1}{4}(15 \pm \sqrt{105}) = 1.19 \text{ or } 6.31.$$

Result (e) is verified in this case since $1.19 < 2 < 3 < 5 < 6.31$.

2.26 *Use the mean value theorem to establish bounds*

(a) *for* $-\ln(1-y)$*, by considering* $\ln x$ *in the range* $0 < 1 - y < x < 1$*,*
(b) *for* $e^y - 1$*, by considering* $e^x - 1$ *in the range* $0 < x < y$*.*

(a) The mean value theorem applied to $\ln x$ within limits $1 - y$ and 1 gives

$$\frac{\ln(1) - \ln(1-y)}{1 - (1-y)} = \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad (*)$$

for some x in the range $1 - y < x < 1$. Now, since $1 - y < x < 1$ it follows that

$$\begin{aligned} & \frac{1}{1-y} > \frac{1}{x} > 1, \\ \Rightarrow\ & \frac{1}{1-y} > \frac{-\ln(1-y)}{y} > 1, \\ \Rightarrow\ & \frac{y}{1-y} > -\ln(1-y) > y. \end{aligned}$$

The second line was obtained by substitution from $(*)$.

(b) The mean value theorem applied to $e^x - 1$ within limits 0 and y gives

$$\frac{e^y - 1 - 0}{y - 0} = e^x \quad \text{for some } x \text{ in the range } 0 < x < y.$$

Now, since $0 < x < y$ it follows that

$$\begin{aligned} & 1 < e^x < e^y, \\ \Rightarrow\ & 1 < \frac{e^y - 1}{y} < e^y, \\ \Rightarrow\ & y < e^y - 1 < ye^y. \end{aligned}$$

Again, the second line was obtained by substitution for x from the mean value theorem result.

> **2.28** *Use Rolle's theorem to deduce that if the equation $f(x) = 0$ has a repeated root x_1 then x_1 is also a root of the equation $f'(x) = 0$.*
>
> (a) *Apply this result to the 'standard' quadratic equation $ax^2 + bx + c = 0$, to show that a necessary condition for equal roots is $b^2 = 4ac$.*
> (b) *Find all the roots of $f(x) = x^3 + 4x^2 - 3x - 18 = 0$, given that one of them is a repeated root.*
> (c) *The equation $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2 = 0$ has a repeated integer root. How many real roots does it have altogether?*

If two roots of $f(x) = 0$ are x_1 and x_2, i.e. $f(x_1) = f(x_2) = 0$, then it follows from Rolle's theorem that there is some x_3 in the range $x_1 \le x_3 \le x_2$ for which $f'(x_3) = 0$. Now let $x_2 \to x_1$ to form the repeated root; x_3 must also tend to the limit x_1, i.e. x_1 is a root of $f'(x) = 0$ as well as of $f(x) = 0$.

(a) A quadratic equation $f(x) = ax^2 + bx + c = 0$ only has two roots and so if they are equal the common root α must also be a root of $f'(x) = 2ax + b = 0$, i.e. $\alpha = -b/2a$. Thus

$$a\frac{b^2}{4a^2} + b\frac{-b}{2a} + c = 0.$$

It then follows that $c - (b^2/4a) = 0$ and that $b^2 = 4ac$.

(b) With $f(x) = x^3 + 4x^2 - 3x - 18$, the repeated root must satisfy

$$f'(x) = 3x^2 + 8x - 3 = (3x - 1)(x + 3) = 0 \quad \text{i.e. } x = \frac{1}{3} \text{ or } x = -3.$$

Trying the two possibilities: $f(\frac{1}{3}) \neq 0$ but $f(-3) = -27 + 36 + 9 - 18 = 0$. Thus $f(x)$ must factorise as $(x + 3)^2(x - b)$, and comparing the constant terms in the two expressions for $f(x)$ immediately gives $b = 2$. Hence, $x = 2$ is the third root.

(c) Here $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2$. As previously, we examine $f'(x) = 0$, i.e. $f'(x) = 4x^3 + 12x^2 + 14x + 6 = 0$. This has to have an integer solution and, by inspection, this is $x = -1$. We can therefore factorise $f(x)$ as the product $(x + 1)^2(a_2x^2 + a_1x + a_0)$. Comparison of the coefficients gives immediately that $a_2 = 1$ and $a_0 = 2$. From the coefficients of x^3 we have $2a_2 + a_1 = 4$; hence $a_1 = 2$. Thus $f(x)$ can be written

$$f(x) = (x + 1)^2(x^2 + 2x + 2) = (x + 1)^2[\,(x + 1)^2 + 1\,].$$

The second factor, containing only positive terms, can have no real zeros and hence $f(x) = 0$ has only two real roots (coincident at $x = -1$).

> **2.30** *Find the following indefinite integrals:*
>
> (a) $\int (4 + x^2)^{-1}\, dx$; (b) $\int (8 + 2x - x^2)^{-1/2}\, dx$ for $2 \le x \le 4$;
> (c) $\int (1 + \sin\theta)^{-1}\, d\theta$; (d) $\int (x\sqrt{1 - x})^{-1}\, dx$ for $0 < x \le 1$.

We make reference to the 12 standard forms given in subsection 2.2.3 and, where relevant, select the appropriate model.

(a) Using model 9,

$$\int \frac{1}{4 + x^2}\, dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + c.$$

(b) We rearrange the integrand in the form of model 12:

$$\int \frac{1}{\sqrt{8 + 2x - x^2}}\, dx = \int \frac{1}{\sqrt{8 + 1 - (x - 1)^2}}\, dx = \sin^{-1}\frac{x - 1}{3} + c.$$

(c) See equation (2.35) and the subsequent text.

$$\begin{aligned}\int \frac{1}{1 + \sin\theta}\, d\theta &= \int \frac{1}{1 + \dfrac{2t}{1 + t^2}}\,\frac{2}{1 + t^2}\, dt \\ &= \int \frac{2}{(1 + t)^2}\, dt \\ &= -\frac{2}{1 + t} + c \\ &= -\frac{2}{1 + \tan\dfrac{\theta}{2}} + c.\end{aligned}$$

(d) To remove the square root, set $u^2 = 1 - x$; then $2u\, du = -dx$ and

$$\begin{aligned}\int \frac{1}{x\sqrt{1 - x}}\, dx &= \int \frac{1}{(1 - u^2)u} \times -2u\, du \\ &= \int \frac{-2}{1 - u^2}\, du \\ &= \int \left(\frac{-1}{1 - u} + \frac{-1}{1 + u}\right) du \\ &= \ln(1 - u) - \ln(1 + u) + c \\ &= \ln\frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} + c.\end{aligned}$$

2.32 *Express $x^2(ax+b)^{-1}$ as the sum of powers of x and another integrable term, and hence evaluate*

$$\int_0^{b/a} \frac{x^2}{ax+b}\,dx.$$

We need to write the numerator in such a way that every term in it that involves x contains a factor $ax + b$. Therefore, write x^2 as

$$x^2 = \frac{x}{a}(ax+b) - \frac{b}{a^2}(ax+b) + \frac{b^2}{a^2}.$$

Then,

$$\begin{aligned}\int_0^{b/a} \frac{x^2}{ax+b}\,dx &= \int_0^{b/a} \left(\frac{x}{a} - \frac{b}{a^2} + \frac{b^2}{a^2(ax+b)}\right) dx \\ &= \left[\frac{x^2}{2a} - \frac{bx}{a^2} + \frac{b^2}{a^3}\ln(ax+b)\right]_0^{b/a} \\ &= \frac{b^2}{a^3}\left(\ln 2 - \frac{1}{2}\right).\end{aligned}$$

An alternative approach, consistent with the wording of the question, is to use the binomial theorem to write the integrand as

$$\frac{x^2}{ax+b} = \frac{x^2}{b}\left(1 + \frac{ax}{b}\right)^{-1} = \frac{x^2}{b}\sum_{n=0}^{\infty}\left(-\frac{ax}{b}\right)^n.$$

Then the integral is

$$\begin{aligned}\int_0^{b/a} \frac{x^2}{ax+b}\,dx &= \frac{1}{b}\int_0^{b/a} \sum_{n=0}^{\infty}(-1)^n \left(\frac{a}{b}\right)^n x^{n+2}\,dx \\ &= \frac{1}{b}\sum_{n=0}^{\infty}(-1)^n \left(\frac{a}{b}\right)^n \frac{1}{n+3}\left(\frac{b}{a}\right)^{n+3} \\ &= \frac{b^2}{a^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+3}.\end{aligned}$$

That these two solutions are the same can be seen by writing $\ln 2 - \frac{1}{2}$ as

$$\begin{aligned}\ln 2 - \frac{1}{2} &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right) - \frac{1}{2} \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+3}.\end{aligned}$$

***2.34** Use logarithmic integration to find the indefinite integrals J of the following:*

(a) $\sin 2x/(1 + 4\sin^2 x)$;
(b) $e^x/(e^x - e^{-x})$;
(c) $(1 + x \ln x)/(x \ln x)$;
(d) $[x(x^n + a^n)]^{-1}$.

To use logarithmic integration each integrand needs to be arranged as a fraction that has the derivative of the denominator appearing in the numerator.

(a) Either by noting that $\sin 2x = 2 \sin x \cos x$ and so is proportional to the derivative of $\sin^2 x$ or by recognising that $\sin^2 x$ can be written in terms of $\cos 2x$ and constants and that $\sin 2x$ is then its derivative, we have

$$\begin{aligned} J &= \int \frac{\sin 2x}{1 + 4\sin^2 x}\, dx \\ &= \int \frac{2 \sin x \cos x}{1 + 4\sin^2 x}\, dx = \frac{1}{4} \ln(1 + 4\sin^2 x) + c, \end{aligned}$$

or

$$J = \int \frac{\sin 2x}{1 + 2(1 - \cos 2x)}\, dx = \frac{1}{4} \ln(3 - 2\cos 2x) + c.$$

These two answers are equivalent since $3 - 2\cos 2x = 3 - 2(1 - 2\sin^2 x) = 1 + 4\sin^2 x$.

(b) This is straightforward if it is noticed that multiplying both numerator and denominator by e^x produces the required form:

$$J = \int \frac{e^x}{e^x - e^{-x}}\, dx = \int \frac{e^{2x}}{e^{2x} - 1}\, dx = \frac{1}{2} \ln(e^{2x} - 1) + c.$$

An alternative, but longer, method is to write the numerator as $\cosh x + \sinh x$ and the denominator as $2 \sinh x$. This leads to $J = \frac{1}{2}(x + \ln \sinh x)$, which can be re-written as

$$J = \tfrac{1}{2}(\ln e^x + \ln \sinh x) = \tfrac{1}{2} \ln(e^x \sinh x) = \tfrac{1}{2} \ln(e^{2x} - 1) + \tfrac{1}{2} \ln \tfrac{1}{2}.$$

The $\frac{1}{2} \ln \frac{1}{2}$ forms part of c.

(c) Here we must first divide the numerator by the denominator to produce two separate terms, and then twice apply the result that $1/z$ is the derivative of $\ln z$:

$$J = \int \frac{1 + x \ln x}{x \ln x}\, dx = \int \left(\frac{1}{x \ln x} + 1 \right) dx = \ln(\ln x) + x + c.$$

(d) To put the integrand in a form suitable for logaritmic integration, we must first multiply both numerator and denominator by nx^{n-1} and then use partial fractions so that each denominator contains x only in the form x^m, of which mx^{m-1} is the derivative.

$$\begin{aligned}
J &= \int \frac{dx}{x(x^n + a^n)} = \int \frac{nx^{n-1}}{nx^n(x^n + a^n)}\, dx \\
&= \frac{1}{na^n} \int \left(\frac{nx^{n-1}}{x^n} - \frac{nx^{n-1}}{x^n + a^n} \right) dx \\
&= \frac{1}{na^n} \left[n \ln x - \ln(x^n + a^n) \right] + c \\
&= \frac{1}{na^n} \ln \left(\frac{x^n}{x^n + a^n} \right) + c.
\end{aligned}$$

2.36 *Find the indefinite integrals J of the following functions involving sinusoids:*

(a) $\cos^5 x - \cos^3 x$;
(b) $(1 - \cos x)/(1 + \cos x)$;
(c) $\cos x \sin x/(1 + \cos x)$;
(d) $\sec^2 x/(1 - \tan^2 x)$.

(a) As the integrand contains only odd powers of $\cos x$, take $\cos x$ out as a common factor and express the remainder in terms of $\sin x$, of which $\cos x$ is the derivative:

$$\begin{aligned}
\cos^5 x - \cos^3 x &= \left[(1 - \sin^2 x)^2 - (1 - \sin^2 x) \right] \cos x \\
&= (\sin^4 x - \sin^2 x) \cos x.
\end{aligned}$$

Hence,

$$J = \int (\sin^4 x - \sin^2 x) \cos x \, dx = \frac{1}{5} \sin^5 x - \frac{1}{3} \sin^3 x + c.$$

A more formal way of expressing this approach is to say 'set $\sin x = u$ with $\cos x \, dx = du$.'

(b) This integral can be found either by writing the numerator and denominator in terms of sinusoidal functions of $x/2$ or by making the substitution $t = \tan(x/2)$. Using first the half-angle identities, we have

$$\begin{aligned}
J &= \int \frac{1 - \cos x}{1 + \cos x}\, dx = \int \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \\
&= \int \tan^2 \frac{x}{2}\, dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx = 2 \tan \frac{x}{2} - x + c.
\end{aligned}$$

The second approach (see subsection 2.2.7) is

$$\begin{aligned}
J &= \int \frac{1 - \dfrac{1-t^2}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}} \, \frac{2\,dt}{1+t^2} \\
&= \int \frac{2t^2}{1+t^2}\, dt \\
&= \int 2\,dt - \int \frac{2}{1+t^2}\, dt \\
&= 2t - 2\tan^{-1} t + c = 2\tan\frac{x}{2} - x + c.
\end{aligned}$$

(c) This integrand, containing only sinusoidal functions, can be converted to an algebraic one by writing $t = \tan(x/2)$ and expressing the functions appearing in the integrand in terms of it,

$$\begin{aligned}
\frac{\cos x \sin x}{1 + \cos x}\, dx &= \frac{\dfrac{1-t^2}{1+t^2}\,\dfrac{2t}{1+t^2}\,\dfrac{2}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}}\, dt \\
&= \frac{2t(1-t^2)}{(1+t^2)^2}\, dt \\
&= 2t\left[\frac{A}{(1+t^2)^2} + \frac{B}{1+t^2}\right] dt,
\end{aligned}$$

with $A + B(1+t^2) = 1 - t^2$, implying that $B = -1$ and $A = 2$.
And so, recalling that $1 + t^2 = \sec^2(x/2) = 1/[\cos^2(x/2)]$,

$$\begin{aligned}
J &= \int \left(\frac{4t}{(1+t^2)^2} - \frac{2t}{1+t^2}\right) dt \\
&= -\frac{2}{1+t^2} - \ln(1+t^2) + c \\
&= -2\cos^2\frac{x}{2} + \ln(\cos^2\frac{x}{2}) + c.
\end{aligned}$$

(d) We can either set $\tan x = u$ or show that the integrand is $\sec 2x$ and then use the result of exercise 2.35; here we use the latter method.

$$\int \frac{\sec^2 x}{1 - \tan^2 x}\, dx = \int \frac{1}{\cos^2 x - \sin^2 x}\, dx = \int \sec 2x\, dx.$$

It then follows from the earlier result that $J = \frac{1}{2}\ln(\sec 2x + \tan 2x) + c$. This can also be written as $\frac{1}{2}\ln[(1 + \tan x)/(1 - \tan x)] + c$.

2.38 *Determine whether the following integrals exist and, where they do, evaluate them:*

(a) $\displaystyle\int_0^\infty \exp(-\lambda x)\,dx$; (b) $\displaystyle\int_{-\infty}^\infty \frac{x}{(x^2+a^2)^2}\,dx$;

(c) $\displaystyle\int_1^\infty \frac{1}{x+1}\,dx$; (d) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$;

(e) $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$; (f) $\displaystyle\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx$.

(a) This is an infinite integral and so we must examine the result of letting the range of a finite integral go to infinity:

$$\int_0^\infty e^{-\lambda x}\,dx = \lim_{R\to\infty}\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^R = \lim_{R\to\infty}\left[\frac{1}{\lambda} - \frac{e^{-\lambda R}}{\lambda}\right].$$

The limit as $R \to \infty$ does exist if $\lambda > 0$ and is then equal to λ^{-1}.

(b) This is also an infinite integral. However, because of the antisymmetry of the integrand, the integral is zero for all finite values of R. It therefore has a limit as $R \to \infty$ of zero, which is consequently the value of the integral.

$$\int_{-\infty}^\infty \frac{x}{(x^2+a^2)^2}\,dx = \lim_{R\to\infty}\left[\frac{-1}{2(x^2+a^2)}\right]_{-R}^R = \lim_{R\to\infty}[0] = 0.$$

(c) The integral is elementary over any finite range $(1, R)$ and so we must examine its behaviour as $R \to \infty$:

$$\int_1^\infty \frac{1}{x+1}\,dx = \lim_{R\to\infty}[\ln(1+x)]_1^R = \lim_{R\to\infty}\ln\frac{1+R}{2} = \infty.$$

The limit is not finite and so the integral does not exist.

(d) The integrand, $1/x^2$ is undefined at $x = 0$ and so we must examine the behaviour of the integral with lower limit ϵ as $\epsilon \to 0$.

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\epsilon\to 0}\left[-\frac{1}{x}\right]_\epsilon^1 = \lim_{\epsilon\to 0}\left(-1+\frac{1}{\epsilon}\right) = \infty.$$

As the limit is not finite the integral does not exist.

(e) Again, a infinite quantity ($\cot 0$) appears in the integrand and the limit test has to be applied.

$$\begin{aligned}\int_0^{\pi/2} \cot\theta\,d\theta &= \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta}\,d\theta \\ &= \lim_{\epsilon\to 0}\,[\,\ln(\sin\theta)\,]_\epsilon^{\pi/2} = \lim_{\epsilon\to 0}[\,0 - \ln(\sin\epsilon)\,] = -(-\infty).\end{aligned}$$

The limit is not finite and so the integral does not exist.

(f) Yet again, the integrand has an infinity (at $x = 1$) and the limit test has to be applied

$$\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx = \lim_{z\to 1}\left[-(1-x^2)^{1/2}\right]_0^z = 0 + 1 = 1.$$

This time the limit does exist; the integral is defined and has value 1.

2.40 *Show, using the following methods, that the indefinite integral of* $x^3/(x+1)^{1/2}$ *is*

$$J = \tfrac{2}{35}(5x^3 - 6x^2 + 8x - 16)(x+1)^{1/2} + c.$$

(a) *Repeated integration by parts.*
(b) *Setting* $x + 1 = u^2$ *and determining* dJ/du *as* $(dJ/dx)(dx/du)$.

(a) Evaluating the successive integrals produced by the repeated integration by parts:

$$\int \frac{x^3}{(x+1)^{1/2}}\,dx = 2x^3\sqrt{x+1} - \int 3x^2\, 2\sqrt{x+1}\,dx,$$
$$\int x^2\sqrt{x+1}\,dx = \frac{2}{3}x^2(x+1)^{3/2} - \int 2x\,\frac{2}{3}(x+1)^{3/2}\,dx,$$
$$\int x(x+1)^{3/2}\,dx = \frac{2}{5}x(x+1)^{5/2} - \int \frac{2}{5}(x+1)^{5/2}\,dx,$$
$$\int (x+1)^{5/2}\,dx = \frac{2}{7}(x+1)^{7/2}.$$

And so, remembering to carry forward the multiplicative factors generated at each stage, we have

$$J = \sqrt{x+1}\left[2x^3 - 4x^2(x+1) + \frac{16}{5}x(x+1)^2 - \frac{32}{35}(x+1)^3\right] + c$$
$$= \frac{2\sqrt{x+1}}{35}\left[\,5x^3 - 6x^2 + 8x - 16\,\right] + c.$$

(b) Set $x + 1 = u^2$, giving $dx = 2u\,du$, to obtain

$$J = \int \frac{(u^2-1)^3}{u}\,2u\,du$$
$$= 2\int (u^6 - 3u^4 + 3u^2 - 1)\,du.$$

This integral is now easily evaluated to give

$$\begin{aligned}
J &= 2\left(\frac{1}{7}u^7 - \frac{3}{5}u^5 + u^3 - u\right) + c \\
&= \frac{2u}{35}(5u^6 - 21u^4 + 35u^2 - 35) + c \\
&= \frac{2u}{35}\,[\,5(x^3 + 3x^2 + 3x + 1) - 21(x^2 + 2x + 1) + 35(x + 1) - 35\,] + c \\
&= \frac{2\sqrt{x+1}}{35}\,[\,5x^3 - 6x^2 + 8x - 16\,] + c.
\end{aligned}$$

i.e. the same final result as for method (a).

2.42 *Define $J(m,n)$, for non-negative integers m and n, by the integral*

$$J(m,n) = \int_0^{\pi/2} \cos^m\theta \sin^n\theta \, d\theta.$$

(a) *Evaluate $J(0,0)$, $J(0,1)$, $J(1,0)$, $J(1,1)$, $J(m,1)$, $J(1,n)$.*
(b) *Using integration by parts prove that, for m and n both > 1,*

$$J(m,n) = \frac{m-1}{m+n}J(m-2,n) \quad \text{and} \quad J(m,n) = \frac{n-1}{m+n}J(m,n-2).$$

(c) *Evaluate (i) $J(5,3)$, (ii) $J(6,5)$, (iii) $J(4,8)$.*

(a) For these special values of m and/or n the integrals are all elementary, as follows.

$$\begin{aligned}
J(0,0) &= \int_0^{\pi/2} 1\, d\theta = \frac{\pi}{2}, \\
J(0,1) &= \int_0^{\pi/2} \sin\theta\, d\theta = 1, \\
J(1,0) &= \int_0^{\pi/2} \cos\theta\, d\theta = 1, \\
J(1,1) &= \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{2}, \\
J(m,1) &= \int_0^{\pi/2} \cos^m\theta \sin\theta\, d\theta = \frac{1}{m+1}, \\
J(1,n) &= \int_0^{\pi/2} \cos\theta \sin^n\theta\, d\theta = \frac{1}{n+1}.
\end{aligned}$$

(b) In order to obtain a reduction formula, we ‘sacrifice’ one of the cosine factors so that it can act as the derivative of a sine function, so allowing $\sin^n\theta$ to be integrated. The two extra powers of $\sin\theta$ generated by the integration by parts are then removed by writing them as $1-\cos^2\theta$.

$$\begin{aligned}
J(m,n) &= \int_0^{\pi/2} \cos^{m-1}\theta \sin^n\theta \cos\theta \, d\theta \\
&= \left[\frac{\cos^{m-1}\theta \sin^{n+1}\theta}{n+1}\right]_0^{\pi/2} \\
&\qquad - \int_0^{\pi/2} \frac{(m-1)\cos^{m-2}\theta(-\sin\theta)\sin^{n+1}\theta}{n+1}\, d\theta \\
&= 0 + \frac{m-1}{n+1}\int_0^{\pi/2} \cos^{m-2}\theta(1-\cos^2\theta)\sin^n\theta\, d\theta \\
&= \frac{m-1}{n+1}J(m-2,n) - \frac{m-1}{n+1}J(m,n). \\
J(m,n) &= \frac{m-1}{m+n}J(m-2,n).
\end{aligned}$$

Similarly, by 'sacrificing' a sine term to act as the derivative of a cosine term,

$$J(m,n) = \frac{n-1}{m+n}J(m,n-2).$$

(c) For these specific cases we apply the reduction formulae in (b) to reduce them to one of the forms evaluated in (a).

$$\text{(i)}\ \ J(5,3) = \frac{2}{8}J(5,1) = \frac{2}{8}\frac{1}{6} = \frac{1}{24},$$

$$\text{(ii)}\ \ J(6,5) = \frac{4}{11}\frac{2}{9}J(6,1) = \frac{4}{11}\frac{2}{9}\frac{1}{7} = \frac{8}{693},$$

$$\text{(iii)}\ \ J(4,8) = \frac{3}{12}\frac{1}{10}J(0,8) = \frac{3}{12}\frac{1}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{7\pi}{2048}.$$

> ***2.44*** *Evaluate the following definite integrals:*
>
> (a) $\int_0^\infty xe^{-x}\,dx$; (b) $\int_0^1 \left[(x^3+1)/(x^4+4x+1)\right] dx$;
> (c) $\int_0^{\pi/2}[a+(a-1)\cos\theta]^{-1}\,d\theta$ *with* $a>\frac{1}{2}$; (d) $\int_{-\infty}^{\infty}(x^2+6x+18)^{-1}\,dx$.

(a) Integrating by parts:

$$\int_0^\infty xe^{-x}\,dx = \left[-xe^{-x}\right]_0^\infty - \int_0^\infty -e^{-x}\,dx = 0 + \left[-e^{-x}\right]_0^\infty = 1.$$

(b) This is a logarithmic integration:

$$\int_0^1 \frac{x^3+1}{x^4+4x+1}\,dx = \frac{1}{4}\int_0^1 \frac{4x^3+4}{x^4+4x+1} = \frac{1}{4}\left[\ln(x^4+4x+1)\right]_0^1 = \frac{1}{4}\ln 6.$$

(c) Writing $t = \tan(\theta/2)$:

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{a+(a-1)\cos\theta}\,d\theta &= \int_0^1 \frac{1}{a+(a-1)\left(\dfrac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2} \\ &= \int_0^1 \frac{2}{2a-1+t^2}\,dt \\ &= \frac{2}{\sqrt{2a-1}}\left[\tan^{-1}\frac{t}{\sqrt{2a-1}}\right]_0^1 \\ &= \frac{2}{\sqrt{2a-1}}\tan^{-1}\frac{1}{\sqrt{2a-1}}.\end{aligned}$$

(d) The denominator has no real zeros ($6^2 < 4 \times 1 \times 18$) and so, completing the square, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+6x+18}\,dx &= \int_{-\infty}^{\infty} \frac{1}{(x+3)^2+9}\,dx \\ &= \frac{1}{3}\left[\tan^{-1}\left(\frac{x+3}{3}\right)\right]_{-\infty}^{\infty} \\ &= \frac{1}{3}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{3}.\end{aligned}$$

2.46 *Find positive constants a, b such that* $ax \le \sin x \le bx$ *for* $0 \le x \le \pi/2$. *Use this inequality to find (to two significant figures) upper and lower bounds for the integral*

$$I = \int_0^{\pi/2} (1+\sin x)^{1/2}\,dx.$$

Use the substitution $t = \tan(x/2)$ *to evaluate* I *exactly.*

Consider $f(x) = (\sin x)/x$. Its derivative is

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{x - \tan x}{x^2}\cos x,$$

which is everwhere negative (or zero) in the given range. This shows that $f(x)$ is a monotonically decreasing function in that range and reaches its lowest value at the end of the range. This value must therefore be $\sin(\pi/2)/(\pi/2)$, i.e. $2/\pi$.

From the standard Maclaurin series for $\sin x$ (subsection 4.6.3)

$$f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots,$$

and the limit of $f(x)$ as $x \to 0$ is 1. In summary,

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1 \quad \text{for} \quad 0 \leq x \leq \frac{\pi}{2}.$$

It then follows that

$$\int_0^{\pi/2} (1 + \frac{2}{\pi}x)^{1/2}\, dx \leq \int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx \leq \int_0^{\pi/2} (1 + x)^{1/2}\, dx,$$

$$\left[\frac{\pi}{2}\,\frac{2}{3}(1 + \frac{2}{\pi}x)^{3/2}\right]_0^{\pi/2} \leq \quad I \quad \leq \left[\frac{2}{3}(1 + x)^{3/2}\right]_0^{\pi/2},$$

$$\frac{\pi}{3}\left[(2)^{3/2} - 1\right] \leq \quad I \quad \leq \frac{2}{3}\left[(1 + \frac{\pi}{2})^{3/2} - 1\right],$$

$$1.91 \leq \quad I \quad \leq \quad 2.08.$$

For an exact evaluation we use the standard half-angle formulae:

$$t = \tan\frac{x}{2}, \qquad \sin x = \frac{2t}{1 + t^2}, \qquad dx = \frac{2}{1 + t^2}\, dt.$$

Substitution of these gives

$$\begin{aligned}
\int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx &= \int_0^1 \left(1 + \frac{2t}{1 + t^2}\right)^{1/2} \frac{2}{1 + t^2}\, dt \\
&= \int_0^1 \frac{2 + 2t}{(1 + t^2)^{3/2}}\, dt \\
&= \int_0^1 \frac{2}{(1 + t^2)^{3/2}}\, dt + 2\left[-\frac{1}{(1 + t^2)^{1/2}}\right]_0^1.
\end{aligned}$$

To evaluate the first integral we turn it back into one involving sinusoidal functions and write $t = \tan\theta$ with $dt = \sec^2\theta\, d\theta$. Then the original integral becomes

$$\begin{aligned}
\int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx &= \int_0^{\pi/4} \frac{2\sec^2\theta}{\sec^3\theta}\, d\theta + 2\left[1 - \frac{1}{\sqrt{2}}\right] \\
&= \int_0^{\pi/4} 2\cos\theta\, d\theta + 2 - \sqrt{2} \\
&= 2[\sin\theta]_0^{\pi/4} + 2 - \sqrt{2} \\
&= \sqrt{2} - 0 + 2 - \sqrt{2} = 2.
\end{aligned}$$

An alternative evaluation can be made by setting $x = (\pi/2) - y$ and then writing $1 + \cos y$ in the form $2\cos^2(y/2)$. This gives the final value of 2 more directly.

Whichever method is used in (b), we note that, as it must (or at least should!) the exact value of the integral lies between our calculated bounds.

> **2.48** *Show that the total length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, which can be parameterised as $x = a\cos^3\theta$, $y = a\sin^3\theta$, is $6a$.*

We first check that $x^{2/3} + y^{2/3} = a^{2/3}$ can be parameterised as $x = a\cos^3\theta$ and $y = a\sin^3\theta$. This is so, since $a^{2/3}\cos^2\theta + a^{2/3}\sin^2\theta = a^{2/3}$ is an identity.

Now the element of length of the curve ds is given by $ds^2 = dx^2 + dy^2$ or, using the parameterisation,

$$\begin{aligned} ds &= \left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{1/2} d\theta \\ &= \left[\left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2\right]^{1/2} d\theta \\ &= 3a\cos\theta\sin\theta\, d\theta. \end{aligned}$$

The total length of the asteroid curve is four times its length in the first quadrant and therefore given by

$$s = 4 \times 3a \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta = 12a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 6a.$$

> **2.50** *The equation of a cardioid in plane polar coordinates is*
>
> $$\rho = a(1 - \sin\phi).$$
>
> *Sketch the curve and find* (i) *its area,* (ii) *its total length,* (iii) *the surface area of the solid formed by rotating the cardioid about its axis of symmetry and* (iv) *the volume of the same solid.*

For a sketch of the 'heart-shaped' (actually more apple-shaped) curve see figure 2.4.

To avoid any possible double counting, integrals will be taken from $\phi = \pi/2$ to $\phi = 3\pi/2$ and symmetry used for scaling up.

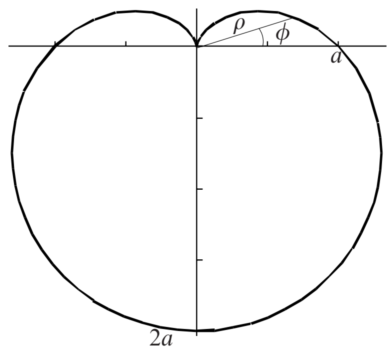

Figure 2.4 The cardioid discussed in exercise 2.50.

(i) Area. In plane polar coordinates this is straightforward.

$$\begin{aligned}\int \frac{1}{2}\rho^2\,d\phi &= 2\int_{\pi/2}^{3\pi/2} \frac{1}{2}a^2(1-\sin\phi)^2\,d\phi \\ &= a^2\int_{\pi/2}^{3\pi/2}(1-2\sin\phi+\sin^2\phi)\,d\phi \\ &= a^2(\pi - 0 + \tfrac{1}{2}\pi) \\ &= \frac{3\pi a^2}{2}.\end{aligned}$$

The third term in the integral was evaluated using the standard result that the average value of the square of a sinusoid over a whole number of quarter cycles is $\frac{1}{2}$.

(ii) Length. Since $ds^2 = d\rho^2 + \rho^2 d\phi^2$, the total length is

$$\begin{aligned}L &= 2\int_{\pi/2}^{3\pi/2}\left[\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2\right]^{1/2} d\phi \\ &= 2\int_{\pi/2}^{3\pi/2}(a^2\cos^2\phi + a^2 - 2a^2\sin\phi + a^2\sin^2\phi)^{1/2}\,d\phi \\ &= 2a\sqrt{2}\int_{\pi/2}^{3\pi/2}(1-\sin\phi)^{1/2}\,d\phi \\ &= 2a\sqrt{2}\int_{0}^{-\pi}(1-\cos\phi')^{1/2}(-d\phi') \quad \text{where} \quad \phi = \frac{1}{2}\pi - \phi'.\end{aligned}$$

Using the trigonometric half-angle formula $1 - \cos\theta = 2\sin^2(\theta/2)$, this integral is easily evaluated to give

$$\begin{aligned} L &= 2a\sqrt{2}\int_{-\pi}^{0} \sqrt{2}\sin\frac{\phi'}{2}\,d\phi' \\ &= 4a\left[-2\cos\frac{\phi'}{2}\right]_{-\pi}^{0} = -8a. \end{aligned}$$

The negative sign is irrelevant and merely reflects the (inappropriate) choice of taking the positive square root of $\sin^2(\phi'/2)$. The total length of the curve is thus $8a$.

(iii) Surface area of the solid of rotation.
The elemental circular strip at any given value of ρ and ϕ has a total length of $2\pi\rho\cos\phi$ and a width ds (on the surface) given by $(ds)^2 = (d\rho)^2 + (\rho d\phi)^2$. This strip contributes an elemental surface area $2\pi\rho\cos\phi\, ds$ and so the total surface area S of the solid is given by

$$\begin{aligned} S &= \int_{\pi/2}^{3\pi/2} 2\pi\rho\cos\phi\left[\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2\right]^{1/2} d\phi \\ &= 2\sqrt{2}\pi a^2\int_{\pi/2}^{3\pi/2}(1-\sin\phi)^{3/2}\cos\phi\,d\phi \quad [\text{using the result from (ii)}] \\ &= 2\sqrt{2}\pi a^2\left[-\frac{2}{5}(1-\sin\phi)^{5/2}\right]_{\pi/2}^{3\pi/2} \\ &= -\frac{32\pi a^2}{5}. \end{aligned}$$

Again, the minus sign is irrelevant and arises because, in the range of ϕ used, the elemental strip radius is actually $-\rho\cos\phi$.

(iv) Volume of the solid of rotation.
The height above the origin of any point is $\rho\sin\phi$ and so, for $\pi/2 \le \phi \le 3\pi/2$, the thickness of any elemental disc is $-d(\rho\sin\phi)$ whilst its area is $\pi\rho^2\cos^2\phi$.

It should be noted that this formulation allows correctly for the 'missing' part of the body of revolution – as it were, for the air that surrounds the 'stalk of the apple'. Whilst ϕ is in the range $\pi/2 \le \phi \le 5\pi/6$ (the upper limit being found by maximising $y = \rho\sin\phi = a(1-\sin\phi)\sin\phi$), negative volume is being added to the solid, representing 'the air'. For $5\pi/6 \le \phi \le \pi$ the solid acquires volume as if there were no air core. For the rest of the range, $\pi \le \phi \le 3\pi/2$, such considerations do not arise.

The required volume is therefore given by the single integral

$$\begin{aligned}
V &= -\int_{\pi/2}^{3\pi/2} \pi\rho^2 \cos^2\phi \, d(\rho \sin\phi) \\
&= -\int_{\pi/2}^{3\pi/2} \pi a^2 (1-\sin\phi)^2 \cos^2\phi \; a(\cos\phi - 2\sin\phi\cos\phi)\, d\phi \\
&= -\pi a^3 \int_{\pi/2}^{3\pi/2} (1 - 2\sin\phi + \sin^2\phi)(1 - 2\sin\phi)\cos^3\phi \, d\phi \\
&= -\pi a^3 \int_{\pi/2}^{3\pi/2} (1 - 4\sin\phi + 5\sin^2\phi - 2\sin^3\phi)(1-\sin^2\phi)\cos\phi\, d\phi \\
&= -\pi a^3 \int_{\pi/2}^{3\pi/2} (1 - 4\sin\phi + 4\sin^2\phi + 2\sin^3\phi - \cdots \\
&\qquad\qquad \cdots - 5\sin^4\phi + 2\sin^5\phi)\cos\phi\, d\phi \\
&= \pi a^3 \left(2 - 0 + \frac{8}{3} + 0 - 2 + 0\right) = \frac{8}{3}\pi a^3.
\end{aligned}$$

3

Complex numbers and hyperbolic functions

> ***3.2*** *By considering the real and imaginary parts of the product* $e^{i\theta}e^{i\phi}$ *prove the standard formulae for* $\cos(\theta+\phi)$ *and* $\sin(\theta+\phi)$.

We apply Euler's equation, $e^{i\theta} = \cos\theta + i\sin\theta$, separately to the two sides of the identity

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$$

and obtain

$$\begin{aligned}\cos(\theta+\phi) + i\sin(\theta+\phi) &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi)\\ &\qquad +i(\cos\theta\sin\phi + \sin\theta\cos\phi).\end{aligned}$$

Equating the real and imaginary parts gives the standard results:

$$\begin{aligned}\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi,\\ \sin(\theta+\phi) &= \cos\theta\sin\phi + \sin\theta\cos\phi.\end{aligned}$$

It is worth noting the relationship between this method of proof and the purely geometrical one given in subsection 1.2.2. In the Argand diagram, $e^{i\theta}$ is represented by a unit vector making an angle θ (A in figure 1.2) with the x-axis. Multiplying by $e^{i\phi}$ corresponds to turning the vector through a further angle ϕ (B in the figure), without any change in length, thus giving the point (P in the figure) that is represented by $e^{i(\theta+\phi)}$.

3.4 *Find the locus in the complex z-plane of points that satisfy the following equations.*

(a) $z - c = \rho\left(\dfrac{1+it}{1-it}\right)$, *where c is complex, ρ is real and t is a real parameter that varies in the range* $-\infty < t < \infty$.

(b) $z = a + bt + ct^2$, *in which t is a real parameter and a, b, and c are complex numbers with b/c real.*

(a) We start by rationalising the complex fraction so that it is easier to see what it represents:

$$\begin{aligned} z - c &= \rho\left(\frac{1+it}{1-it}\right) \qquad\qquad -\infty < t < \infty \\ &= \rho\frac{1+it}{1-it}\,\frac{1+it}{1+it} \\ &= \rho\frac{(1-t^2)+2it}{1+t^2}. \end{aligned}$$

The real and imaginary parts of the RHS now have the forms of cosine and sine functions, respectively, as expressed by the $\tan(x/2)$ formulae (see subsection 2.2.7). The equation can therefore be written as

$$z - c = \rho(\cos 2\theta + i\sin 2\theta) = \rho e^{2i\theta},$$

where $\theta = \tan^{-1} t$. Since $-\infty < t < \infty$, θ must lie in the range $-\pi/2 < \theta < \pi/2$ and $-\pi < 2\theta < \pi$. Thus the locus of z is a circle centred on $z = c$ and of radius ρ.

(b) If we write $a = a_r + ia_i$, etc. then, since t is real, $x + iy = z = a + bt + ct^2$ gives

$$\begin{aligned} x &= a_r + b_r t + c_r t^2, \\ y &= a_i + b_i t + c_i t^2. \end{aligned}$$

Multiplying these equations by c_i and c_r, respectively, and subtracting eliminates the t^2 term and gives

$$c_i x - c_r y = a_r c_i - a_i c_r + t(c_i b_r - c_r b_i).$$

Now, since the ratio b/c is real the phases of b and c are equal or differ by π. Either way, $b_i c_r = c_i b_r$ with the consequence that the term in parentheses in the above equation vanishes and

$$y = \frac{c_i}{c_r}x + a_i - \frac{c_i a_r}{c_r},$$

i.e. the locus is a straight line of slope $\tan(\arg c)$ [or, equivalently, $\tan(\arg b)$]. The same conclusion can be reached by eliminating the t term.

3.6 *Find the equations in terms of x and y of the sets of points in the Argand diagram that satisfy the following:*

(a) $\operatorname{Re} z^2 = \operatorname{Im} z^2$;
(b) $(\operatorname{Im} z^2)/z^2 = -i$;
(c) $\arg[z/(z-1)] = \pi/2$.

(a) Straightforward substitution of $z = x + iy$ gives

$$\begin{aligned}
\operatorname{Re} z^2 &= \operatorname{Im} z^2, \\
\operatorname{Re}\,(x^2 - y^2 + 2ixy) &= \operatorname{Im}\,(x^2 - y^2 + 2ixy), \\
x^2 - y^2 &= 2xy, \\
y^2 + 2xy - x^2 &= 0, \\
y &= (-1 \pm \sqrt{2})x.
\end{aligned}$$

This is a pair of straight lines through the origin. Since the product of their slopes, $(-1+\sqrt{2})(-1-\sqrt{2})$, is equal to -1, the lines are orthogonal.

Geometrically, it is clear that the condition is equivalent to z^2 lying on the line of positive slope $\pi/4$. Thus z must lie on one of the two lines with slopes $\pi/8$ and $-3\pi/8$; that the tangents of these angles are $\sqrt{2}-1$ and $-\sqrt{2}-1$ confirms this conclusion.

(b) Since $\operatorname{Im} z^2$ is necessarily real, z^2 will have to be purely imaginary. Proceeding as in part (a):

$$\begin{aligned}
\operatorname{Im} z^2 &= -iz^2, \\
2xy &= -i(x^2 - y^2 + 2ixy), \\
0 &= x^2 - y^2, \\
y &= \pm x.
\end{aligned}$$

This is the pair of orthogonal lines that bisects the angles between the x- and y-axes.

We note that the imposed condition can only be satified non-trivially because of the particular constants involved; had the equation been, say, $(\operatorname{Im} z^2)/z^2 = -3i$, the real parts of the equality would not have cancelled and there would have been no solution; for the form $(\operatorname{Im} z^2) = -3iz^2$, the only solution would have been $z = 0$.

(c) Rearranging the given condition,

$$\arg \frac{z}{z-1} = \frac{\pi}{2},$$

we obtain

$$z = (z-1)\lambda e^{i\pi/2} = \lambda(z-1)i \quad \text{with} \quad \lambda > 0,$$
$$x + iy = i\lambda(x-1) - \lambda y.$$

Thus $x = -\lambda y$ and $y = \lambda(x-1)$, and so

$$x(x-1) = -y^2$$
$$(x - \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}.$$

This is a circle of radius $\frac{1}{2}$ centred on $(\frac{1}{2}, 0)$. Since $\lambda > 0$ and the circle lies entirely in $0 \le x \le 1$, both expressions for y, namely $-\lambda^{-1}x$ and $\lambda(x-1)$, must be negative. Hence the locus is the part of the circle that lies in $y < 0$. Plotting the points 0, 1 and z in the complex plane shows the relationship of this result to the classical geometric result about the 'angle in a semi-circle being a right angle'.

3.8 *The two sets of points $z = a$, $z = b$, $z = c$, and $z = A$, $z = B$, $z = C$ are the corners of two similar triangles in the Argand diagram. Express in terms of $a, b, \ldots, C$*

(a) *the equalities of corresponding angles, and*
(b) *the constant ratio of corresponding sides,*

in the two triangles.

By noting that any complex quantity can be expressed as

$$z = |z| \exp(i \arg z),$$

deduce that

$$a(B-C) + b(C-A) + c(A-B) = 0.$$

(a) The angle α between the two sides of the triangle that meet at $z = a$ is the difference between the arguments of $b - a$ and $c - a$. This, in turn, is equal to the argument of their ratio, i.e.

$$\alpha = \arg \frac{b-a}{c-a}.$$

Thus the equality of corresponding angles in the similar triangles is expressed by

$$\arg \frac{b-a}{c-a} = \arg \frac{B-A}{C-A}$$

and similar relations.

(b) The constant ratio of corresponding sides is expressed by

$$\left|\frac{b-a}{c-a}\right| = \left|\frac{B-A}{C-A}\right|.$$

Now, using the fact that $z = |z| \exp(i \arg z)$, we can write

$$\begin{aligned}\frac{b-a}{c-a} &= \left|\frac{b-a}{c-a}\right| \exp\left(i \arg \frac{b-a}{c-a}\right) \\ &= \left|\frac{B-A}{C-A}\right| \exp\left(i \arg \frac{B-A}{C-A}\right) = \frac{B-A}{C-A},\end{aligned}$$

where the two results obtained previously have been used to justify the second equality. Hence,

$$(C-A)b - (C-A)a = (B-A)c - (B-A)a.$$

Cancelling the term aA that appears on both sides of the equality and then rearranging gives

$$\begin{aligned} & b(C-A) - aC + c(A-B) + aB = 0, \\ \Rightarrow \quad & b(C-A) + a(B-C) + c(A-B) = 0,\end{aligned}$$

as stated in the question.

3.10 *The most general type of transformation between one Argand diagram, in the z-plane, and another, in the Z-plane, that gives one and only one value of Z for each value of z (and conversely) is known as the* general bilinear transformation *and takes the form*

$$z = \frac{aZ+b}{cZ+d}.$$

(a) *Confirm that the transformation from the Z-plane to the z-plane is also a general bilinear transformation.*

(b) *Recalling that the equation of a circle can be written in the form*

$$\left|\frac{z-z_1}{z-z_2}\right| = \lambda, \qquad \lambda \neq 1,$$

show that the general bilinear transformation transforms circles into circles (or straight lines). What is the condition that z_1, z_2 and λ must satisfy if the transformed circle is to be a straight line?

(a) To test whether this is so, we must make Z the subject of the transformation equation. Starting from the original form and rearranging:

$$\begin{aligned} z &= \frac{aZ+b}{cZ+d}, \\ czZ + zd &= aZ + b, \\ Z(cz-a) &= -dz + b, \\ Z &= \frac{-dz+b}{cz-a} \qquad\qquad (*), \end{aligned}$$

i.e. another general bilinear transformation, though with different, but related, parameters.

(b) Given the circle

$$\left| \frac{z-z_1}{z-z_2} \right| = \lambda, \quad \lambda \neq 1,$$

in the z-plane, it transforms into the curve given in the Z-plane by

$$\left| \frac{\dfrac{aZ+b}{cZ+d} - z_1}{\dfrac{aZ+b}{cZ+d} - z_2} \right| = \lambda,$$

This equation can be manipulated to make the multipliers of Z unity, as follows:

$$\left| \frac{(a-z_1c)Z + b - z_1d}{(a-z_2c)Z + b - z_2d} \right| = \lambda,$$

$$\frac{|cz_1 - a|}{|cz_2 - a|} \left| \frac{-Z + \dfrac{b-z_1d}{cz_1-a}}{-Z + \dfrac{b-z_2d}{cz_2-a}} \right| = \lambda,$$

$$\left| \frac{Z-Z_1}{Z-Z_2} \right| = \left| \frac{cz_2-a}{cz_1-a} \right| \lambda = \mu.$$

Thus, the transformed curve is also a circle (or a straight line). It is a straight line if Z is always equidistant from Z_1 and Z_2, i.e. if $\mu = 1$. The condition for this is

$$|a - cz_1| = \lambda |a - cz_2|.$$

We note, from (*), that Z_1 and Z_2 are the points into which z_1 and z_2 are carried by the transformation.

3.12 *Denote the nth roots of unity by* $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$.

(a) *Prove that*

$$rm\ (i)\ \sum_{r=0}^{n-1} \omega_n^r = 0, \qquad \text{(ii)}\ \prod_{r=0}^{n-1} \omega_n^r = (-1)^{n+1}.$$

(b) *Express* $x^2 + y^2 + z^2 - yz - zx - xy$ *as the product of two factors, each linear in* x, y *and* z, *with coefficients dependent on the third roots of unity (and those of the* x *terms arbitrarily taken as real).*

(a) In order to establish properties of the sums and products of the nth roots of unity we need to express their common defining property as a polynomial equation; this is $\omega^n - 1 = 0$. Now writing the polynomial as the product of factors typified by $(\omega - \omega_n^r)$ we have

$$\begin{aligned} \omega^n - 1 &= (\omega - 1)(\omega - \omega_n)(\omega - \omega_n^2) \cdots (\omega - \omega_n^{n-1}) \\ &= \omega^n - \omega^{n-1} \sum_{r=0}^{n-1} \omega_n^r + \cdots + (-1)^n \prod_{r=0}^{n-1} \omega_n^r. \end{aligned}$$

Equating coefficients of

(i) ω^{n-1}, $\quad 0 = \sum_{r=0}^{n-1} \omega_n^r$,

(ii) constants, $\quad -1 = (-1)^n \prod_{r=0}^{n-1} \omega_n^r$.

Hence the stated results.

(b) Writing the given expression f in the required form with the x-coefficients both taken as $+1$, we have

$$\begin{aligned} f &= x^2 + y^2 + z^2 - yz - zx - xy \\ &= (x + \alpha y + \beta z)(x + \gamma y + \delta z), \quad \text{say,} \\ &= x^2 + \alpha\gamma y^2 + \beta\delta z^2 + (\alpha + \gamma)xy + (\beta + \delta)xz + (\alpha\delta + \beta\gamma)yz. \end{aligned}$$

The cube roots of unity have the explicit properties:

$$\begin{aligned} 1 + \omega_3 + \omega_3^2 &= 0, \\ 1 \times \omega_3 \times \omega_3^2 &= 1. \end{aligned}$$

So, if we choose $\alpha = \omega_3$ then, from the coefficient of y^2 we must have $\gamma = \omega_3^2$. This also makes $\alpha + \gamma$, the coefficient of the xy term, equal to -1 as required.

Turning now to the choices for β and δ, we cannot take $\beta = \omega_3$ since the coefficient of z^2 would then imply that $\delta = \omega_3^2$; this makes the coefficient of yz wrong (2 rather than -1). Therefore, take $\beta = \omega_3^2$ leading to the result $\delta = \omega_3$. This choice makes the coefficients of both z^2 and yz correct and leads to a factorisation of the form

$$x^2 + y^2 + z^2 - yz - zx - xy = (x + \omega_3 y + \omega_3^2 z)(x + \omega_3^2 y + \omega_3 z).$$

We note that the factors can be made to have the coefficents of y (say) equal to unity by multiplying the first by ω_3^2 and the second by ω_3; the net effect is to multiply by ω_3^3, i.e. by unity, and so the LHS is unaffected.

3.14 *The complex position vectors of two parallel interacting equal fluid vortices moving with their axes of rotation always perpendicular to the z-plane are z_1 and z_2. The equations governing their motions are*

$$\frac{dz_1^*}{dt} = -\frac{i}{z_1 - z_2}, \qquad \frac{dz_2^*}{dt} = -\frac{i}{z_2 - z_1}.$$

Deduce that (a) $z_1 + z_2$, (b) $|z_1 - z_2|$ *and* (c) $|z_1|^2 + |z_2|^2$ *are all constant in time, and hence describe the motion geometrically.*

(a) To obtain the time derivative of $z_1 + z_2$ we add the two equations and take the complex conjugate of the result.

$$\begin{aligned}\frac{d(z_1 + z_2)}{dt} &= \left[\frac{d(z_1^* + z_2^*)}{dt}\right]^* \\ &= \left(-\frac{i}{z_1 - z_2} - \frac{i}{z_2 - z_1}\right)^* \\ &= 0 \quad \text{i.e. } z_1 + z_2 \text{ is constant.}\end{aligned}$$

(b) It is easier to consider the time derivative of the square of $|z_1 - z_2|$, expressed as $(z_1 - z_2)(z_1^* - z_2^*)$:

$$\begin{aligned}\frac{d(|z_1 - z_2|^2)}{dt} &= \frac{d}{dt}\left[(z_1 - z_2)(z_1^* - z_2^*)\right] \\ &= (z_1 - z_2)\left(\frac{dz_1^*}{dt} - \frac{dz_2^*}{dt}\right) + (z_1^* - z_2^*)\left(\frac{dz_1}{dt} - \frac{dz_2}{dt}\right) \\ &= (z_1 - z_2)\left(-\frac{i}{z_1 - z_2} + \frac{i}{z_2 - z_1}\right) \\ &\quad + (z_1^* - z_2^*)\left(\frac{i}{z_1^* - z_2^*} - \frac{i}{z_2^* - z_1^*}\right) \\ &= -2i + 2i = 0,\end{aligned}$$

i.e. $|z_1 - z_2|^2$ is constant, and so, therefore, is $|z_1 - z_2|$.

(c) We write $2|z_1|^2+2|z_2|^2$ as $|z_1+z_2|^2+|z_1-z_2|^2$. Since both the latter terms have been shown to be constants of the motion, $|z_1|^2+|z_2|^2$ must also be constant in time.

Since the geometrical centre of the pair of vertices is fixed [result (a)], as is their distance apart [result (b)], they must move in circular motion about a fixed point with the vortices at the opposite ends of a diameter.

3.16 *The polynomial $f(z)$ is defined by*

$$f(z) = z^5 - 6z^4 + 15z^3 - 34z^2 + 36z - 48.$$

(a) *Show that the equation $f(z) = 0$ has roots of the form $z = \lambda i$ where λ is real, and hence factorize $f(z)$.*
(b) *Show further that the cubic factor of $f(z)$ can be written in the form $(z + a)^3 + b$, where a and b are real, and hence solve the equation $f(z) = 0$ completely.*

(a) Substitute $z = \lambda i$ in

$$f(z) = z^5 - 6z^4 + 15z^3 - 34z^2 + 36z - 48,$$

to obtain

$$f(\lambda i) = i(\lambda^5 - 15\lambda^3 + 36\lambda) + (-6\lambda^4 + 34\lambda^2 - 48).$$

For λ to be a root, both parts of $f(\lambda i)$ must be zero, i.e.

$$\lambda = 0 \quad \text{or} \quad \lambda^2 = \frac{15 \pm \sqrt{225 - 144}}{2} = 12 \text{ or } 3,$$

and

$$3\lambda^4 - 17\lambda^2 + 24 = 0 \quad \text{i.e.} \quad \lambda^2 = \frac{17 \pm \sqrt{289 - 288}}{6} = 3 \text{ or } \frac{16}{6}.$$

Only $\lambda^2 = 3$ satisfies both. Thus two of the (five) roots are $z = \pm\sqrt{3}i$ and $(z - i\sqrt{3})(z + i\sqrt{3})$ are factors of $f(z)$.
By eye (or by equating coefficients or by long division),

$$f(z) = (z^2 + 3)(z^3 - 6z^2 + 12z - 16).$$

(b) If

$$\begin{aligned} z^3 - 6z^2 + 12z - 16 &= (z + a)^3 + b \\ &= z^3 + 3az^2 + 3a^2z + a^3 + b, \end{aligned}$$

then $a = -2$ and $b = -8$ provides a consistent solution. Thus, the three remaining roots are given by $(z-2)^3 - 8 = 0$, yielding

$$z = 2 + 2 = 4; \quad z = 2 + 2e^{2\pi i/3} = 1 + i\sqrt{3}; \quad z = 2 + 2e^{4\pi i/3} = 1 - i\sqrt{3}.$$

These, together with $z = \pm\sqrt{3}i$, are the five roots of the original equation.

3.18 *By considering* $(1 + \exp i\theta)^n$, *prove that*

$$\sum_{r=0}^{n} {}^nC_r \cos r\theta = 2^n \cos^n(\theta/2)\cos(n\theta/2),$$

$$\sum_{r=0}^{n} {}^nC_r \sin r\theta = 2^n \cos^n(\theta/2)\sin(n\theta/2),$$

where ${}^nC_r = n!/[r!(n-r)!]$.

To express $1 + \exp i\theta$ in terms of its real and imaginary parts, we first use Euler's equation and then the half-angle formulae:

$$1 + e^{i\theta} = 1 + \cos\theta + i\sin\theta = 2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

Thus,

$$\left(1 + e^{i\theta}\right)^n = \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n = 2^n\left(\cos\frac{\theta}{2}\right)^n\left(e^{i\theta/2}\right)^n.$$

But, we also have from the binomial expansion that

$$(1 + e^{i\theta})^n = 1 + ne^{i\theta} + \cdots + {}^nC_r e^{ir\theta} + \cdots + e^{in\theta}.$$

Equating the real parts of the two equal expressions yields the result

$$\begin{aligned}\sum_{r=0}^{n} {}^nC_r \cos r\theta &= 1 + n\cos\theta + \cdots + {}^nC_r \cos r\theta + \cdots + \cos n\theta \\ &= 2^n\left(\cos\frac{\theta}{2}\right)^n \cos\frac{n\theta}{2}.\end{aligned}$$

Similarly, by equating the imaginary parts, we obtain

$$\begin{aligned}\sum_{r=0}^{n} {}^nC_r \sin r\theta &= n\sin\theta + \cdots + {}^nC_r \sin r\theta + \cdots + \sin n\theta \\ &= 2^n\left(\cos\frac{\theta}{2}\right)^n \sin\frac{n\theta}{2}.\end{aligned}$$

3.20 *Express* $\sin^4\theta$ *entirely in terms of the trigonometric functions of multiple angles and deduce that its average value over a complete cycle is* $\frac{3}{8}$.

We first express $\sin\theta$ in terms of complex exponentials and then compute its fourth power using a binomial expansion:

$$\sin^4\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^4$$
$$= \frac{1}{2^4}\left(e^{i4\theta} - 4e^{i2\theta} + 6 - 4e^{-i2\theta} + e^{-i4\theta}\right).$$

We now collect together the terms containing $e^{im\theta}$ and $e^{-im\theta}$ and write them in terms of sinusoids:

$$\sin^4\theta = \tfrac{1}{16}(2\cos 4\theta - 8\cos 2\theta + 6)$$
$$= \tfrac{1}{8}\cos 4\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}.$$

Clearly, the average values of the first two terms over a complete cycle are both zero; so that of $\sin^4\theta$ is $\frac{3}{8}$.

Strictly speaking, the two final lines of equations are not necessary once it is noted that $e^{im\theta}$ has zero average value for all m *except* $m = 0$. The reader may like to show that the average value of $\sin^{2p}\theta$, with p a positive integer, is $(2p)!/[\,2^{2p}(p!)^2\,]$.

3.22 *Prove the following results involving hyperbolic functions.*

(a) *That*

$$\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right).$$

(b) *That, if* $y = \sinh^{-1} x$,

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

(a) When trying to prove equalities, it is generally better to start with the more complicated explicit expression and try to simplify it, than to start with a simpler explicit expression and have to guess how best to write it in a more complicated way. We therefore start with RHS of the putative equality:

$$\begin{aligned} f(x,y) &= 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \\ &= 2\,\tfrac{1}{2}\left(e^{(x+y)/2} - e^{-(x+y)/2}\right)\tfrac{1}{2}\left(e^{(x-y)/2} - e^{-(x-y)/2}\right) \\ &= \tfrac{1}{2}\left(e^{2x/2} - e^{2y/2} - e^{-2y/2} + e^{-2x/2}\right) \\ &= \tfrac{1}{2}\left(e^{x} + e^{-x} - e^{y} - e^{-y}\right) \\ &= \cosh x - \cosh y, \end{aligned}$$

thus establishing the stated result.

(b) To establish the unknown derivative of this inverse function, we first convert it into a function for which we do know the derivative. With $y = \sinh^{-1} x$, we have $x = \sinh y$ and consequently that $dx/dy = \cosh y$. Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cosh y} = \frac{1}{(1+x^2)^{1/2}}, \\ \frac{d^2y}{dx^2} &= -\frac{x}{(1+x^2)^{3/2}}. \end{aligned}$$

For the second equality in the first line we have used the identity $\cosh^2 z = 1 + \sinh^2 z$.

Hence

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\frac{x}{(1+x^2)^{1/2}} + \frac{x}{(1+x^2)^{1/2}} = 0,$$

as stated.

3.24 *Use the definitions and properties of hyperbolic functions to do the following:*

(a) *Solve* $\cosh x = \sinh x + 2\,\text{sech}\, x$.
(b) *Show that the real solution* x *of* $\tanh x = \text{cosech}\, x$ *can be written in the form* $x = \ln(u + \sqrt{u})$*. Find an explicit value for* u*.*
(c) *Evaluate* $\tanh x$ *when* x *is the real solution of* $\cosh 2x = 2\cosh x$*.*

(a) Expressing each term of

$$\cosh x = \sinh x + 2\,\text{sech}\, x$$

in exponential form:

$$\begin{aligned}\tfrac{1}{2}(e^x + e^{-x}) &= \tfrac{1}{2}(e^x - e^{-x}) + \frac{4}{e^x + e^{-x}},\\ e^{-x} &= \frac{4}{e^x + e^{-x}},\\ 1 + e^{-2x} &= 4.\end{aligned}$$

Taking logarithms of this, after simplification, yields

$$x = \tfrac{1}{2}(-\ln 3) = -\ln\sqrt{3} = -0.5493.$$

(b) Expressing the defining equality in terms of sinh and cosh functions,

$$\tanh x = \operatorname{cosech} x \quad \Rightarrow \quad \sinh^2 x = \cosh x.$$

Writing $\cosh x$ as u, this equation can be put in the form $u^2 - 1 = u$. It follows from substituting this into the standard logarithmic expression for an inverse hyperbolic cosine that

$$x = \cosh^{-1} u = \ln(\sqrt{u^2 - 1} + u) = \ln(\sqrt{u} + u).$$

Since $u^2 - u - 1 = 0$, we also have that

$$u = \frac{1 \pm \sqrt{5}}{2}, \qquad \text{(i.e. the golden mean and minus its reciprocal!)}$$

with the positive sign being taken to make $\cosh x > 0$, as is required if x is to be real.

(c) Using a double angle formula for hyperbolic functions to replace $\cosh 2x$ in

$$\cosh 2x = 2\cosh x,$$

we have

$$\begin{aligned}2\cosh^2 x - 1 &= 2\cosh x,\\ \cosh x &= \frac{1 \pm \sqrt{3}}{2},\end{aligned}$$

with the positive sign needed for x to be real. It then follows that

$$\sinh x = \pm\left(\frac{1 + 2\sqrt{3} + 3}{4} - 1\right)^{1/2} = \pm\left(\frac{\sqrt{3}}{2}\right)^{1/2},$$

$$\Rightarrow \quad \tanh x = \frac{\pm 2\left(\frac{\sqrt{3}}{2}\right)^{1/2}}{1 + \sqrt{3}} = \pm\frac{(12)^{1/4}}{1 + \sqrt{3}}.$$

3.26 *In the theory of special relativity, the relationship between the position and time coordinates of an event as measured in two frames of reference that have parallel* x*-axes can be expressed in terms of hyperbolic functions. If the coordinates are* x *and* t *in one frame and* x' *and* t' *in the other then the relationship take the form*

$$x' = x\cosh\phi - ct\sinh\phi,$$
$$ct' = -x\sinh\phi + ct\cosh\phi.$$

Express x *and* ct *in terms of* x', ct' *and* ϕ *and show that*

$$x^2 - (ct)^2 = (x')^2 - (ct')^2.$$

We need to solve

$$x' = x\cosh\phi - ct\sinh\phi,$$
$$ct' = -x\sinh\phi + ct\cosh\phi.$$

for x and ct in turn.

Multiplying the first equation by $\cosh\phi$ and the second by $\sinh\phi$ and adding yields

$$x'\cosh\phi + ct'\sinh\phi = x\cosh^2\phi - x\sinh^2\phi = x.$$

Multiplying the first equation by $\sinh\phi$ and the second by $\cosh\phi$ and adding yields

$$x'\sinh\phi + ct'\cosh\phi = -ct\sinh^2\phi + ct\cosh^2\phi = ct.$$

Thus the inverse expressions are the same as the original ones except that ϕ is replaced by $-\phi$.

To show the stated equality in the two frames of (the Lorentz invariant) $x^2-(ct)^2$, we simply resubstitute for x and ct:

$$\begin{aligned} x^2 - (ct)^2 &= (x'\cosh\phi + ct'\sinh\phi)^2 - (x'\sinh\phi + ct'\cosh\phi)^2 \\ &= x'^2(\cosh^2\phi - \sinh^2\phi) + (ct')^2(\sinh^2\phi - \cosh^2\phi) \\ &= x'^2 + (ct')^2(-1) = x'^2 - (ct')^2. \end{aligned}$$

Thus, this form, '$x^2 - (ct)^2$' has a value that is independent of ϕ and so has the same value in all frames of reference; it is called a scalar invariant.

3.28 *The principal value of the logarithmic function of a complex variable is defined to have its argument in the range* $-\pi < \arg z \leq \pi$. *By writing* $z = \tan w$ *in terms of exponentials show that*

$$\tan^{-1} z = \frac{1}{2i} \ln \left(\frac{1+iz}{1-iz} \right).$$

Use this result to evaluate

$$\tan^{-1} \left(\frac{2\sqrt{3} - 3i}{7} \right).$$

We express $\tan w$ in terms of exponential functions by first writing it as $\sin w / \cos w$:

$$\begin{aligned}
z = \tan w &= \frac{-i(e^{iw} - e^{-iw})}{e^{iw} + e^{-iw}}, \\
z(e^{iw} + e^{-iw}) &= -ie^{iw} + ie^{-iw}, \\
(z+i)e^{2iw} &= -z + i, \\
e^{2iw} &= \frac{1+iz}{1-iz}, \\
\tan^{-1} z = w &= \frac{1}{2i} \ln \frac{1+iz}{1-iz}.
\end{aligned}$$

Now setting $z = (2\sqrt{3} - 3i)/7$ and recalling that $\ln z = \ln |z| + i \arg z$ gives

$$\begin{aligned}
\tan^{-1} \frac{2\sqrt{3} - 3i}{7} &= \frac{1}{2i} \ln \frac{7 + i2\sqrt{3} + 3}{7 - i2\sqrt{3} - 3} \\
&= \frac{1}{2i} \ln \left(\frac{10 + i2\sqrt{3}}{4 - i2\sqrt{3}} \, \frac{4 + i2\sqrt{3}}{4 + i2\sqrt{3}} \right) \\
&= \frac{1}{2i} \ln \frac{28 + i28\sqrt{3}}{16 + 12} \\
&= \frac{1}{2i} \ln(1 + i\sqrt{3}) \\
&= \frac{1}{2i} \left(\ln 2 + i\frac{\pi}{3} \right) = \frac{\pi}{6} - i \ln \sqrt{2}.
\end{aligned}$$

4

Series and limits

4.2 *If you invest £1000 on the first day of each year, and interest is paid at 5% on your balance at the end of each year, how much money do you have after 25 years?*

An investment A (= £1000) made at the start of the pth year is worth Ar^{26-p} at the end of the 25th year, where $r = 1.05$. The total value of the investment at the end of 25 years is therefore

$$\sum_{p=1}^{p=25} Ar^{26-p} = Ar \sum_{q=0}^{q=24} r^q = \frac{Ar(r^{25} - 1)}{r - 1},$$

where we have set $25 - p = q$ and used the formula for the sum of a finite geometric series. Inserting numerical values yields £50 113 as the total value.

4.4 *Show that for testing the convergence of the series*

$$x + y + x^2 + y^2 + x^3 + y^3 + \cdots,$$

where $0 < x < y < 1$, the D'Alembert ratio test fails but the Cauchy root test is successful.

The ratio of successive terms has one of the two forms

$$\frac{x^{m+1}}{y^m} = x\left(\frac{x}{y}\right)^m \to 0 \quad \text{as} \quad m \to \infty,$$

or

$$\frac{y^m}{x^m} = \left(\frac{y}{x}\right)^m \to \infty \quad \text{as} \quad m \to \infty.$$

Since the ratio does not tend to a unique limit, the D'Alembert test fails.

However, since $x < y$, the nth term u_n of the series (whether $x^{(n+1)/2}$ or $y^{n/2}$) is $\leq y^{n/2}$. Thus $(u_n)^{1/n} \leq (y^{n/2})^{1/n} = y^{1/2} < 1$, and the Cauchy root test proves the convergence of the series.

That the series does converge is clear, as it is the sum, $x/(1-x) + y/(1-y)$, of two (absolutely) convergent series; but that is not the point of the question!

4.6 *By grouping and rearranging terms of the absolutely convergent series*

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

show that

$$S_\text{o} = \sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} = \frac{3S}{4}.$$

From the given result we pick out the terms making up the wanted series and note that what remains (the sum of the inverse squares of the even integers) is a multiple of the originally given series.

$$\begin{aligned} S &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \\ &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{2^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) = S_\text{o} + \tfrac{1}{4}S, \end{aligned}$$

and hence the stated result.

4.8 *The $N+1$ complex numbers ω_m are given by $\omega_m = \exp(2\pi i m/N)$ for $m = 0, 1, 2, \ldots, N$.*

(a) *Evaluate the following:*

$$\text{(i)} \sum_{m=0}^{N} \omega_m, \quad \text{(ii)} \sum_{m=0}^{N} \omega_m^2, \quad \text{(iii)} \sum_{m=0}^{N} \omega_m x^m.$$

(b) *Use these results to evaluate*

$$\text{(i)} \sum_{m=0}^{N} \left[\cos\left(\frac{2\pi m}{N}\right) - \cos\left(\frac{4\pi m}{N}\right)\right], \quad \text{(ii)} \sum_{m=0}^{3} 2^m \sin\left(\frac{2\pi m}{3}\right).$$

(a)(i) This sum is a geometric series with common ratio $e^{2\pi i/N}$. Thus,

$$\sum_{m=0}^{N} \omega_m = e^{2\pi i 0/N} \frac{1 - e^{2\pi i(N+1)/N}}{1 - e^{2\pi i/N}} = 1 \frac{1 - e^{2\pi i/N}}{1 - e^{2\pi i/N}} = 1.$$

However, if $N = 1$ then the common ratio is unity and direct computation is needed:

$$\sum_{m=0}^{1} \omega_m = e^0 + e^{2\pi i} = 2.$$

(a)(ii) By a similar calculation to that in (i),

$$\sum_{m=0}^{N} \omega_m^2 = e^{4\pi i 0/N} \frac{1 - e^{4\pi i(N+1)/N}}{1 - e^{4\pi i/N}},$$

i.e. is unity unless $N = 1$ or $N = 2$, when the common ratio is unity.

For $N = 1$ there are two terms each equal to 1 and the sum equals 2.
For $N = 2$ there are three terms each equal to 1 and the sum equals 3.

(a)(iii) If $x = 1$ then the calculation is as in (i); we assume that $x \neq 1$. This sum is then a geometric series with common ratio $xe^{2\pi i/N}$. Thus,

$$\sum_{m=0}^{N} \omega_m x^m = x^0 e^{2\pi i 0/N} \frac{1 - x^{N+1} e^{2\pi i(N+1)/N}}{1 - xe^{2\pi i/N}} = \frac{1 - x^{N+1} e^{2\pi i/N}}{1 - xe^{2\pi i/N}}.$$

If $N = 1$ then this reduces to $(1 - x^2)/(1 - x)$, i.e. to $1 + x$.

(b)(i) We recognise $\cos(2\pi m/N)$ as the real part of ω_m and, by squaring the definition of ω_m, recognise $\cos(4\pi m/N)$ as the real part of ω_m^2. And so

$$\sum_{m=0}^{N} \left[\cos\left(\frac{2\pi m}{N}\right) - \cos\left(\frac{4\pi m}{N}\right) \right] = \text{Re} \sum_{m=0}^{N} (\omega_m - \omega_m^2).$$

From the previous results, this has the value $2 - 2 = 0$ for $N = 1$, $1 - 3 = -2$ for $N = 2$ and $1 - 1 = 0$ for all $N \geq 3$.

(b)(ii) Taking $N = 3$

$$\sum_{m=0}^{3} 2^m \sin\left(\frac{2\pi m}{3}\right) = \text{Im} \sum_{m=0}^{3} \omega_m 2^m$$
$$= \text{Im} \frac{1 - 2^4 e^{2\pi i/3}}{1 - 2e^{2\pi i/3}}$$
$$= \text{Im} \frac{1 + 8 - i8\sqrt{3}}{1 + 1 - i\sqrt{3}}$$
$$= \text{Im} \frac{(9 - i8\sqrt{3})(2 + i\sqrt{3})}{7}$$
$$= \frac{-16\sqrt{3} + 9\sqrt{3}}{7}$$
$$= -\sqrt{3}.$$

In the second line we used the result from part (a)(iii).

> ***4.10*** *Determine whether the following series converge (θ and p are positive real numbers):*
>
> $$\text{(a)} \ \sum_{n=1}^{\infty} \frac{2\sin n\theta}{n(n+1)}, \quad \text{(b)} \ \sum_{n=1}^{\infty} \frac{2}{n^2}, \quad \text{(c)} \ \sum_{n=1}^{\infty} \frac{1}{2n^{1/2}},$$
>
> $$\text{(d)} \ \sum_{n=2}^{\infty} \frac{(-1)^n(n^2+1)^{1/2}}{n \ln n}, \quad \text{(e)} \ \sum_{n=1}^{\infty} \frac{n^p}{n!}.$$

(a) As shown in the text, the series $\sum_{n=1}^{N} \frac{1}{n(n+1)}$ has a partial sum of $\frac{N}{N+1}$. This tends to the limit 1 as $N \to \infty$. Thus the series is (absolutely) convergent. The nth term of the given series is $\leq \frac{1}{n(n+1)}$ in magnitude. Thus, the given series is absolutely convergent and therefore also convergent.

(b) If, in the quotient test, we take $u_n = \frac{1}{n(n+1)}$ and $v_n = \frac{2}{n^2}$ then $\rho = \frac{1}{2}$. Since this is finite but non-zero and $\sum u_n$ converges, so must $\sum v_n$.

(c) We have already shown that $\sum n^{-1}$ diverges. Every term of the series $u_n = n^{-1/2}$ is not smaller than the corresponding term in $\sum n^{-1}$. It follows that $\sum n^{-1/2}$ must also diverge.

(d) As $n \to \infty$ the terms decrease to zero (albeit slowly) as $1/\ln n$. This together with the alternating sign of consecutive terms in enough to establish convergence using the alternating series test.

(e) The ratio of successive terms is

$$\frac{(n+1)^p}{(n+1)!}\,\frac{n!}{n^p} = \frac{\left(1+\frac{1}{n}\right)^p}{n+1} \to 0 \quad \text{for all } p > 0.$$

Thus, by d'Alembert's ratio test, the series converges.

4.12 *Determine whether the following series are convergent:*

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{n^{1/2}}{(n+1)^{1/2}}, \quad \text{(b)}\ \sum_{n=1}^{\infty} \frac{n^2}{n!}, \quad \text{(c)}\ \sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^{n/2}}, \quad \text{(d)}\ \sum_{n=1}^{\infty} \frac{n^n}{n!}.$$

(a) The individual terms tend to unity, rather than zero, as $n \to \infty$ and so the series must diverge.

(b) The succesive term ratio is

$$\frac{(n+1)^2}{(n+1)!}\,\frac{n!}{n^2} = \frac{1}{n+1}\left(1+\frac{1}{n}\right)^2 \to 0 \quad \textit{as} \quad n \to \infty.$$

Thus the series is convergent.

(c) The nth root of u_n is $(\ln n)/n^{1/2}$ which $\to 0$ as $n \to \infty$. Thus the series is convergent by the Cauchy root test.

(d) The succesive term ratio is

$$\frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n} = \frac{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}}{(n+1)\,n^n} = \left(1+\frac{1}{n}\right)^n \to e \quad \text{as} \quad n \to \infty.$$

This ratio is > 1 and so the series diverges.

4.14 *Obtain the positive values of x for which the following series converges:*

$$\sum_{n=1}^{\infty} \frac{x^{n/2}e^{-n}}{n}.$$

Using the Cauchy root test:

$$\rho = \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{x^{n/2}e^{-n}}{n}\right)^{1/n} = \frac{x^{1/2}}{e}.$$

For convergence $\rho < 1$ and this requires $x < e^2$. Direct substitution shows that that the series diverges at the limit $x = e^2$.

4.16 *An extension to the proof of the integral test (subsection 4.3.2) shows that, if $f(x)$ is positive, continuous and monotonically decreasing, for $x \geq 1$, and the series $f(1) + f(2) + \cdots$ is convergent, then its sum does not exceed $f(1) + L$, where L is the integral*

$$\int_1^\infty f(x)\, dx.$$

Use this result to show that the sum $\zeta(p)$ of the Riemann zeta series $\sum n^{-p}$, with $p > 1$, is not greater than $p/(p-1)$.

The function $f(x)$ appropriate to the Riemann zeta series is $f(x) = x^{-p}$.

$$\int_1^\infty \frac{1}{x^p}\, dx = \left[-\frac{1}{p-1} \frac{1}{x^{p-1}} \right]_1^\infty = \frac{1}{p-1}.$$

This implies that

$$\sum_{n=1}^\infty \frac{1}{n^p} \leq \frac{1}{1^p} + \frac{1}{p-1} = \frac{p}{p-1}.$$

4.18 *Illustrate result* (iv) *of section 4.4 concerning Cauchy products by considering the double summation*

$$S = \sum_{n=1}^\infty \sum_{r=1}^n \frac{1}{r^2(n+1-r)^3}.$$

By examining the points in the nr-plane over which the double summation is to be carried out, show that S can be written as

$$S = \sum_{n=r}^\infty \sum_{r=1}^\infty \frac{1}{r^2(n+1-r)^3}.$$

Deduce that $S \leq 3$.

As shown in figure 4.1, the original summation runs along lines (shown solid) parallel to the r-axis. The same mesh of points can be covered by lines running parallel to the n-axis; these are shown broken in the figure. Thus we have

$$S = \sum_{n=1}^\infty \sum_{r=1}^n \frac{1}{r^2(n+1-r)^3} = \sum_{n=r}^\infty \sum_{r=1}^\infty \frac{1}{r^2(n+1-r)^3}.$$

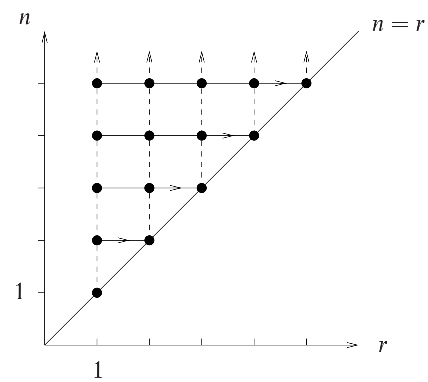

Figure 4.1 The summation points used in exercise 4.18.

If we now set $n+1-r$ equal to s, the double summation becomes

$$S = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r^2 s^3} = \zeta(2)\zeta(3).$$

In view of the result of exercise 4.16, $\zeta(2) \le 2$ and $\zeta(3) \le 3/2$. Consequently $S \le 3$.

4.20 *Identify the series*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n-1)!},$$

and then by integration and differentiation deduce the values S of the following series:

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{(2n)!};\qquad \text{(b)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{(2n+1)!};$$

$$\text{(c)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n\pi^{2n}}{4^n(2n-1)!};\qquad \text{(d)}\ \sum_{n=0}^{\infty}\frac{(-1)^{n}(n+1)}{(2n)!}.$$

Writing out the first few terms of the given series,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n-1)!} &= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots \\ &= x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= x\sin x, \qquad (*)\end{aligned}$$

allows ready identification of the series.

(a) Differentiate both sides of (∗):

$$\sin x + x\cos x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n x^{2n-1}}{(2n-1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4n^2 x^{2n-1}}{(2n)!}.$$

Now set $x = 1$ to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(2n)!} = \tfrac{1}{4}(\sin 1 + \cos 1) = 0.345.$$

(b) Integrate both sides of (∗):

$$\int^{x} x\sin x\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)(2n-1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n x^{2n+1}}{(2n+1)!}.$$

Now, the LHS can be explicitly integrated by parts to yield $\sin x - x\cos x$. Setting $x = 1$ in both this and the derived series then gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(2n+1)!} = \tfrac{1}{2}(\sin 1 - \cos 1) = 0.151.$$

(c) This is clearly similar to the RHS of the equation obtained in (a) if both its denominator and numerator are divided by $2n$ and x is set to $\pi/2$. If this is done then the equation in (a) reads

$$\sin \tfrac{1}{2}\pi + \tfrac{1}{2}\pi \cos \tfrac{1}{2}\pi = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n\, \pi^{2n-1}}{(2n-1)!\, 2^{2n-1}}.$$

After multiplying both sides by $\frac{1}{2}\pi$ and noting that $\cos\frac{1}{2}\pi = 0$, this can be rearranged as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n\pi^{2n}}{4^n (2n-1)!} = \tfrac{1}{2} \times 1 \times \tfrac{1}{2}\pi = \tfrac{1}{4}\pi.$$

(d) It is not obvious how to obtain this sum, but the lowered starting value for the summation index suggests a redefinition of it with $n \to n-1$. To achieve this result the equation in (a) needs to be differentiated again:

$$\begin{aligned} 2\cos x - x\sin x &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n(2n-1) x^{2n-2}}{(2n-1)!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n\, x^{2n-2}}{(2n-2)!} \\ &= \sum_{s=0}^{\infty} \frac{(-1)^{s} 2(s+1) x^{2s}}{(2s)!}, \end{aligned}$$

where in the last line we have defined s as $n-1$. Finally, setting $x = 1$ and rearranging gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{(2n)!} = \tfrac{1}{2}(2\cos 1 - \sin 1) = 0.120.$$

4.22 *Find the Maclaurin series for*

$$\text{(a)}\ \ \ln\left(\frac{1+x}{1-x}\right), \qquad \text{(b)}\ \ (x^2+4)^{-1}, \qquad \text{(c)}\ \ \sin^2 x.$$

(a) Using the Maclaurin series for $\ln(1+x)$,

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\
&= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) \\
&= 2x + 2\frac{x^3}{3} + \cdots \\
&= 2\sum_{n\ \text{odd}} \frac{x^n}{n}.
\end{aligned}$$

This series is convergent only for $-1 < x < 1$.

(b) Here the binomial expansion can be employed directly. This saves the trouble of having to find the expansion coefficients by differentiation.

$$\begin{aligned}
(x^2+4)^{-1} &= \frac{1}{4}\left(1 + \frac{x^2}{4}\right)^{-1} \\
&= \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} \\
&= \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n}.
\end{aligned}$$

(c) We calculate the derivatives of $\sin^2 x$ at $x = 0$.

$$\begin{aligned}
y &= \sin^2 x, \\
y' &= 2\sin x\cos x = \sin 2x, \\
y'' &= 2\cos 2x, \\
y''' &= -4\sin 2x, \\
\vdots &= \quad \vdots \\
y^{(2m)} &= (-1)^{m+1}2^{2m-1}\cos 2x, \\
y^{(2m+1)} &= (-1)^m 2^{2m}\sin 2x.
\end{aligned}$$

At $x = 0$ all odd derivatives vanish and $\cos 2x = 1$ in the even ones. So the Maclaurin expansion is

$$\sin^2 x = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}2^{2m-1}x^{2m}}{(2m)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2x)^{2n}}{2(2n)!}.$$

This result could also have been obtained by writing $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and using the known Maclaurin series for a cosine function.

4.24 *Find the first three non-zero terms in the Maclaurin series for the following functions:*

(a) $(x^2 + 9)^{-1/2}$, (b) $\ln[(2 + x)^3]$, (c) $\exp(\sin x)$,
(d) $\ln(\cos x)$, (e) $\exp[-(x - a)^{-2}]$, (f) $\tan^{-1} x$.

(a) We use the binomial expansion directly to avoid the need to find derivatives, but must first get the term inside the parentheses into the form $1 + \alpha x^2$.

$$\begin{aligned}
(x^2 + 9)^{-1/2} &= \frac{1}{3}\left(1 + \frac{x^2}{9}\right)^{-1/2} \\
&= \frac{1}{3}\left[1 - \frac{x^2}{18} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\,\frac{x^4}{81} + \cdots\right] \\
&= \frac{1}{3} - \frac{x^2}{54} + \frac{x^4}{648} + \cdots .
\end{aligned}$$

(b) This function needs a small amount of manipulation before the expansion of $\ln(1 + \delta)$ is invoked.

$$\begin{aligned}
\ln[(2 + x)^3] &= 3\ln\left[2\left(1 + \frac{x}{2}\right)\right] \\
&= 3\ln 2 + 3\left[\frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \cdots\right] \\
&= \ln 8 + \frac{3x}{2} - \frac{3x^2}{8} + \cdots .
\end{aligned}$$

(c) Write $f(x) = \exp(\sin x)$. Then

$$\begin{aligned} f(0) &= 1 \\ f' &= \cos x \, e^{\sin x}, \qquad f'(0) = 1, \\ f'' &= -\sin x \, e^{\sin x} + \cos^2 x \, e^{\sin x}, \qquad f''(0) = 1 \end{aligned}$$

Hence, by the normal Taylor expansion about $x = 0$,

$$f(x) = 1 + x + \frac{x^2}{2!} + \cdots .$$

(d) Here the calculation of the derivatives needed for the Taylor expansion rapidly becomes complicated. Further, as three non-vanishing terms are needed, quite high derivatives are involved.

$$\begin{aligned} y &= \ln(\cos x), \quad y(0) = 0, \\ y' &= -\frac{\sin x}{\cos x} = -\tan x, \quad y'(0) = 0, \\ y'' &= -\sec^2 x, \quad y''(0) = -1, \\ y''' &= -2\sec^2 x \tan x, \quad y'''(0) = 0, \\ y^{(4)} &= -4\sec^2 x \tan^2 x - 2\sec^4 x, \quad y^{(4)}(0) = -2, \\ y^{(5)} &= -8\sec^2 x \tan^3 x - 16\sec^4 x \tan x, \quad y^{(5)}(0) = 0. \end{aligned}$$

Calculating $y^{(6)}$ is complicated but the only term that does not vanish at $x = 0$ comes from differentiating the $\tan x$ factor in the final product in $y^{(5)}$. This term will be $-16\sec^6 x$ which makes the value of $y^{(6)}(0)$ equal to -16. Hence, finally,

$$y(x) = -\frac{x^2}{2!} - 2\frac{x^4}{4!} - 16\frac{x^6}{6!} + \cdots = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + \cdots .$$

(e) Write $f(x) = \exp[-(x-a)^{-2}]$.

$$\begin{aligned} f(x) &= \exp[-(x-a)^{-2}], \quad f(0) = \exp(-a^{-2}), \\ f' &= \frac{2f}{(x-a)^3}, \quad f'(0) = -\frac{2}{a^3}\exp(-a^{-2}), \\ f'' &= \frac{2f'}{(x-a)^3} - \frac{6f}{(x-a)^4}, \quad f''(0) = \left(\frac{4}{a^6} - \frac{6}{a^4}\right)\exp(-a^{-2}). \end{aligned}$$

Thus, from the Taylor expansion, the Maclaurin series is

$$f(x) = \exp(-a^{-2})\left[1 - \frac{2x}{a^3} - \frac{3a^2 - 2}{a^6}x^2 + \cdots\right].$$

(f) If $f(x) = \tan^{-1} x$ then

$$f' = \frac{1}{1+x^2}, \quad f'' = -\frac{2x}{(1+x^2)^2}, \quad f''' = \frac{6x^2 - 2}{(1+x^2)^3}$$

$$f^{(4)} = \frac{24x - 24x^3}{(1+x^2)^4}, \quad f^{(5)} = \frac{24 - \cdots}{(1+x^2)^5}.$$

At $x = 0$ only the odd derivatives are non-zero (as they must be since $\tan^{-1} x$ is an odd function of x) and

$$\begin{aligned} \tan^{-1} x &= x - 2\frac{x^3}{3!} + 24\frac{x^5}{5!} + \cdots \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots . \end{aligned}$$

> ***4.26*** *Determine whether the following functions $f(x)$ are* (i) *continuous, and* (ii) *differentiable at $x = 0$:*
>
> (a) $f(x) = \exp(-|x|)$;
> (b) $f(x) = (1 - \cos x)/x^2$ *for* $x \neq 0$, $f(0) = \frac{1}{2}$;
> (c) $f(x) = x \sin(1/x)$ *for* $x \neq 0$, $f(0) = 0$;
> (d) $f(x) = [4 - x^2]$, *where* $[y]$ *denotes the integer part of* y.

(a) Taking $\Delta > 0$

$$\exp(-|\Delta|) = 1 - \Delta + \frac{\Delta^2}{2!} - \cdots ,$$

whilst for $\Delta < 0$

$$\exp(-|\Delta|) = 1 + \Delta + \frac{\Delta^2}{2!} + \cdots .$$

In the limit $\Delta \to 0$ both series tend to the common limit 1. Thus the function is continuous at $x = 0$.

However, the derivative at $x = 0$ when $\Delta > 0$ is given by

$$\lim_{\Delta \to 0} \frac{1 - \Delta + \frac{\Delta^2}{2!} - \cdots - 1}{\Delta},$$

whilst that for $\Delta < 0$ is

$$\lim_{\Delta \to 0} \frac{1 + \Delta + \frac{\Delta^2}{2!} + \cdots - 1}{\Delta}.$$

The former limit has value -1 whilst the latter is $+1$. These are not equal and so $\exp(-|x|)$ is not differentiable at $x = 0$.

(b) Whether x is positive or negative,

$$\frac{1-\cos x}{x^2} = \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}{x^2}$$
$$= \frac{1}{2} - \frac{x^2}{4!} + \cdots .$$

As $x \to 0$ from either positive or negative values this expression tends to the value $\frac{1}{2}$. As this is the defined value of the function at $x = 0$, the function is continuous.

Proceeding as in (a), the first derivative of $f(x)$ is given for any Δ by

$$\lim_{\Delta \to 0} \frac{\frac{1}{2} - \frac{\Delta^2}{4!} + \cdots - \frac{1}{2}}{\Delta}.$$

This has the value zero, whether Δ is positive or negative. Thus, the function is differentiable at $x = 0$.

(c) Consider the modulus of $x \sin(1/x)$. This is $\leq |x|$ for all x and, as the latter $\to 0$ as $x \to 0$, $|x \sin(1/x)|$ must also $\to 0$ as $x \to 0$. Thus the function is continuous at $x = 0$.

However,

$$\frac{f(x)-f(0)}{x-0} = \frac{x\sin(1/x)-0}{x} = \sin\left(\frac{1}{x}\right).$$

This oscillates (increasingly rapidly) as $x \to 0$ between ± 1 and does not tend to a limit. Therefore the function is not differentiable at $x = 0$.

(d) For $|x| \leq 1$ the function $[4 - x^2]$ has the (constant) value 3. However, at $x = 0$ it has the value 4. Thus the fuction is not continuous at $x = 0$. It also follows that it cannot be differentiable there.

> ***4.28*** *Evaluate the following limits:*
>
> $$\text{(a)} \lim_{x\to 0} \frac{\sin 3x}{\sinh x}, \qquad \text{(b)} \lim_{x\to 0} \frac{\tan x - \tanh x}{\sinh x - x},$$
> $$\text{(c)} \lim_{x\to 0} \frac{\tan x - x}{\cos x - 1}, \quad \text{(d)} \lim_{x\to 0} \left(\frac{\operatorname{cosec} x}{x^3} - \frac{\sinh x}{x^5}\right).$$

Using L'Hôpital's rule whenever both the numerator and denominator of a fraction become zero in the limit, we have for cases (a)-(c):

(a) $$\lim_{x\to 0} \frac{\sin 3x}{\sinh x} = \lim_{x\to 0} \frac{3\cos 3x}{\cosh x} = 3.$$

(b)
$$\begin{aligned}
L &= \lim_{x\to 0} \frac{\tan x - \tanh x}{\sinh x - x} \\
&= \lim_{x\to 0} \frac{\sec^2 x - \operatorname{sech}^2 x}{\cosh x - 1} \\
&= \lim_{x\to 0} \frac{2\sec^2 x \tan x + 2\operatorname{sech}^2 x \tanh x}{\sinh x} \\
&= \lim_{x\to 0} \frac{2(\tan x + \tan^3 x + \tanh x - \tanh^3 x)}{\sinh x} \\
&= \lim_{x\to 0} \frac{2(\sec^2 x + 3\tan^2 x \sec^2 x + \operatorname{sech}^2 x - 3\tanh^2 x \operatorname{sech}^2 x)}{\cosh x} \\
&= 4.
\end{aligned}$$

(c)
$$\lim_{x\to 0} \frac{\tan x - x}{\cos x - 1} = \lim_{x\to 0} \frac{\sec^2 x - 1}{-\sin x} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{-\cos x} = 0.$$

(d) In order to avoid undetermined combinations such as $x \operatorname{cosec} x$, we arrange the function so that all terms in it tend to 0 or a finite quantity as $x \to 0$. For this particular case this also allows us to use known McLaurin expansions of the factors involved.

[Inspection of the function shows that both fractions behave as x^{-4} as $x \to 0$, but that their difference will be less divergent. Finding the limit (if it is finite) using L'Hôpital's rule will take up to six differentiations; hence our preference for Maclaurin expansions.]

$$\begin{aligned}
L &= \frac{\operatorname{cosec} x}{x^3} - \frac{\sinh x}{x^5} \\
&= \frac{x^2 - \sin x \sinh x}{x^5 \sin x} \\
&= \frac{x^2 - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}{x^5\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} \\
&= \frac{1 - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)}{x^4\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)} \\
&= \frac{(1-1) + x^2\left(\frac{1}{3!} - \frac{1}{3!}\right) + x^4\left(-\frac{1}{5!} - \frac{1}{5!} + \frac{1}{(3!)^2}\right) + \cdots}{x^4\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)},
\end{aligned}$$

which has as its limit, when $x \to 0$, the value

$$\frac{1}{(3!)^2} - \frac{2}{5!} = \frac{20 - 12}{720} = \frac{1}{90}.$$

4.30 *Use Taylor expansions to three terms to find approximations to* (a) $\sqrt[4]{17}$, *and* (b) $\sqrt[3]{26}$.

(a) For $f(x) = x^{1/4}$,

$$f'(x) = \frac{1}{4}\frac{1}{x^{3/4}} \quad \text{and} \quad f''(x) = -\frac{3}{16}\frac{1}{x^{7/4}}.$$

Thus, writing 17 as $16 + 1$,

$$\begin{aligned}(17)^{1/4} &= (16)^{1/4} + 1\,\frac{1}{4}\,\frac{1}{(16)^{3/4}} - \frac{1}{2}\,1^2\,\frac{3}{16}\,\frac{1}{(16)^{7/4}} + \cdots \\ &= 2 + \frac{1}{32} - \frac{3}{2 \times 16 \times 16 \times 8} + \cdots \\ &= 2.030\,518.\end{aligned}$$

The more accurate answer is 2.030 543.

(b) For $f(x) = x^{1/3}$,

$$f'(x) = \frac{1}{3}\frac{1}{x^{2/3}} \quad \text{and} \quad f''(x) = -\frac{2}{9}\frac{1}{x^{5/3}}.$$

Thus, writing 26 as $27 + (-1)$,

$$\begin{aligned}(26)^{1/3} &= (27)^{1/3} + (-1)\,\frac{1}{3}\,\frac{1}{(27)^{2/3}} - \frac{1}{2}\,(-1)^2\,\frac{2}{9}\,\frac{1}{(27)^{5/3}} + \cdots \\ &= 3 - \frac{1}{27} - \frac{1}{9 \times 27 \times 9} + \cdots \\ &= 2.962\,506.\end{aligned}$$

The more accurate answer is 2.962 496.

4.32 *Evaluate*

$$\lim_{x\to 0}\left[\frac{1}{x^3}\left(\operatorname{cosec} x - \frac{1}{x} - \frac{x}{6}\right)\right].$$

In order to evaluate the limit L, we avoid ill-defined products by multiplying both numerator and denominator by $x \sin x$. (See also part (d) of exercise 4.28)

$$\begin{aligned}
L &= \lim_{x\to 0} \frac{1}{x^3}\left(\operatorname{cosec} x - \frac{1}{x} - \frac{x}{6}\right) \\
&= \lim_{x\to 0} \frac{1}{x^4 \sin x}\left(x - \sin x - \frac{x^2}{6}\sin x\right) \\
&= \lim_{x\to 0} \frac{1}{x^4 \sin x}\left[x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \frac{x^2}{6}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\right] \\
&= \lim_{x\to 0} \frac{1}{x^4 \sin x}\left[x(1-1) + x^3\left(\frac{1}{3!} - \frac{1}{6}\right) + x^5\left(-\frac{1}{5!} + \frac{1}{6\times 3!}\right) + \cdots\right] \\
&= \lim_{x\to 0}\left(\frac{x}{\sin x}\,\frac{-6+20}{720} + \cdots\right) \\
&= \frac{7}{360}.
\end{aligned}$$

4.34 *In a very simple model of a crystal, point-like atomic ions are regularly spaced along an infinite one-dimensional row with spacing R. Alternate ions carry equal and opposite charges ±e. The potential energy of the ith ion in the electric field due to the jth ion is*

$$\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}},$$

where q_i, q_j are the charges on the ions and r_{ij} is the distance between them.

Write down a series giving the total contribution V_i of the ith ion to the overall potential energy. Show that the series converges, and, if V_i is written as

$$V_i = \frac{\alpha e^2}{4\pi\epsilon_0 R},$$

find a closed-form expression for α, the Madelung constant for this (unrealistic) lattice.

The ion that is nR distant from the ith ion has charge $(-1)^n q_i$ and contributes

$$\frac{(-1)^n q_i}{4\pi\epsilon_0\,(nR)}$$

to the potential at the ith ion. The infinite sum of terms of this form converges by the alternating sign test and the total potential energy of the ith ion is

$$q_i \sum_{n=-\infty}^{-1} \frac{(-1)^n q_i}{4\pi\epsilon_0\,|n|R} + q_i \sum_{n=1}^{\infty} \frac{(-1)^n q_i}{4\pi\epsilon_0\, nR} = \frac{2q_i^2}{4\pi\epsilon_0 R}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = \frac{2e^2 \ln 2}{4\pi\epsilon_0 R}.$$

The value of α is thus $-2\ln 2$.

4.36 *In quantum theory a certain method (the Born approximation) gives the (so-called) amplitude $f(\theta)$ for the scattering of a particle of mass m through an angle θ by a uniform potential well of depth V_0 and radius b (i.e. the potential energy of the particle is $-V_0$ within a sphere of radius b and zero elsewhere) as*

$$f(\theta) = \frac{2mV_0}{\hbar^2 K^3}(\sin Kb - Kb\cos Kb).$$

Here $\hbar$ is the Planck constant divided by 2π, the energy of the particle is $\hbar^2k^2/(2m)$ and K is $2k\sin(\theta/2)$.

Use l'Hôpital's rule to evaluate the amplitude at low energies, i.e. when k and hence K tend to zero, and so determine the low-energy total cross-section.

[*Note: the differential cross-section is given by $|f(\theta)|^2$ and the total cross-section by the integral of this over all solid angles, i.e. $2\pi\int_0^\pi |f(\theta)|^2 \sin\theta \, d\theta$.*]

We need to determine the value L given by

$$\begin{aligned} L &= \lim_{K\to 0} \frac{\sin Kb - Kb\cos Kb}{K^3} \\ &= \lim_{K\to 0} \frac{\left(Kb - \dfrac{(Kb)^3}{3!} + \cdots\right) - Kb\left(1 - \dfrac{(Kb)^2}{2!} + \dfrac{(Kb)^4}{4!} - \cdots\right)}{K^3} \\ &= \lim_{K\to 0} \frac{(Kb)^3\left(-\dfrac{1}{3!} + \dfrac{1}{2!} + O(K^2b^2)\right)}{K^3} \\ &= \tfrac{1}{3}b^3. \end{aligned}$$

Thus $f(\theta) = 2mV_0b^3/(3\hbar)^2$. This result is real and independent of θ (indicating that the scattering is isotropic). Consequently, the total cross-section is simply 4π times the square of $f(\theta)$, i.e. $4\pi[2mV_0b^3/(3\hbar)^2]^2$.

5

Partial differentiation

5.2 *Determine which of the following are exact differentials:*

(a) $(3x + 2)y\,dx + x(x + 1)\,dy$;
(b) $y \tan x\,dx + x \tan y\,dy$;
(c) $y^2(\ln x + 1)\,dx + 2xy \ln x\,dy$;
(d) $y^2(\ln x + 1)\,dy + 2xy \ln x\,dx$;
(e) $[x/(x^2 + y^2)]\,dy - [y/(x^2 + y^2)]\,dx$.

If $df = A\,dx + B\,dy$ then a necessary and sufficient condition that df is exact is that $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$. We apply this test in each case.

(a) $$\frac{\partial[(3x + 2)y]}{\partial y} = 3x + 2,$$
$$\frac{\partial[x(x + 1)]}{\partial x} = 2x + 1, \text{ unequal} \Rightarrow \text{not exact.}$$

(b) $$\frac{\partial(y \tan x)}{\partial y} = \tan x, \quad \frac{\partial(x \tan y)}{\partial x} = \tan y, \quad \text{unequal} \Rightarrow \text{not exact.}$$

(c) $$\frac{\partial[y^2(\ln x + 1)]}{\partial y} = 2y(\ln x + 1),$$
$$\frac{\partial(2xy \ln x)}{\partial x} = 2y \ln x + 2xy\frac{1}{x} = 2y(\ln x + 1), \quad \text{equal} \Rightarrow \text{exact.}$$

(d) $$\frac{\partial[y^2(\ln x + 1)]}{\partial x} = \frac{y^2}{x}, \quad \frac{\partial(2xy \ln x)}{\partial y} = 2x \ln x, \text{ unequal} \Rightarrow \text{not exact.}$$

(e) $$\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

as was shown in exercise 5.1(c). The equality of these two expressions shows that the differential is exact.

5.4 *Show that*

$$df = y(1 + x - x^2)\,dx + x(x+1)\,dy$$

is not an exact differential.

Find the differential equation that a function $g(x)$ must satisfy if $d\phi = g(x)df$ is to be an exact differential. Verify that $g(x) = e^{-x}$ is a solution of this equation and deduce the form of $\phi(x, y)$.

If $df = A\,dx + B\,dy$ then a necessary and sufficient condition that df is exact is that $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$. We apply this test.

$$\frac{\partial[y(1 + x - x^2)]}{\partial y} = 1 + x - x^2, \qquad \frac{\partial[x(x+1)]}{\partial x} = 2x + 1.$$

These are not equal and so the differential is not exact.

If $g(x)df$ is to be exact we must have

$$\begin{aligned}\frac{\partial[gy(1 + x - x^2)]}{\partial y} &= \frac{\partial[gx(x+1)]}{\partial x},\\ g(1 + x - x^2) &= g'x(x+1) + g(2x+1),\\ g(-x - x^2) &= g'x(x+1),\\ -g &= g'.\end{aligned}$$

Clearly $g(x) = e^{-x}$, with $g'(x) = -e^{-x}$, satisfies this equation. The more general solution $g(x) = Ae^{-x}$ would do just as well from the point of view of making $d\phi$ an exact differential.

Accepting the stated form (with $A = 1$), we must now have that

$$x(x+1)e^{-x} = \frac{\partial\phi}{\partial y}, \quad \Rightarrow \quad \phi(x, y) = xy(x+1)e^{-x} + h(x).$$

$$y(1 + x - x^2)e^{-x} = \frac{\partial\phi}{\partial x} = -xy(x+1)e^{-x} + (2x+1)ye^{-x} + h'(x).$$

The terms involving e^{-x} all cancel and thus $h'(x) = 0$, implying $h(x) = k$ (a constant) and $\phi(x, y) = xy(x+1)e^{-x} + k$.

5.6 *A possible equation of state for a gas takes the form*

$$pV = RT \exp\left(-\frac{\alpha}{VRT}\right),$$

in which α *and* R *are constants. Calculate expressions for*

$$\left(\frac{\partial p}{\partial V}\right)_T, \qquad \left(\frac{\partial V}{\partial T}\right)_p, \qquad \left(\frac{\partial T}{\partial p}\right)_V,$$

and show that their product is -1*, as stated in section 5.4.*

The required differentiations are most easily carried out after taking the logarithms of both sides of the original equation (for typographical convenience we use P instead of p):

$$\ln P + \ln V - \ln R - \ln T = -\frac{\alpha}{VRT}. \qquad (*)$$

First, differentiating $(*)$ with respect to V with T held fixed:

$$\frac{1}{P}\left(\frac{\partial P}{\partial V}\right)_T + \frac{1}{V} = \frac{\alpha}{V^2RT},$$

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\alpha}{V^2RT} - \frac{1}{V}\right)P$$

$$= \frac{P(\alpha - VRT)}{V^2RT}.$$

Next, differentiating $(*)$ with respect to T with P held fixed:

$$\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P - \frac{1}{T} = \frac{\alpha}{VRT^2} + \frac{\alpha}{V^2RT}\left(\frac{\partial V}{\partial T}\right)_P,$$

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{\dfrac{\alpha}{VRT^2} + \dfrac{1}{T}}{\dfrac{1}{V} - \dfrac{\alpha}{V^2RT}}$$

$$= \frac{V(\alpha + VRT)}{T(VRT - \alpha)}.$$

Finally, differentiating $(*)$ with respect to P with V held fixed.

$$\frac{1}{P} - \frac{1}{T}\left(\frac{\partial T}{\partial P}\right)_V = \frac{\alpha}{VRT^2}\left(\frac{\partial T}{\partial P}\right)_V,$$

$$\left(\frac{\partial T}{\partial P}\right)_V = \frac{1}{P\left(\dfrac{\alpha}{VRT^2} + \dfrac{1}{T}\right)}$$

$$= \frac{VRT^2}{P(\alpha + VRT)}.$$

These are the three partial derivatives and their product,

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V,$$

is

$$\frac{P(\alpha - VRT)}{V^2RT} \frac{V(\alpha + VRT)}{T(VRT - \alpha)} \frac{VRT^2}{P(\alpha + VRT)} = -1,$$

as expected.

5.8 *In the xy-plane, new coordinates s and t are defined by*

$$s = \tfrac{1}{2}(x + y), \qquad t = \tfrac{1}{2}(x - y).$$

Transform the equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0$$

into the new coordinates and deduce that its general solution can be written

$$\phi(x, y) = f(x + y) + g(x - y),$$

where $f(u)$ and $g(v)$ are arbitrary functions of u and v respectively.

To make the transformation we need the partial derivatives

$$\frac{\partial s}{\partial x} = \frac{1}{2}, \quad \frac{\partial s}{\partial y} = \frac{1}{2}, \quad \frac{\partial t}{\partial x} = \frac{1}{2}, \quad \frac{\partial t}{\partial y} = -\frac{1}{2}.$$

Now, using the chain rule, we obtain for the partial differential operators

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x} = \frac{1}{2}\frac{\partial}{\partial s} + \frac{1}{2}\frac{\partial}{\partial t},$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\frac{\partial t}{\partial y} = \frac{1}{2}\frac{\partial}{\partial s} - \frac{1}{2}\frac{\partial}{\partial t}.$$

Thus, we have

$$\frac{\partial^2}{\partial x^2} = \frac{1}{4}\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)$$
$$= \frac{1}{4}\left(\frac{\partial^2}{\partial s^2} + 2\frac{\partial^2}{\partial t \partial s} + \frac{\partial^2}{\partial t^2}\right),$$
$$\frac{\partial^2}{\partial y^2} = \frac{1}{4}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)$$
$$= \frac{1}{4}\left(\frac{\partial^2}{\partial s^2} - 2\frac{\partial^2}{\partial t \partial s} + \frac{\partial^2}{\partial t^2}\right),$$

and so, by subtraction of these two operators,

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial t \partial s}.$$

On writing $\phi(x, y)$ as $\psi(s, t)$, the original equation becomes

$$\frac{\partial^2 \psi}{\partial t \partial s} = 0,$$

with a first integral (with respect to t) of

$$\frac{\partial \psi}{\partial s} = f_1(s),$$

where $f_1(s)$ is any arbitrary function of s (but not of t). A second integration, with respect to s, gives

$$\psi = \int^s f_1(u)\, du + f_2(t),$$

where $f_2(t)$ is any arbitrary function of t (but not of s).

Thus $\phi(x, y) = \psi(s, t)$ can be written as the sum of two arbitrary functions, one of s and the other of t, or, equivalently, of two (slightly different, because of the factors of $\frac{1}{2}$) arbitrary functions of $x + y$ and $x - y$, respectively.

5.10 *If $x = e^u \cos\theta$ and $y = e^u \sin\theta$, show that*

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial \theta^2} = (x^2 + y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right),$$

where $f(x, y) = \phi(u, \theta)$.

The four partial derivatives needed to make the change of variables are:

$$\frac{\partial x}{\partial u} = e^u \cos\theta = x, \qquad \frac{\partial x}{\partial \theta} = -e^u \sin\theta = -y,$$
$$\frac{\partial y}{\partial u} = e^u \sin\theta = y, \qquad \frac{\partial y}{\partial \theta} = e^u \cos\theta = x,$$

giving (using the chain rule) the connections between the differential operators in the two sets of coordinates as

$$\frac{\partial}{\partial u} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Now,

$$\begin{aligned}
\frac{\partial^2}{\partial u^2} &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \\
&= x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + yx\frac{\partial^2}{\partial y\partial x} + xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y}. \\
\frac{\partial^2}{\partial \theta^2} &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\
&= y^2\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x} - xy\frac{\partial^2}{\partial y\partial x} - yx\frac{\partial^2}{\partial x\partial y} + x^2\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}.
\end{aligned}$$

Adding these two operators:

$$\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial \theta^2} = (x^2+y^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + xy\left(2\frac{\partial^2}{\partial x\partial y} - 2\frac{\partial^2}{\partial x\partial y}\right).$$

Thus,

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial \theta^2} = (x^2+y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right),$$

where $f(x,y) = \phi(u,\theta)$.

5.12 *Show that*

$$f(x,y) = x^3 - 12xy + 48x + by^2, \qquad b \neq 0,$$

has two, one, or zero stationary points according to whether $|b|$ *is less than, equal to, or greater than* 3.

At a stationary point, the total differential df must be zero whatever values the infinitesimal changes dx and dy take. This condition requires that

$$\begin{aligned}
0 &= \frac{\partial f}{\partial x} = 3x^2 - 12y + 48, \\
0 &= \frac{\partial f}{\partial y} = -12x + 2by.
\end{aligned}$$

From the second of these $y = 6x/b$ (with $b \neq 0$), and the first equation can be written as

$$x^2 - \frac{24}{b}x + 16 = 0.$$

This is a quadratic equation for x and has two, one or zero real roots according to whether

$$\left(\frac{24}{b}\right)^2 \text{ is greater than, or equal to, or less than } 4 \times 1 \times 16,$$

$$\text{i.e. } \left|\frac{24}{b}\right| \text{ is greater than, or equal to, or less than } 8,$$

$$\text{i.e. } |\,b\,| \text{ is less than, or equal to, or greater than } 3.$$

> ***5.14*** *Find the stationary points of the function*
>
> $$f(x, y) = x^3 + xy^2 - 12x - y^2$$
>
> *and identify their nature.*

As explained in the solution to exercise 5.12, stationary points occur when df is zero whatever values the infinitesimal changes dx and dy take. For the present question this implies that

$$0 = \frac{\partial f}{\partial x} = 3x^2 + y^2 - 12,$$
$$0 = \frac{\partial f}{\partial y} = 2xy - 2y.$$

From the second equation, either $y = 0$ or $x = 1$ and, correspondingly, from the first $x = \pm 2$ or $y = \pm 3$.

To determine the nature of the stationary points, we must calculate the second derivatives. The required derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = 2y, \quad \frac{\partial^2 f}{\partial y^2} = 2(x - 1).$$

At $(2, 0)$, $\dfrac{\partial^2 f}{\partial x^2} = 12$, $\dfrac{\partial^2 f}{\partial x \partial y} = 0$ and $\dfrac{\partial^2 f}{\partial y^2} = 2$. Since both unmixed second derivatives are positive and $2 \times 12 > 0$ this is a minimum with value -16.

At $(-2, 0)$, $\dfrac{\partial^2 f}{\partial x^2} = -12$, $\dfrac{\partial^2 f}{\partial x \partial y} = 0$ and $\dfrac{\partial^2 f}{\partial y^2} = -6$. Since both unmixed second derivatives are negative and $-6 \times -12 > 0$ this is a maximum with value 16.

At $(1, \pm 3)$, $\dfrac{\partial^2 f}{\partial x^2} = 6$, $\dfrac{\partial^2 f}{\partial x \partial y} = \pm 6$ and $\dfrac{\partial^2 f}{\partial y^2} = 0$. Since $0 \times 6 \not> 36$ these are saddle points with the common value -11.

5.16 *The temperature of a point (x, y, z) on the unit sphere is given by*

$$T(x, y, z) = 1 + xy + yz.$$

By using the method of Lagrange multipliers find the temperature of the hottest point on the sphere.

It is clear that the larger the absolute values that x, y and z can take, the larger the maximum temperature can be; the hottest point(s) of the sphere will therefore occur on its surface, i.e. the coordinates of the hottest point must satisfy $x^2 + y^2 + z^2 - 1 = 0$.

We incorporate this constraint using a Lagrange multiplier and consider

$$f(x, y, z) = T(x, y, z) + \lambda(x^2 + y^2 + z^2 - 1) = 1 + xy + yz + \lambda(x^2 + y^2 + z^2 - 1).$$

Its stationary values are given by

$$0 = \frac{\partial f}{\partial x} = y + 2\lambda x,$$
$$0 = \frac{\partial f}{\partial y} = x + z + 2\lambda y,$$
$$0 = \frac{\partial f}{\partial z} = y + 2\lambda z.$$

From symmetry $x = z$, leading to

$$0 = y + 2\lambda x \quad \text{and} \quad 0 = 2x + 2\lambda y.$$

Elimination of λ between these two equations gives $y^2 = 2x^2$, and then substitution for y and z in $x^2 + y^2 + z^2 = 1$ yields $x^2 = \frac{1}{4}$ and $x (= z) = \pm\frac{1}{2}$.

The four possible hottest spots are therefore $(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2})$. Direct substitution shows that a maximum of $T = 1 + \frac{1}{\sqrt{2}}$ occurs at $\pm(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$. [The two other stationary points give temperature minima.]

5.18 *Two horizontal corridors, $0 \le x \le a$ with $y \ge 0$, and $0 \le y \le b$ with $x \ge 0$, meet at right angles. Find the length L of the longest ladder (considered as a stick) that may be carried horizontally around the corner.*

Let the ends of the ladder touch the outside walls of the corner at the points $(a+\xi, 0)$ and $(0, b+\eta)$. Then the square of the length of the ladder is

$$L^2 = (a+\xi)^2 + (b+\eta)^2.$$

The longest ladder will touch the inside corner at (a, b) and simple geometry then requires that

$$\frac{\eta}{a} = \frac{b}{\xi}.$$

Thus we need to maximise L (or L^2) subject to $\eta\xi = ab$. We therefore consider

$$f(\xi, \eta) = (a+\xi)^2 + (b+\eta)^2 + \lambda\eta\xi.$$

Its stationary values occur when

$$0 = \frac{\partial f}{\partial \xi} = 2(a+\xi) + \lambda\eta,$$
$$0 = \frac{\partial f}{\partial \eta} = 2(b+\eta) + \lambda\xi.$$

Thus $2\xi(a+\xi) - 2\eta(b+\eta) = 0$; together with $\eta\xi = ab$, this gives as an equation for ξ,

$$2\xi a + 2\xi^2 - \frac{2ab}{\xi}\left(b + \frac{ab}{\xi}\right) = 0,$$
$$\Rightarrow \quad (\xi^3 - ab^2)(\xi + a) = 0.$$

The only physical solution to this is $\xi = (ab^2)^{1/3}$; the corresponding value of η is $ab/\xi = a^{2/3}b^{1/3}$. Then

$$\begin{aligned} L^2 &= \left(a + a^{1/3}b^{2/3}\right)^2 + \left(b + a^{2/3}b^{1/3}\right)^2 \\ &= a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2 \\ &= \left(a^{2/3} + b^{2/3}\right)^3. \end{aligned}$$

Thus the longest ladder that may be carried horizontally round the corner is of length $\left(a^{2/3} + b^{2/3}\right)^{3/2}$.

5.20 *Show that the envelope of all concentric ellipses that have their axes along the x- and y-coordinate axes and that have the sum of their semi-axes equal to a constant L is the same curve (an astroid) as that found in the worked example in section 5.10.*

The equation of a typical ellipse with semi-axis a in the x-direction is

$$f(x, y, a) = \frac{x^2}{a^2} + \frac{y^2}{(L-a)^2} - 1 = 0. \qquad (*)$$

To find the envelope of all the ellipses we set $\partial f/\partial a = 0$. This gives

$$\frac{\partial f}{\partial a} = -\frac{2x^2}{a^3} + \frac{2y^2}{(L-a)^3} = 0.$$

Re-arranging this equation so as to provide expressions that can be used to eliminate a from ($*$) we obtain

$$\frac{x^2}{y^2} = \frac{a^3}{(L-a)^3} \quad \Rightarrow \quad \frac{a}{L-a} = \frac{x^{2/3}}{y^{2/3}},$$

yielding

$$a = \frac{x^{2/3}}{x^{2/3} + y^{2/3}} L \quad \text{and} \quad L - a = \frac{y^{2/3}}{x^{2/3} + y^{2/3}} L.$$

Substituting these values into $f(x, y, a) = 0$ gives the equation of the envelope as

$$\frac{x^2(x^{2/3} + y^{2/3})^2}{x^{4/3}L^2} + \frac{y^2(x^{2/3} + y^{2/3})^2}{y^{4/3}L^2} = 1,$$
$$(x^{2/3} + y^{2/3})^2(x^{2/3} + y^{2/3}) = L^2,$$
$$x^{2/3} + y^{2/3} = L^{2/3},$$

i.e. an astroid.

5.22 *Prove that the envelope of the circles whose diameters are those chords of a given circle that pass through a fixed point on its circumference, is the cardioid*

$$r = a(1 + \cos\theta).$$

Here a is the radius of the given circle and (r, θ) are the polar coordinates of the envelope. Take as the system parameter the angle ϕ between a chord and the polar axis from which θ is measured.

The fixed circle, shown in figure 5.1, has diameter OD. Its chord OQ is a diameter of a typical member of the family of circles generated as Q is varied.

Since OQ is a diameter of the typical circle with parameter ϕ the angle OPQ is a right angle. The radial polar coordinate of P is therefore $r = c\cos(\theta - \phi)$. Similarly, since angle OQD is also a right angle, $c = 2a\cos\phi$.

Thus the equation of a typical circle is

$$f(r, \theta, \phi) = r - 2a\cos\phi\cos(\theta - \phi) = 0.$$

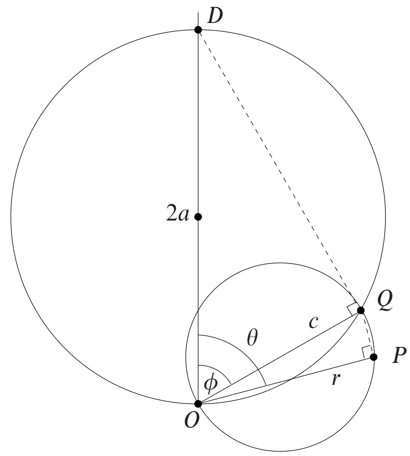

Figure 5.1 The fixed circle in exercise 5.22 has diameter OD and a general chord OQ. A typical member of the family of circles passes through O and has OQ as a diameter. The family is generated as Q is varied.

The envelope to this family of circles, in which ϕ is a parameter that is fixed for each circle, is given by $\partial f/\partial\phi = 0$, i.e.

$$2a\sin\phi\cos(\theta-\phi) - 2a\cos\phi\sin(\theta-\phi) = 0.$$

Using the compound-angle formula for $\sin(A+B)$, this can be simplified to

$$\sin(\phi - (\theta - \phi)) = 0,$$

and thus $\theta = 2\phi$. This gives the point on the circle with parameter ϕ at which the envelope touches it. The formal second solution, $\theta = 2\phi - \pi$, leads to a negative value for r and can be discarded.

The equation of the envelope itself is therefore obtained by eliminating ϕ between this condition and the equation of the circle:

$$\begin{aligned} r &= 2a\cos\tfrac{1}{2}\theta\cos(\theta - \tfrac{1}{2}\theta) \\ &= 2a\cos^2\tfrac{1}{2}\theta \\ &= a(1+\cos\theta). \end{aligned}$$

This curve is a cardioid, and a sketch of one is shown in figure 2.4, as part of the answer to exercise 2.50.

5.24 *In order to make a focussing mirror that concentrates parallel axial rays to one spot (or conversely forms a parallel beam from a point source) a parabolic shape should be adopted. If a mirror that is part of a circular cylinder or sphere were used, the light would be spread out along a curve. This curve is known as a* caustic *and is the envelope of the rays reflected from the mirror. Denoting by θ the angle which a typical incident axial ray makes with the normal to the mirror at the place where it is reflected, the geometry of reflection (the angle of incidence equals the angle of reflection) is shown in figure 5.2.*

Show that a parametric specification of the caustic is

$$x = R\cos\theta\left(\tfrac{1}{2} + \sin^2\theta\right), \qquad y = R\sin^3\theta,$$

where R is the radius of curvature of the mirror. The curve is, in fact, part of an epicycloid.

Denoting the points where the ray strikes the mirror and later crosses the axis by P and Q respectively, we see, by applying the sine rule to the triangle OPQ, that

$$\frac{OQ}{\sin\theta} = \frac{R}{\sin 2\theta}.$$

Thus, taking O as the origin, the equation of the reflected ray is

$$y = \tan 2\theta\left(x - \frac{R\sin\theta}{\sin 2\theta}\right).$$

Putting this into the standard form $f(x, y, \theta) = 0$, setting $\partial f/\partial\theta$ equal to zero, and then eliminating y from the two resulting equations gives

$$\begin{aligned}
0 &= f(x, y, \theta) = y\cos 2\theta - x\sin 2\theta + R\sin\theta,\\
0 &= \frac{\partial f}{\partial\theta} = -2y\sin 2\theta - 2x\cos 2\theta + R\cos\theta,\\
0 &= -x(\sin^2 2\theta + \cos^2 2\theta) + R\sin\theta\sin 2\theta + \tfrac{1}{2}R\cos\theta\cos 2\theta.
\end{aligned}$$

From the last of these

$$x = R\cos\theta\left[2\sin^2\theta + \tfrac{1}{2}(1 - 2\sin^2\theta)\right] = R\cos\theta(\tfrac{1}{2} + \sin^2\theta),$$

and re-substitution in $f(x, y, \theta) = 0$ then yields

$$\begin{aligned}
y\cos 2\theta &= R\cos\theta(\tfrac{1}{2} + \sin^2\theta)\sin 2\theta - R\sin\theta\\
&= R\cos^2\theta\sin\theta + 2R\cos^2\theta\sin^3\theta - R\sin\theta\\
&= R\sin^3\theta(2\cos^2\theta - 1).\\
\Rightarrow \quad y &= R\sin^3\theta.
\end{aligned}$$

These expressions for x and y are the stated parametric specification of the caustic.

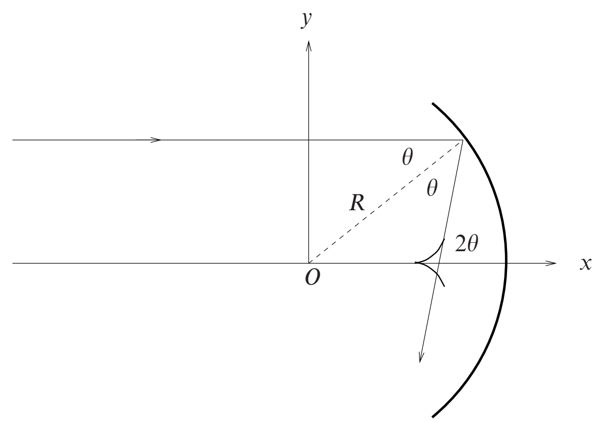

Figure 5.2 The reflecting mirror discussed in exercise 5.24.

5.26 *Functions $P(V, T)$, $U(V, T)$ and $S(V, T)$ are related by*

$$T\,dS = d\,U + P\,dV,$$

where the symbols have the same meaning as in the previous question. P is known from experiment to have the form

$$P = \frac{T^4}{3} + \frac{T}{V},$$

in appropriate units. If

$$U = \alpha V T^4 + \beta T,$$

where α, β, are constants (or at least do not depend on T, V), deduce that α must have a specific value but β may have any value. Find the corresponding form of S.

Writing all the other quantities as functions of T and V, we express the total differential dS as

$$dS = \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial S}{\partial T}\right)_V dT$$

and similarly for $d\,U$.

The given relationship then becomes

$$T\left(\frac{\partial S}{\partial V}\right)_T dV + T\left(\frac{\partial S}{\partial T}\right)_V dT = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT + P\,dV,$$

from which we can deduce that

$$T\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T + P \quad \text{and} \quad T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V.$$

These two equations give us explicit expressions for $\left(\frac{\partial S}{\partial V}\right)_T$ and $\left(\frac{\partial S}{\partial T}\right)_V$.

Differentiating them again, the first with respect to T and the second with respect to V, and then using the fact that $\partial^2 S/\partial T \partial V$ and $\partial^2 S/\partial V \partial T$ must be equal, allows us to write

$$\left(\frac{\partial}{\partial T}\left[\frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{P}{T}\right]\right)_V = \left(\frac{\partial}{\partial V}\left[\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V\right]\right)_T,$$
$$\left(\frac{\partial}{\partial T}\left[\alpha T^3 + \frac{T^3}{3} + \frac{1}{V}\right]\right)_V = \left(\frac{\partial}{\partial V}\left[4\alpha V T^2 + \frac{\beta}{T}\right]\right)_T,$$

i.e.

$$3\alpha T^2 + T^2 = 4\alpha T^2.$$

This necessary equality implies that we must have $\alpha = 1$, but it imposes no constraint on β.

Having now found explicit expressions for the two partial derivatives of S, integrating them will give two forms for $S(V, T)$. The two forms must be made to be mutually compatible, if they are not already so, by appropriate choices for the arbitrary functions of the non-integrated variable ($f(V)$ and $g(T)$, below) that are introduced by the integrations:

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V = 4VT^2 + \frac{\beta}{T},$$
$$\Rightarrow \quad S(V, T) = \tfrac{4}{3}VT^3 + \beta \ln T + f(V).$$

and

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{P}{T} = T^3 + \frac{T^3}{3} + \frac{1}{V},$$
$$\Rightarrow \quad S(V, T) = \tfrac{4}{3}VT^3 + \ln V + g(T).$$

Clearly, $f(V)$ must be identified with $\ln V$ and $g(T)$ with $\beta \ln T$ to give as the full expression for S

$$S(V, T) = \tfrac{4}{3}VT^3 + \beta \ln T + \ln V + c,$$

where c is a constant and, as shown earlier, β is arbitrary.

5.28 *The entropy $S(H,T)$, the magnetisation $M(H,T)$ and the internal energy $U(H,T)$ of a magnetic salt placed in a magnetic field of strength H at temperature T are connected by the equation*

$$T\,dS = dU - H\,dM.$$

By considering $d(U - TS - HM)$ prove that

$$\left(\frac{\partial M}{\partial T}\right)_H = \left(\frac{\partial S}{\partial H}\right)_T.$$

For a particular salt

$$M(H,T) = M_0[1 - \exp(-\alpha H/T)].$$

Show that, at a fixed temperature, if the applied field is increased from zero to a strength such that the magnetization of the salt is $\frac{3}{4}M_0$ then the salt's entropy decreases *by an amount*

$$\frac{M_0}{4\alpha}(3 - \ln 4).$$

Given that $T\,dS = d\,U - H\,dM$, consider dF where $F = U - TS - HM$.

$$\begin{aligned} dF &= d(U - TS - HM) \\ &= d\,U - T\,dS - S\,dT - H\,dM - M\,dH \\ &= -S\,dT - M\,dH. \end{aligned}$$

It follows that

$$\left(\frac{\partial F}{\partial T}\right)_H = -S \quad \text{and} \quad \left(\frac{\partial F}{\partial H}\right)_T = -M.$$

Differentiating the first equation with respect to H and the second with respect to T yields

$$-\left(\frac{\partial S}{\partial H}\right)_T = \frac{\partial^2 F}{\partial H \partial T} = \frac{\partial^2 F}{\partial T \partial H} = -\left(\frac{\partial M}{\partial T}\right)_H,$$

thus establishing the equality stated in the question.

With

$$M(H,T) = M_0\left(1 - e^{-\alpha H/T}\right),$$

$$\left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H = -\frac{M_0 \alpha H}{T^2} e^{-\alpha H/T}.$$

If the final field strength is H_1 then $e^{-\alpha H_1/T} = \frac{1}{4}$, or $H_1 = \alpha^{-1} T \ln 4$. The change in entropy $\Delta S = S(H, T) - S(0, T)$ is given by

$$\begin{aligned}
\Delta S &= \int_0^{H_1} -\frac{M_0 \alpha H}{T^2} e^{-\alpha H/T}\, dH \\
&= -\frac{M_0 \alpha}{T^2} \left\{ \left[-\frac{HT}{\alpha} e^{-\alpha H/T} \right]_0^{H_1} + \int_0^{H_1} \frac{T}{\alpha} e^{-\alpha H/T}\, dH \right\} \\
&= -\frac{M_0 \alpha}{T^2} \left[-\frac{T^2}{4\alpha^2} \ln 4 + \left(\frac{T}{\alpha} \right)^2 \frac{3}{4} \right] \\
&= -\frac{M_0}{4\alpha}(3 - \ln 4),
\end{aligned}$$

i.e. the salt's entropy *decreases* by the stated amount.

> ***5.30*** *The integral*
>
> $$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx$$
>
> *has the value* $(\pi/\alpha)^{1/2}$. *Use this result to evaluate*
>
> $$J(n) = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx,$$
>
> *where n is a positive integer. Express your answer in terms of factorials.*

We first observe that differentiating the given result with respect to α will introduce a factor of $-x^2$ into the integrand; doing so repeatedly will enable a factor of $(-1)^n x^{2n}$ to be generated. We therefore define a function $I(n, \alpha)$ by

$$I(n, \alpha) = \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\, dx,$$

with $I(0, \alpha) = \sqrt{\pi/\alpha}$. The required $J(n)$ will be equal to $I(n, 1)$.

We now carry out the n differentiations on the explicitly stated form of $I(0, \alpha)$ to generate an explicit form for $I(n, \alpha)$.

$$\begin{aligned}
I(n, \alpha) &= (-1)^n \frac{d^n I(0, \alpha)}{d\alpha^n} \\
&= (-1)^n \frac{d^n (\sqrt{\pi} \alpha^{-1/2})}{d\alpha^n} \\
&= (-1)^n \left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) \cdots \left(-\frac{2n-1}{2} \right) \frac{\sqrt{\pi}}{\alpha^{n+1/2}} \\
&= \frac{(2n)!}{2^n\, 2^n\, n!} \frac{\sqrt{\pi}}{\alpha^{n+1/2}}.
\end{aligned}$$

Now, setting $\alpha = 1$, we obtain

$$J(n) = I(n, 1) = \frac{(2n)!\,\sqrt{\pi}}{4^n\, n!}.$$

5.32 *The functions $f(x,t)$ and $F(x)$ are defined by*

$$f(x,t) = e^{-xt},$$
$$F(x) = \int_0^x f(x,t)\, dt.$$

Verify by explicit calculation that

$$\frac{dF}{dx} = f(x,x) + \int_0^x \frac{\partial f(x,t)}{\partial x}\, dt.$$

For the LHS

$$F(x) = \int_0^x e^{-xt}\, dt = \left[-\frac{e^{-xt}}{x}\right]_0^x = \frac{1 - e^{-x^2}}{x},$$

and hence

$$\frac{dF}{dx} = 2e^{-x^2} - \frac{1 - e^{-x^2}}{x^2}.$$

For the RHS, we start from

$$\frac{\partial f(x,t)}{\partial x} = -te^{-xt},$$

and so obtain

$$\begin{aligned}
\int_0^x \frac{\partial f(x,t)}{\partial x}\, dt &= -\int_0^x te^{-xt}\, dt \\
&= -\left[-\frac{te^{-xt}}{x}\right]_0^x - \int_0^x \frac{e^{-xt}}{x}\, dt \\
&= \frac{xe^{-x^2}}{x} - 0 - \left[-\frac{e^{-xt}}{x^2}\right]_0^x \\
&= e^{-x^2} - \frac{1 - e^{-x^2}}{x^2}.
\end{aligned}$$

Further $f(x,x) = e^{-x^2}$, and so

$$f(x,x) + \int_0^x \frac{\partial f(x,t)}{\partial x}\, dt = 2e^{-x^2} - \frac{1 - e^{-x^2}}{x^2} = \frac{dF}{dx},$$

as stated.

> ***5.34*** *Find the derivative with respect to x of the integral*
>
> $$I(x) = \int_x^{3x} \exp xt \, dt.$$

Using the extension to Leibnitz' rule, we have

$$\begin{aligned}
I(x) &= \int_x^{3x} e^{xt} \, dt, \\
\frac{dI}{dx} &= \int_x^{3x} te^{xt} \, dt + 3\, e^{3x^2} - e^{x^2} \\
&= \left[\frac{te^{xt}}{x}\right]_x^{3x} - \int_x^{3x} \frac{e^{xt}}{x} \, dt + 3\, e^{3x^2} - e^{x^2} \\
&= 3\, e^{3x^2} - e^{x^2} - \left[\frac{e^{xt}}{x^2}\right]_x^{3x} + 3\, e^{3x^2} - e^{x^2} \\
&= 6\, e^{3x^2} - 2e^{x^2} - \frac{1}{x^2}(e^{3x^2} - e^{x^2}).
\end{aligned}$$

6

Multiple integrals

> ***6.2*** *Evaluate the volume integral of $x^2 + y^2 + z^2$ over the rectangular parallelepiped bounded by the six surfaces $x = \pm a$, $y = \pm b$, $z = \pm c$.*

This is a straightforward triple integral; the order of performing the integrations is arbitrary and for this integrand no particular one offers any special advantage.

$$\begin{aligned}
I &= \int_{-a}^{a} dx \int_{-b}^{b} dy \int_{-c}^{c} (x^2 + y^2 + z^2)\, dz \\
&= 2 \int_{-a}^{a} dx \int_{-b}^{b} (cx^2 + cy^2 + \tfrac{1}{3}c^3)\, dy \\
&= 4 \int_{-a}^{a} \left(bcx^2 + \tfrac{1}{3}b^3c + \tfrac{1}{3}bc^3\right)\, dx \\
&= 8 \left(\tfrac{1}{3}a^3bc + \tfrac{1}{3}ab^3c + \tfrac{1}{3}abc^3\right) \\
&= \tfrac{8}{3}abc(a^2 + b^2 + c^2).
\end{aligned}$$

As would be expected, the result is symmetric in a, b and c.

> ***6.4*** *Evaluate the surface integral of $f(x, y)$ over the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$ for the functions*
>
> $$\text{(a) } f(x, y) = \frac{x}{x^2 + y^2}, \qquad \text{(b) } f(x, y) = (b - y + x)^{-3/2}.$$

(a) It is not clear which order of integration is to be preferred; integrating first with respect to x will produce a logarithmic function whilst doing so with respect to y will generate an inverse tangent. We arbitrarily choose the former.

$$\begin{aligned} I &= \int_0^b dy \int_0^a \frac{x}{x^2 + y^2}\, dx \\ &= \int_0^b \left[\tfrac{1}{2}\ln(x^2 + y^2)\right]_0^a dy \\ &= \frac{1}{2}\int_0^b \ln\left(\frac{a^2 + y^2}{y^2}\right) dy. \end{aligned}$$

In order to carry out the y integration we use the device of introducing an additional factor '1' into the integrand and then integrate by parts. By choosing this '1' as the term to be integrated, we obtain

$$\begin{aligned} I &= \frac{1}{2}\left[y \ln\left(\frac{a^2 + y^2}{y^2}\right)\right]_0^b - \frac{1}{2}\int_0^b y\left(\frac{2y}{a^2 + y^2} - \frac{2y}{y^2}\right) dy \\ &= \frac{b}{2}\ln\left(\frac{a^2 + b^2}{b^2}\right) + \int_0^b \frac{a^2}{a^2 + y^2}\, dy \\ &= \frac{b}{2}\ln\left(\frac{a^2 + b^2}{b^2}\right) + a\tan^{-1}\left(\frac{b}{a}\right). \end{aligned}$$

Not surprisingly, the inverse tangent we avoided initially by our choice of integration order has popped up again!

(b) This integrand could be made to look more symmetric by writing $z = b - y$, but it is no more difficult to integrate it as it stands. We arbitrarily choose to perform the x-integration first.

$$\begin{aligned} \int_0^b dy \int_0^a \frac{1}{(b - y + x)^{3/2}}\, dx &= \int_0^b \left[\frac{-2}{(b - y + x)^{1/2}}\right]_{x=0}^{x=a} dy \\ &= 2\int_0^b \left[-\frac{1}{(b + a - y)^{1/2}} + \frac{1}{(b - y)^{1/2}}\right] dy \\ &= 2\left[2(b + a - y)^{1/2} - 2(b - y)^{1/2}\right]_0^b \\ &= 4\left[a^{1/2} - (b + a)^{1/2} + b^{1/2}\right]. \end{aligned}$$

In view of the opening comment, the symmetry of the answer with respect to a and b was to be expected.

> ***6.6*** *The function*
>
> $$\Psi(r) = A\left(2 - \frac{Zr}{a}\right)e^{-Zr/2a}$$
>
> *gives the form of the quantum mechanical wavefunction representing the electron in a hydrogen-like atom of atomic number Z when the electron is in its first allowed spherically symmetric excited state. Here r is the usual spherical polar coordinate, but, because of the spherical symmetry, the coordinates θ and ϕ do not appear explicitly in Ψ. Determine the value that A (assumed real) must have if the wavefunction is to be correctly normalised, i.e. the volume integral of $|\Psi|^2$ over all space is equal to unity.*

To evaluate the integral of $|\Psi|^2$ over all space we use spherical polar coordinates and, in this spherically symmetrical case, a volume element of $4\pi r^2\,dr$.

$$\begin{aligned}
\int |\Psi|^2\,dV &= A^2\int_0^\infty 4\pi r^2\left(2-\frac{Zr}{a}\right)^2 e^{-Zr/a}\,dr \\
&= 4\pi A^2\int_0^\infty\left(4r^2 - \frac{4Zr^3}{a} + \frac{Z^2r^4}{a^2}\right)e^{-Zr/a}\,dr \\
&= 4\pi A^2\left(4\frac{2!\,a^3}{Z^3} - 4\frac{3!\,Za^4}{aZ^4} + \frac{4!\,Z^2a^5}{a^2Z^5}\right) \\
&= \frac{32\pi A^2a^3}{Z^3}.
\end{aligned}$$

Thus if the wavefunction is to be correctly normalised A must be taken as

$$A = \pm\frac{Z^{3/2}}{\sqrt{32\pi}\,a^{3/2}}.$$

> ***6.8*** *A planar figure is formed from uniform wire and consists of two semicircular arcs, each with its own closing diameter, joined so as to form a letter 'B'. The figure is freely suspended from its top left-hand corner. Show that the straight edge of the figure makes an angle θ with the vertical given by* $\tan\theta = (2+\pi)^{-1}$.

For each semi-circle, denote its radius by a, the linear density of the wire by ρ and the distance of its centre of gravity from its straight edge by d. Further, let the distance of the centre of gravity of the whole figure from its straight edge be $\bar{x}$. Then, since rotating a semi-circle about its straight edge produces a sphere, by Pappus' second theorem,

$$4\pi a^2 = 2\pi d \times \pi a, \qquad \Rightarrow \qquad d = \frac{2a}{\pi}.$$

Then, for the centre of gravity of the letter 'B',

$$\bar{x} = \frac{\left(2\pi a\rho \times \dfrac{2a}{\pi}\right) + (4a\rho \times 0)}{2\pi a\rho + 4a\rho} = \frac{4a}{2\pi + 4}.$$

When the wire letter is supended its centre of gravity will lie below the suspension point and the straight edge will make an angle θ with the vertical given by $\tan\theta = \bar{x}/2a$. Thus $\theta = \tan^{-1}[\,1/(\pi+2)\,]$.

6.10 *A thin uniform circular disc has mass M and radius a.*

(a) *Prove that its moment of inertia about an axis perpendicular to its plane and passing through its centre is* $\frac{1}{2}Ma^2$.
(b) *Prove that the moment of inertia of the same disc about a diameter is* $\frac{1}{4}Ma^2$.

This is an example of the general result for planar bodies that the moment of inertia of the body about an axis perpendicular to the plane is equal to the sum of the moments of inertia about two perpendicular axes lying in the plane: in an obvious notation

$$I_z = \int r^2\,dm = \int (x^2+y^2)\,dm = \int x^2\,dm + \int y^2\,dm = I_y + I_x.$$

Denote the mass per unit area of the disc by σ. Then, using plane polar coordinates (ρ, ϕ) or Cartesian coordinates (x, y), as appropriate, we find the moments of inertia of the disc about axes (a) perpendicular to its plane, and (b) about the y-axis as follows.

(a)
$$I_\perp = \int_0^a \sigma\rho^2\, 2\pi\rho\, d\rho = \frac{2\pi\sigma a^4}{4} = \tfrac{1}{2}Ma^2.$$

(b)
$$\begin{aligned} I_\parallel &= \int_{-a}^{a} \sigma x^2\, 2(a^2-x^2)^{1/2}\,dx \\ &= 4\sigma \int_0^a x^2(a^2-x^2)^{1/2}\,dx \\ &= 4\sigma \int_{\pi/2}^{0} a^2\cos^2\phi\; a\sin\phi\;(-a\sin\phi\,d\phi) \\ &= \sigma a^4 \int_0^{\pi/2} \sin^2 2\phi\; d\phi = \sigma a^4\, \tfrac{1}{2}\,\tfrac{\pi}{2} = \tfrac{1}{4}Ma^2. \end{aligned}$$

In the third line we set x equal to $a\cos\phi$.

6.12 *The shape of an axially symmetric hard-boiled egg, of uniform density ρ_0, is given in spherical polar coordinates by $r = a(2 - \cos\theta)$, where θ is measured from the axis of symmetry.*

(a) *Prove that the mass M of the egg is $M = \frac{40}{3}\pi\rho_0 a^3$.*
(b) *Prove that the egg's moment of inertia about its axis of symmetry is $\frac{342}{175}Ma^2$.*

(a) We need to consider slices of the egg perpendicular to the polar axis, the thickness of a typical slice being dz where $z = r\cos\theta$ and consequently

$$\begin{aligned} dz &= d(r\cos\theta) \\ &= d(2a\cos\theta - a\cos^2\theta) \\ &= 2a\sin\theta(\cos\theta - 1)\,d\theta \end{aligned}$$

Writing $\cos\theta$ as c in places to save space, the element of mass lying between z and $z + dz$ is

$$\begin{aligned} dm &= \pi\rho_0(r\sin\theta)^2\,dz \\ &= \pi\rho_0\,a^2\sin^2\theta\,(2-c)^2\,2a\sin\theta(c-1)\,d\theta \\ &= 2\pi\rho_0 a^3\,(1-c^2)(2-c)^2(1-c)\,dc. \end{aligned}$$

This now has to be integrated between $c = 1$ and $c = -1$. Only those terms in the integrand which are even powers of c will give a non-zero contribution. Omitting the multiplicative constants, the integrand is

$$(1 - c - c^2 + c^3)(4 - 4c + c^2) = 4 + c^2 - 5c^4 + \text{ odd powers of } c.$$

Consequently, the value of the volume integral is

$$M = 2\pi\rho_0 a^3\,2\left(4 + \frac{1}{3} - \frac{5}{5}\right) = \frac{40}{3}\pi\rho_0 a^3.$$

(b) The moment of inertia of a single slice of mass dm about the axis of symmetry is $\frac{1}{2}(r\sin\theta)^2\,dm$ and again this has to be integrated between $\theta = 0$ and $\theta = \pi$.

$$\begin{aligned} dI &= \tfrac{1}{2}(r\sin\theta)^2\,dm \\ &= \tfrac{1}{2}a^2(2-c)^2(1-c^2)\,2\pi\rho_0 a^3(1-c^2)(2-c)^2(1-c)\,dc \\ &= \pi\rho_0 a^5\,(1-c^2)^2(2-c)^4(1-c)\,dc \end{aligned}$$

The c-dependent terms in this integrand are

$$\begin{aligned}
f(c) &= (1-c^2)^2(2-c)^4(1-c) \\
&= (1-c)(1-2c^2+c^4)(16-32c+24c^2-8c^3+c^4) \\
&= (1-c-2c^2+2c^3+c^4-c^5)(16-32c+24c^2-8c^3+c^4) \\
&= 16+c^2(24+32-32)+c^4(1+8-48-64+16) \\
&\quad +c^6(-2-16+24+32)+c^8(1+8)+\text{ odd powers of } c.
\end{aligned}$$

As previously, this has to be integrated with respect to c between -1 and $+1$ with the odd powers of c contributing nothing. Thus the total moment of inertia is

$$\begin{aligned}
I &= \pi\rho_0 a^5\, 2\left(16+\frac{24}{3}-\frac{87}{5}+\frac{38}{7}+\frac{9}{9}\right) \\
&= \frac{912}{35}\pi\rho_0 a^5 \\
&= \frac{342}{175}Ma^2.
\end{aligned}$$

6.14 *By expressing both the integrand and the surface element in spherical polar coordinates, show that the surface integral*

$$\int \frac{x^2}{x^2+y^2}\,dS$$

over the surface $x^2+y^2=z^2$, $0\le z\le 1$, *has the value* $\pi/\sqrt{2}$.

The surface S is an inverted cone of unit height and half angle $\pi/4$. Since a cone is a coordinate surface ($\theta=$ constant) in spherical polar coordinates, we change to that system.

In these coordinates the surface is given by $0\le r\le\sqrt{2}$, $\theta=\pi/4$ and $0\le\phi\le 2\pi$. The integrand is $[r^2\sin^2(\pi/4)\cos^2\phi]/[r^2\sin^2(\pi/4)]$ whilst the surface element is $dS = r\sin(\pi/4)\,d\phi\,dr$. Thus the integral I is given by

$$\begin{aligned}
I &= \int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\sqrt{2}}\frac{r}{\sqrt{2}}\,dr \\
&= \frac{2\pi}{2}\left[\frac{r^2}{2\sqrt{2}}\right]_0^{\sqrt{2}} \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

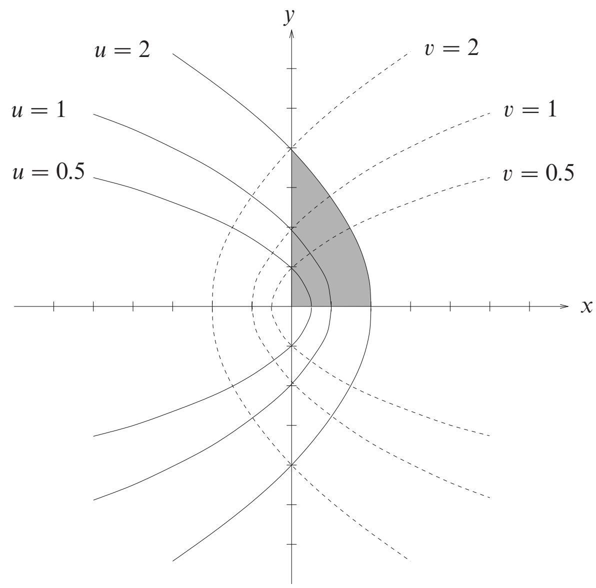

Figure 6.1 The parabolic coordinate curves discussed in exercise 6.16.

6.16 *Sketch the two families of curves*

$$y^2 = 4u(u - x), \qquad y^2 = 4v(v + x),$$

where u and v are parameters.

By transforming to the uv-plane, evaluate the integral of $y/(x^2 + y^2)^{1/2}$ over that part of the quadrant $x > 0$, $y > 0$ bounded by the lines $x = 0$, $y = 0$ and the curve $y^2 = 4a(a - x)$.

Sketches of typical curves are shown in figure 6.1. Each family is a set of non-intersecting parabolas with the x-axis as the axis of symmetry. However, each u-curve meets each v-curve in two places and vice versa.

The area over which the integral is to be taken has the points $(0, 0)$, $(a, 0)$ and $(0, 2a)$ as its 'corners' and is shown shaded in the figure for the case $a = 2$. We transform to the uv-plane where:

(i) The boundary $x = 0$, $y > 0$ becomes $4u^2 = y^2 = 4v^2$, i.e. $u = v$.
(ii) The boundary $x > 0$, $y = 0$ becomes $v = 0$.
(iii) The boundary $y^2 = 4a(a - x)$ becomes $u = a$.

In this plane the integration region is thus a right-angled triangle with vertices at $(0,0)$, $(a,0)$ and (a,a).

The equations of the transformation can be rewritten as $x = u - v$ and $y = 2\sqrt{uv}$, making the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & \sqrt{\dfrac{v}{u}} \\ -1 & \sqrt{\dfrac{u}{v}} \end{vmatrix} = \sqrt{\frac{u}{v}} + \sqrt{\frac{v}{u}}.$$

The integral can therefore be transformed to one over the triangular region and evaluated as follows.

$$\begin{aligned} I &= \int_S \frac{y}{x^2+y^2}\,dx\,dy \\ &= \int_0^a du \int_0^u \frac{2\sqrt{uv}}{(u^2-2uv+v^2+4uv)^{1/2}}\left(\sqrt{\frac{u}{v}}+\sqrt{\frac{v}{u}}\right)dv \\ &= \int_0^a du \int_0^u \frac{2(u+v)}{u+v}\,dv \\ &= \int_0^a 2u\,du \\ &= 2\left[\frac{u^2}{2}\right]_0^a = a^2. \end{aligned}$$

6.18 *Sketch the domain of integration for the integral*

$$I = \int_0^1 \int_{x=y}^{1/y} \frac{y^3}{x}\exp[y^2(x^2+x^{-2})]\,dx\,dy$$

and characterise its boundaries in terms of new variables $u = xy$ *and* $v = y/x$. *Show that the Jacobian for the change from* (x,y) *to* (u,v) *is equal to* $(2v)^{-1}$, *and hence evaluate* I.

The integration area is shown shaded in figure 6.2. In terms of the new variables, $u = xy$ and $v = y/x$, the original variables are $x = (u/v)^{1/2}$ and $y = (uv)^{1/2}$.
(i) The boundary $y = 0$, $0 < x < \infty$ becomes both $u = 0$ and $v = 0$.
(ii) The boundary $y = x$ becomes $v = 1$.
(iii) The boundary $x = 1/y$ becomes $u = 1$.

The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{2(uv)^{1/2}}\frac{u^{1/2}}{2v^{1/2}} - \frac{(-1)u^{1/2}}{2v^{3/2}}\frac{v^{1/2}}{2u^{1/2}} = \frac{1}{2v}.$$

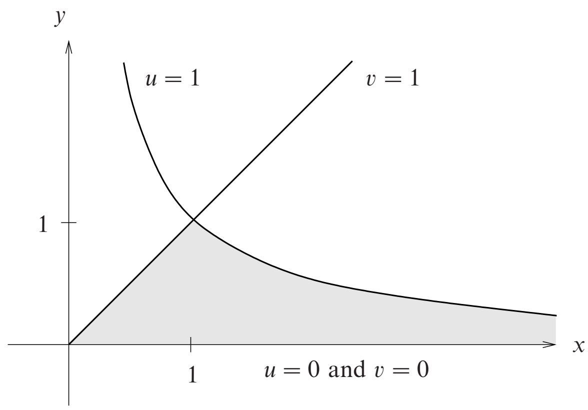

Figure 6.2 The integration area for exercise 6.18 is shown shaded.

Making the change of variables and then integrating gives

$$\begin{aligned} I &= \int_0^1 \int_{x=y}^{x=1/y} \frac{y^3}{x} \exp\left[y^2(x^2 + x^{-2})\right]\, dx\, dy \\ &= \int_0^1 \int_0^1 (uv)^{3/2} \left(\frac{v}{u}\right)^{1/2} \exp\left[uv\left(\frac{u}{v} + \frac{v}{u}\right)\right] \frac{1}{2v}\, du\, dv \\ &= \int_0^1 \int_0^1 \frac{uv}{2} \exp(u^2 + v^2)\, du\, dv \\ &= \frac{1}{2} \int_0^1 u \exp(u^2)\, du \int_0^1 v \exp(v^2)\, dv \\ &= \tfrac{1}{8} \left[\exp(u^2)\right]_0^1 \left[\exp(v^2)\right]_0^1 \\ &= \tfrac{1}{8}(e-1)^2. \end{aligned}$$

> ***6.20*** *Define a coordinate system u, v whose origin coincides with that of the usual x, y system and whose u-axis coincides with the x-axis, whilst the v-axis makes an angle α with it. By considering the integral $I = \int \exp(-r^2)\, dA$, where r is the radial distance from the origin, over the area defined by $0 \le u < \infty$, $0 \le v < \infty$, prove that*
>
> $$\int_0^\infty \int_0^\infty \exp(-u^2 - v^2 - 2uv\cos\alpha)\, du\, dv = \frac{\alpha}{2\sin\alpha}.$$

As can be seen from figure 6.3, the coordinates of a general point P that lies in

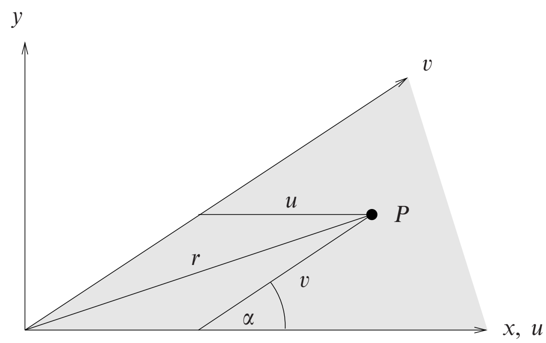

Figure 6.3 The coordinate system for exercise 6.20.

the area defined by positive values for u and v are related in the two systems by

$$x = v\cos\alpha + u \quad \text{and} \quad y = v\sin\alpha.$$

The Jacobian of a coordinate transformation between the two systems is therefore

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} = \sin\alpha.$$

We note that this value for the Jacobian does not depend upon the actual position of P.

Now, because of azimuthal symmetry, the integral of $\exp(-r^2)$ over the region of positive u and v (shown shaded in the figure) is $\alpha/2\pi$ of the same integral taken over the whole of the xy-space. This latter is

$$\int e^{-r^2}\,dA = \int\int e^{-(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = (\sqrt{\pi})^2.$$

Expressed in terms of u and v, $r^2 = x^2 + y^2 = u^2 + v^2 + 2uv\cos\alpha$. As shown above, $dx\,dy = \sin\alpha\,du\,dv$ and so the integral over the shaded region takes the form

$$\int_0^\infty \int_0^\infty \exp(-u^2 - v^2 - 2uv\cos\alpha)\sin\alpha\,du\,dv.$$

This integral therefore has the value $(\alpha/2\pi) \times \pi$ and the stated result about the integral in the question follows when both the itegral and its value are divided by the constant $\sin\alpha$.

6.22 *The distances of the variable point P, which has coordinates x, y, z, from the fixed points $(0, 0, 1)$ and $(0, 0, -1)$ are denoted by u and v respectively. New variables ξ, η, ϕ are defined by*

$$\xi = \tfrac{1}{2}(u + v), \qquad \eta = \tfrac{1}{2}(u - v),$$

and ϕ is the angle between the plane $y = 0$ and the plane containing the three points. Prove that the Jacobian $\partial(\xi, \eta, \phi)/\partial(x, y, z)$ has the value $(\xi^2 - \eta^2)^{-1}$ and that

$$\int\int\int_{\text{all space}} \frac{(u - v)^2}{uv} \exp\left(-\frac{u + v}{2}\right) dx\, dy\, dz = \frac{16\pi}{3e}.$$

From straightforward algebraic geometry, u and v are given by the positive square roots of

$$u^2 = x^2 + y^2 + (z - 1)^2 \quad \text{and} \quad v^2 = x^2 + y^2 + (z + 1)^2.$$

The new variables and their ranges are

$$\begin{aligned} \xi &= \tfrac{1}{2}(u + v) \quad \text{over} \quad 1 \le \xi < \infty, \\ \eta &= \tfrac{1}{2}(u - v) \quad \text{over} \quad -1 \le \eta < 1, \\ \phi &= \tan^{-1}\frac{y}{x} \quad \text{over} \quad 0 \le \phi < 2\pi. \end{aligned}$$

We start by calculating

$$\frac{\partial \xi}{\partial x} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{x}{u} + \frac{x}{v}\right) = \frac{x\xi}{uv}.$$

Similarly for $\partial\xi/\partial y$, $\partial\eta/\partial x$ and $\partial\eta/\partial y$.

The other required derivatives are

$$\begin{aligned} \frac{\partial \xi}{\partial z} &= \frac{1}{2}\left(\frac{z - 1}{u} + \frac{z + 1}{v}\right) = \frac{z\xi + \eta}{uv}, \\ \frac{\partial \eta}{\partial z} &= \frac{1}{2}\left(\frac{z - 1}{u} - \frac{z + 1}{v}\right) = \frac{-z\eta - \xi}{uv}, \\ \frac{\partial \phi}{\partial x} &= \frac{-y}{x^2}\,\frac{1}{1 + \frac{y^2}{x^2}} = -\frac{y}{x^2 + y^2}, \\ \frac{\partial \phi}{\partial y} &= \frac{1}{x}\,\frac{1}{1 + \frac{y^2}{x^2}} = \frac{x}{x^2 + y^2}. \end{aligned}$$

Collecting these together gives the Jacobian as

$$\frac{\partial(\xi,\eta,\phi)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{x\xi}{uv} & -\dfrac{x\eta}{uv} & -\dfrac{y}{x^2+y^2} \\ \dfrac{y\xi}{uv} & -\dfrac{y\eta}{uv} & \dfrac{x}{x^2+y^2} \\ \dfrac{z\xi+\eta}{uv} & -\dfrac{z\eta+\xi}{uv} & 0 \end{vmatrix}$$

$$= \frac{-1}{(uv)^2}\,\frac{1}{x^2+y^2}\begin{vmatrix} x\xi & x\eta & -y \\ y\xi & y\eta & x \\ z\xi+\eta & z\eta+\xi & 0 \end{vmatrix}$$

$$= \frac{-1}{(uv)^2}\,\frac{1}{x^2+y^2}\begin{vmatrix} 0 & x\eta & -y \\ 0 & y\eta & x \\ \dfrac{\eta^2-\xi^2}{\eta} & z\eta+\xi & 0 \end{vmatrix}$$

$$= \frac{-(\eta^2-\xi^2)(x^2+y^2)}{(uv)^2(x^2+y^2)} = \frac{\xi^2-\eta^2}{(uv)^2}.$$

But $uv = \xi^2 - \eta^2$ and so the Jacobian has the value $(\xi^2-\eta^2)^{-1}$. To obtain the third line of the above evaluation of the Jacobian we subtracted ξ/η times the 2nd column from the 1st column.

We now express the given integral in terms of ξ and η:

$$I = \int\int\int_{\text{all space}} \frac{(u-v)^2}{uv}\exp\left(-\frac{u+v}{2}\right)\,dx\,dy\,dz$$

$$= \int_0^{2\pi} d\phi \int_1^{\infty} d\xi \int_{-1}^{1} \frac{4\eta^2}{\xi^2-\eta^2}e^{-\xi}\,\frac{1}{(\xi^2-\eta^2)^{-1}}\,d\eta$$

$$= 8\pi \int_1^{\infty} e^{-\xi}\,d\xi \int_{-1}^{1} \eta^2\,d\eta$$

$$= 8\pi\, e^{-1}\,\frac{2}{3} = \frac{16\pi}{3e},$$

as stated in the question.

7

Vector algebra

> ***7.2*** *A unit cell of diamond is a cube of side A with carbon atoms at each corner, at the centre of each face and, in addition, at positions displaced by $\frac{1}{4}A(\mathbf{i}+\mathbf{j}+\mathbf{k})$ from each of those already mentioned;* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *are unit vectors along the cube axes. One corner of the cube is taken as the origin of coordinates. What are the vectors joining the atom at $\frac{1}{4}A(\mathbf{i}+\mathbf{j}+\mathbf{k})$ to its four nearest neighbours? Determine the angle between the carbon bonds in diamond.*

The four nearest neighbours are positioned at

$$A(0,0,0),\ A(\tfrac{1}{2},\tfrac{1}{2},0),\ A(0,\tfrac{1}{2},\tfrac{1}{2}) \text{ and } A(\tfrac{1}{2},0,\tfrac{1}{2}).$$

The corresponding vectors joining them to the atom at $A(\frac{1}{4},\frac{1}{4},\frac{1}{4})$ are

$$\frac{A}{4}(-1,-1,-1),\quad \frac{A}{4}(1,1,-1),\quad \frac{A}{4}(-1,1,1),\quad \frac{A}{4}(1,-1,1).$$

The length of each vector is $\sqrt{3}A/4$ and so the angle between any two bonds (say the first and second) is

$$\theta = \cos^{-1}\frac{\frac{A^2}{16}(-1-1+1)}{\left(\frac{\sqrt{3}A}{4}\right)^2} = \cos^{-1}\left(\frac{-1}{3}\right) = 109.5^\circ.$$

> ***7.4*** *Find the angle between the position vectors to the points* $(3,-4,0)$ *and* $(-2,1,0)$ *and find the direction cosines of a vector perpendicular to both.*

If θ is the angle between the vectors $\mathbf{a} = (3,-4,0)$ and $\mathbf{b} = (-2,1,0)$ then its cosine is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{a\,b} = \frac{-6-4+0}{5\sqrt{5}} = \frac{-2}{\sqrt{5}},$$

giving $\theta = 153.4°$.

A vector perpendicular to both **a** and **b** is their cross product

$$\mathbf{a} \times \mathbf{b} = (0 - 0,\ 0 - 0,\ 3 - 8) = (0, 0, -5).$$

The normalised cross product is $(0, 0, -1)$ whose components therefore are the required direction cosines. Clearly $(0, 0, 1)$ is an equally valid vector perpendicular to both **a** and **b**.

> ***7.6*** *Use vector methods to prove that the lines joining the mid-points of the opposite edges of a tetrahedron OABC meet at a point and that this point bisects each of the lines.*

Let the vertices of the tetrahedron have vector positions **0**, **a**, **b** and **c**. The mid-points of the pair of opposite sides OA and BC are $\frac{1}{2}(\mathbf{0} + \mathbf{a})$ and $\frac{1}{2}(\mathbf{b} + \mathbf{c})$, respectively. The mid-point of the line joining these two points is, similarly, $\frac{1}{2}[\frac{1}{2}(\mathbf{0} + \mathbf{a}) + \frac{1}{2}(\mathbf{b} + \mathbf{c})] = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

From the symmetry of this expression it is clear that the same result would be obtained by considering the pair of sides OB and AC, or the pair of sides OC and AB. Thus the lines joining the mid-points of all pairs of opposite edges meet at this one point, which bisects each of them.

> ***7.8*** *Prove, by writing it out in component form, that*
>
> $$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a},$$
>
> *and deduce the result, stated in (7.25), that the operation of forming the vector product is non-associative.*

We compute only the x-component of each side of the equation. The corresponding results for other components can be obtained by cyclic permutation of x, y and z.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x) \\
[(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}]_x &= (a_z b_x - a_x b_z) c_z - (a_x b_y - a_y b_x) c_y \\
&= b_x (a_z c_z + a_y c_y) - a_x (b_z c_z + b_y c_y) \\
&= b_x (a_z c_z + a_y c_y + a_x c_x) - a_x (b_x c_x + b_z c_z + b_y c_y) \\
&= [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}]_x .
\end{aligned}$$

To obtain the penultimate line we both added and subtracted $a_x b_x c_x$ on the

RHS. This establishes the result for the x-component and hence for all three components.

We have shown that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Now consider

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -(\mathbf{b} \cdot \mathbf{a})\mathbf{c} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}.$$

The last terms on the RHSs of the two equations are not equal, showing that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

7.10 *For four arbitrary vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$, *evaluate*

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$$

in two different ways and so prove that

$$\mathbf{a}\,[\,\mathbf{b}, \mathbf{c}, \mathbf{d}\,] - \mathbf{b}\,[\,\mathbf{c}, \mathbf{d}, \mathbf{a}\,] + \mathbf{c}\,[\,\mathbf{d}, \mathbf{a}, \mathbf{b}\,] - \mathbf{d}\,[\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,] = \mathbf{0}.$$

Show that this reduces to the normal Cartesian representation of the vector $\mathbf{d}$, *i.e.* $d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}$, *if* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are taken as* $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$, *the Cartesian base vectors.*

Firstly, treating the given expression as the triple vector product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$,

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{b}\,[\,\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})] - \mathbf{a}\,[\,\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})]\\ &= \mathbf{b}\,[\,\mathbf{c}, \mathbf{d}, \mathbf{a}\,] - \mathbf{a}\,[\,\mathbf{b}, \mathbf{c}, \mathbf{d}\,].\end{aligned}$$

Secondly, treating the given expression as the triple vector product of $\mathbf{a} \times \mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$,

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{c}\,[\,\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})] - \mathbf{d}\,[\,\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})]\\ &= \mathbf{c}\,[\,\mathbf{d}, \mathbf{a}, \mathbf{b}\,] - \mathbf{d}\,[\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,]\end{aligned}$$

Now, equating these two expressions gives the stated result; explicitly,

$$\mathbf{a}\,[\,\mathbf{b}, \mathbf{c}, \mathbf{d}\,] - \mathbf{b}\,[\,\mathbf{c}, \mathbf{d}, \mathbf{a}\,] + \mathbf{c}\,[\,\mathbf{d}, \mathbf{a}, \mathbf{b}\,] - \mathbf{d}\,[\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,] = 0.$$

Setting $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$ and $\mathbf{c} = \mathbf{k}$ reduces the above equation to

$$\mathbf{i}\,[\,\mathbf{j}, \mathbf{k}, \mathbf{d}\,] - \mathbf{j}\,[\,\mathbf{k}, \mathbf{d}, \mathbf{i}\,] + \mathbf{k}\,[\,\mathbf{d}, \mathbf{i}, \mathbf{j}\,] - \mathbf{d}\,[\,\mathbf{i}, \mathbf{j}, \mathbf{k}\,] = 0,$$

which, since $[\,\mathbf{i}, \mathbf{j}, \mathbf{k}\,] = 1$, reduces to

$$\mathbf{d} = \mathbf{i}\,d_x - \mathbf{j}\,(-d_y) + \mathbf{k}\,d_z = d_x\,\mathbf{i} + d_y\,\mathbf{j} + d_z\,\mathbf{k}.$$

7.12 *The plane P_1 contains the points A, B and C, which have position vectors* $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 7\mathbf{i} + 2\mathbf{j}$ *and* $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ *respectively. Plane P_2 passes through A and is orthogonal to the line BC, whilst plane P_3 passes through B and is orthogonal to the line AC. Find the coordinates of* $\mathbf{r}$, *the point of intersection of the three planes.*

Since both $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ lie in P_1, a normal to that plane is in the direction of

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (10, 0, 0) \times (5, 1, 2) = (0, -20, 10).$$

The equation of P_1 is therefore of the form $-2y + z = c$ and, since A lies on it, $c = -4$.

The specification for P_2 takes the form

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{c}) = 0 \quad \text{or} \quad \mathbf{r} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}).$$

Thus

$$(x, y, z) \cdot (5, -1, -2) = (-3, 2, 0) \cdot (5, -1, -2) = -17 \quad \text{or} \quad 5x - y - 2z = -17.$$

For P_3

$$(\mathbf{r} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a}) = 0 \quad \text{or} \quad \mathbf{r} \cdot (\mathbf{c} - \mathbf{a}) = \mathbf{b} \cdot (\mathbf{c} - \mathbf{a}),$$

leading to

$$(x, y, z) \cdot (5, 1, 2) = (7, 2, 0) \cdot (5, 1, 2) = 37 \quad \text{or} \quad 5x + y + 2z = 37.$$

Solving the three equations for P_1, P_2 and P_3 simultaneously gives the coordinates of the point of intersection. By adding the equations for P_2 and P_3 we obtain $x = 2$. Then using either of these equations and that for P_1 yields $y = 7$ and $z = 10$.

7.14 *Two fixed points, A and B, in three-dimensional space have position vectors* $\mathbf{a}$ *and* $\mathbf{b}$. *Identify the plane P given by*

$$(\mathbf{a} - \mathbf{b}) \cdot \mathbf{r} = \tfrac{1}{2}(a^2 - b^2),$$

where a and b are the magnitudes of $\mathbf{a}$ *and* $\mathbf{b}$.
Show also that the equation

$$(\mathbf{a} - \mathbf{r}) \cdot (\mathbf{b} - \mathbf{r}) = 0$$

describes a sphere S of radius $|\mathbf{a} - \mathbf{b}|/2$. *Deduce that the intersection of P and S is also the intersection of two spheres, centred on A and B and each of radius* $|\mathbf{a} - \mathbf{b}|/\sqrt{2}$.

The normal to the plane P is clearly in the direction $\mathbf{a}-\mathbf{b}$ and furthermore the point $\mathbf{r}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$ satisfies

$$(\mathbf{a}-\mathbf{b})\cdot\tfrac{1}{2}(\mathbf{a}+\mathbf{b})=\tfrac{1}{2}(a^2-b^2),$$

and so lies in the plane. The plane must therefore be orthogonal to the line joining A to B and pass through its mid-point.

From the given equation

$$\begin{aligned}(\mathbf{a}-\mathbf{r})\cdot(\mathbf{b}-\mathbf{r})&=0,\\ \mathbf{r}\cdot\mathbf{r}-(\mathbf{a}+\mathbf{b})\cdot\mathbf{r}+\mathbf{a}\cdot\mathbf{b}&=0,\\ \left[\mathbf{r}-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\right]^2&=-\mathbf{a}\cdot\mathbf{b}+[\tfrac{1}{2}(\mathbf{a}+\mathbf{b})]^2\\ &=[\tfrac{1}{2}(\mathbf{a}-\mathbf{b})]^2.\end{aligned}$$

Thus the equation describes a sphere S of radius $\frac{1}{2}|\mathbf{a}-\mathbf{b}|$ centred on the point $\frac{1}{2}(\mathbf{a}+\mathbf{b})$. It has AB as a diameter.

Now consider the (circular) intersection of P and S, given by solving their equations simultaneously:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{r}-\mathbf{b}\cdot\mathbf{r}&=\tfrac{1}{2}(a^2-b^2),\\ r^2+\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{r}-\mathbf{b}\cdot\mathbf{r}&=0.\end{aligned}$$

Subtracting them gives

$$\begin{aligned}r^2+\mathbf{a}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{r}&=\tfrac{1}{2}(b^2-a^2),\\ (\mathbf{r}-\mathbf{a})^2&=\tfrac{1}{2}(b^2-a^2)-\mathbf{a}\cdot\mathbf{b}+a^2\\ &=\tfrac{1}{2}(b^2+a^2)-\mathbf{a}\cdot\mathbf{b}\\ &=\tfrac{1}{2}(\mathbf{b}-\mathbf{a})^2.\end{aligned}$$

Adding them gives

$$\begin{aligned}r^2+\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{r}&=\tfrac{1}{2}(a^2-b^2),\\ (\mathbf{r}-\mathbf{b})^2&=\tfrac{1}{2}(a^2-b^2)-\mathbf{a}\cdot\mathbf{b}+b^2\\ &=\tfrac{1}{2}(b^2+a^2)-\mathbf{a}\cdot\mathbf{b}\\ &=\tfrac{1}{2}(\mathbf{a}-\mathbf{b})^2.\end{aligned}$$

The two deduced equations satisfied by the points that lie on the intersection of P and S are those of spheres of equal radius $|\mathbf{a}-\mathbf{b}|/\sqrt{2}$, one centred on A and the other on B. Thus the intersection of the plane and sphere is also the intersection of two equal (larger) spheres whose centres are A and B.

__7.16__ The vectors **a**, **b** *and* **c** *are coplanar and related by*

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = 0,$$

where λ, μ, ν are not all zero. Show that the condition for the points with position vectors $\alpha\mathbf{a}$, $\beta\mathbf{b}$ and $\gamma\mathbf{c}$ to be collinear is

$$\frac{\lambda}{\alpha} + \frac{\mu}{\beta} + \frac{\nu}{\gamma} = 0.$$

We assume that **a**, **b** and **c** are not simply multiples of each other.

For collinearity of the three points we must have

$$\gamma\mathbf{c} = \theta\alpha\mathbf{a} + (1-\theta)\beta\mathbf{b}$$

for some θ. Thus

$$\lambda\mathbf{a} + \mu\mathbf{b} + \frac{\theta\alpha\nu}{\gamma}\mathbf{a} + \frac{(1-\theta)\beta\nu}{\gamma}\mathbf{b} = \mathbf{0},$$

implying that

$$\lambda + \frac{\theta\alpha\nu}{\gamma} = 0 \qquad \text{and} \qquad \mu + \frac{(1-\theta)\beta\nu}{\gamma} = 0.$$

Eliminating θ and then dividing through by $\beta\gamma$ gives

$$\gamma\mu + \beta\nu - \beta\nu\left(\frac{-\lambda\gamma}{\alpha\nu}\right) = 0 \quad \Rightarrow \quad \frac{\mu}{\beta} + \frac{\nu}{\gamma} + \frac{\lambda}{\alpha} = 0,$$

which is therefore a necessary condition for the collinearity of the three points.

__7.18__ Four points X_i, $i = 1,2,3,4$, taken for simplicity as all lying within the octant $x, y, z \geq 0$, have position vectors $\mathbf{x}_i$. Convince yourself that the direction of vector $\mathbf{x}_n$ lies within the sector of space defined by the directions of the other three vectors if

$$\min_{\text{over } j}\left[\frac{\mathbf{x}_i \cdot \mathbf{x}_j}{|\mathbf{x}_i||\mathbf{x}_j|}\right],$$

considered for $i = 1,2,3,4$ in turn, takes its maximum value for $i = n$, i.e. n equals that value of i for which the largest of the set of angles which $\mathbf{x}_i$ makes with the other vectors is found to be the lowest. Determine whether any of the four points with coordinates

$$X_1 = (3,2,2), \quad X_2 = (2,3,1), \quad X_3 = (2,1,3), \quad X_4 = (3,0,3)$$

lies within the tetrahedron defined by the origin and the other three points.

Suppose that, for some n, x_n lies within the sector defined by the other three vectors. Then each of the other three vectors must make a larger angle with at least one of the other remaining two than it does with x_n. Since a larger angle between unit vectors corresponds to a smaller value of their scalar product, s_{ij}, this requirement can be expressed as in the question. Clearly, at most one of the vectors can satisfy the geometrical condition; if none of the vectors does so then the same scalar product will appear as the minimum for two different values of i and be the largest such minimum.

For the given points the table of scalar products is as follows.

s_{ij}	X_1	X_2	X_3	X_4	Minimum
X_1	1	0.907	0.907	0.857	0.857
X_2	0.907	1	0.714	0.567	0.567
X_3	0.907	0.714	1	0.945	0.714
X_4	0.857	0.567	0.945	1	0.567

The largest minimum occurs uniquely in the line corresponding to X_1 whose direction is therefore contained in the sector defined by the directions of X_2, X_3 and X_4.

To establish whether X_1 lies inside the tetrahedron defined by the origin, X_2, X_3 and X_4, we need to determine whether or not it lies on the same side of the plane P, defined by X_2, X_3 and X_4, as the origin.
The normal to P is given by

$$(\mathbf{X}_3 - \mathbf{X}_2) \times (\mathbf{X}_4 - \mathbf{X}_2) = (0, -2, 2) \times (1, -3, 2) = (2, 2, 2) = \frac{2}{\sqrt{3}}(1, 1, 1)$$

and therefore, since it contains X_2, the equation of the plane is

$$f(x, y, z) = x + y + z - 6 = 0.$$

At the origin the value of $f(x, y, z)$ is $-6 < 0$, whilst at X_1 it is $f(3, 2, 2) = 7-6 > 0$; therefore the origin and X_1 are on opposite sides of P and it follows that X_1 does not lie inside the tetrahedron.

7.20 *Three non-coplanar vectors* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$, *have as their respective reciprocal vectors the set* $\mathbf{a}'$, $\mathbf{b}'$ *and* $\mathbf{c}'$. *Show that the normal to the plane containing the points* $k^{-1}\mathbf{a}$, $l^{-1}\mathbf{b}$ *and* $m^{-1}\mathbf{c}$ *is in the direction of the vector* $k\mathbf{a}' + l\mathbf{b}' + m\mathbf{c}'$.

The plane containing $k^{-1}\mathbf{a}$, $l^{-1}\mathbf{b}$ and $m^{-1}\mathbf{c}$ is $\mathbf{r} = \alpha k^{-1}\mathbf{a} + \beta l^{-1}\mathbf{b} + \gamma m^{-1}\mathbf{c}$, where the scalar quantities α, β and γ satisfy the relationship $\alpha + \beta + \gamma = 1$.

The normal to this plane is in the direction

$$\begin{aligned}\mathbf{n} &= (l^{-1}\mathbf{b} - k^{-1}\mathbf{a}) \times (m^{-1}\mathbf{c} - k^{-1}\mathbf{a}) \\ &= (lm)^{-1}(\mathbf{b} \times \mathbf{c}) - (km)^{-1}(\mathbf{a} \times \mathbf{c}) - (lk)^{-1}(\mathbf{b} \times \mathbf{a}) \\ &= \frac{[\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,]}{klm}(k\mathbf{a}' + l\mathbf{b}' + m\mathbf{c}').\end{aligned}$$

To obtain the last line we used the definitions of the reciprocal vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$, as given in section 7.9.

7.22 *In subsection 7.6.2 we showed how the moment or torque of a force about an axis could be represented by a vector in the direction of the axis. The magnitude of the vector gives the size of the moment and the sign of the vector gives the sense. Similar representations can be used for angular velocities and angular momenta.*

(a) *The magnitude of the angular momentum about the origin of a particle of mass m moving with velocity* $\mathbf{v}$ *on a path that is a perpendicular distance d from the origin is given by* $m|\mathbf{v}|d$. *Show that if* $\mathbf{r}$ *is the position of the particle then the vector* $\mathbf{J} = \mathbf{r} \times m\mathbf{v}$ *represents the angular momentum.*

(a) *Now consider a rigid collection of particles (or a solid body) rotating about an axis through the origin, the angular velocity of the collection being represented by* $\boldsymbol{\omega}$.

(i) *Show that the velocity of the ith particle is*

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

and that the total angular momentum $\mathbf{J}$ *is*

$$\mathbf{J} = \sum_i m_i[r_i^2\,\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i].$$

(ii) *Show further that the component of* $\mathbf{J}$ *along the axis of rotation can be written as* $I\boldsymbol{\omega}$, *where* I, *the moment of inertia of the collection about the axis or rotation, is given by*

$$I = \sum_i m_i \rho_i^2.$$

Interpret ρ_i *geometrically.*

(iii) *Prove that the total kinetic energy of the particles is* $\frac{1}{2}I\omega^2$.

(a) The magnitude of the angular momentum is $m|\mathbf{v}|d$ (see figure 7.1(a)) and, as drawn in the figure, its sense is downwards. Now consider $\mathbf{J} = \mathbf{r} \times m\mathbf{v}$. This also has magnitude $J = m|\mathbf{v}|r\sin\theta = m|\mathbf{v}|d$, and, as shown in the figure, is directed downwards; it is therefore a vector expression for the angular momentum.

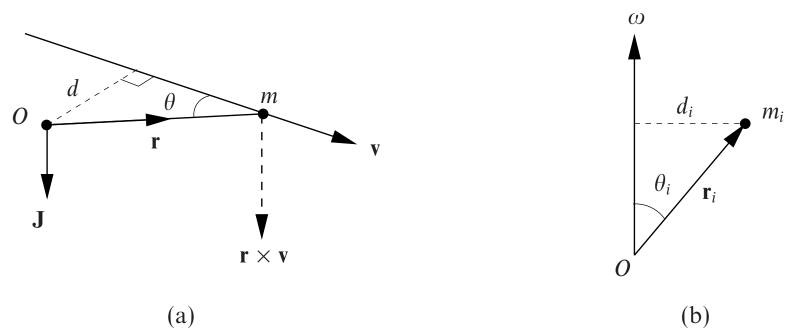

Figure 7.1 The vectors discussed in exercise 7.22. (a) The vector representation of angular momentum, $\mathbf{J} = \mathbf{r} \times m\mathbf{v}$. (b) The linear velocity of the ith particle in a rotating rigid body is given by $\boldsymbol{\omega}\times\mathbf{r}_i$.

(b)(i) As can be seen from figure (b), the velocity of the ith particle has magnitude ωd_i and is directed into the plane of the paper. Its velocity is therefore represented vectorially by $\mathbf{v}_i = \boldsymbol{\omega}\times\mathbf{r}_i$, since $d_i = r_i \sin\theta_i$.

Its angular momentum about the axis of $\boldsymbol{\omega}$ is, from part (a), given by

$$\mathbf{J}_i = \mathbf{r}_i \times m_i\mathbf{v}_i = \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i).$$

The total angular momentum of the whole collection is consequently

$$\begin{aligned}\mathbf{J} &= \sum_i \mathbf{J}_i \\ &= \sum_i m_i[\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \\ &= \sum_i m_i \left[\boldsymbol{\omega} r_i^2 - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i\right]. \qquad \text{(see exercise 7.8)}\end{aligned}$$

(b)(ii) The component of $\mathbf{J}$ along the direction of $\boldsymbol{\omega}$ is

$$\begin{aligned}\frac{\mathbf{J}\cdot\boldsymbol{\omega}}{\omega} &= \frac{1}{\omega}\sum_i m_i \left[r_i^2\omega^2 - (\mathbf{r}_i \cdot \boldsymbol{\omega})^2\right] \\ &= \omega \sum_i m_i \left[r_i^2 - \left(\frac{\mathbf{r}_i \cdot \boldsymbol{\omega}}{\omega}\right)^2\right].\end{aligned}$$

This is of the form $I\omega$, where

$$I = \sum_i m_i \left[r_i^2 - \left(\frac{\mathbf{r}_i \cdot \boldsymbol{\omega}}{\omega}\right)^2\right] = \sum_i m_i \rho_i^2.$$

Here ρ_i, which is independent of the magnitude of $\boldsymbol{\omega}$, is given by

$$\begin{aligned}\rho_i^2 &= r_i^2 - \left(\frac{\mathbf{r}_i \cdot \boldsymbol{\omega}}{\omega}\right)^2 \\ &= r_i^2 - r_i^2 \cos^2\theta_i \\ &= r_i^2 \sin^2\theta_i,\end{aligned}$$

i.e. ρ_i is the distance of the ith particle from the axis of rotation [denoted by d_i in figure (b)].

(b)(iii) The total kinetic energy of the particles is the sum of their individual kinetic energies, and so

$$\begin{aligned}T &= \sum_i \tfrac{1}{2} m_i v_i^2 \\ &= \frac{1}{2}\sum_i m_i(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot(\boldsymbol{\omega}\times\mathbf{r}_i) \\ &= \frac{1}{2}\sum_i m_i\left[r_i^2\omega^2 - (\mathbf{r}_i\cdot\boldsymbol{\omega})^2\right] \\ &= \frac{1}{2}I\omega^2.\end{aligned}$$

To obtain the penultimate line we used the result of exercise 7.9.

7.24 *Without carrying out any further integration, use the results of the previous exercise (the parallel axis theorem), the worked example in subsection 6.3.4 and exercise 6.10 to prove that the moment of inertia of a uniform rectangular lamina, of mass M and sides a and b, about an axis perpendicular to its plane and passing through the point* $(\alpha a/2, \beta b/2)$, *with* $-1 \le \alpha, \beta \le 1$, *is*

$$\frac{M}{12}[a^2(1+3\alpha^2) + b^2(1+3\beta^2)].$$

In the worked example the moment of inertia (MI) about a side of length b was found to be $\frac{1}{3}Ma^2$.

By the parallel axis theorem the MI about a parallel axis through the centre of gravity, O, of the lamina is $\frac{1}{3}Ma^2 - M(\frac{1}{2}a)^2 = \frac{1}{12}Ma^2$.

By symmetry, the MI about an an axis passing through O and parallel to a side of length a will have the corresponding value $\frac{1}{12}Mb^2$.

By the perpendicular axes theorem established in exercise 6.10, the MI about an axis *normal* to the lamina and passing through O is equal to $\frac{1}{12}(a^2+b^2)$.

A second use of the parallel axis theorem then gives the MI about an axis perpendicular to the lamina and passing through $(\alpha a/2, \beta b/2)$ as

$$\frac{1}{12}M(a^2+b^2)+M\left[\left(\frac{\alpha a}{2}\right)^2+\left(\frac{\beta b}{2}\right)^2\right]=\frac{M}{12}\left[a^2(1+3\alpha^2)+b^2(1+3\beta^2)\right].$$

7.26 *Systems that can be modelled as damped harmonic oscillators are widespread; pendulum clocks, car shock absorbers, tuning circuits in television sets and radios, and collective electron motions in plasmas and metals are just a few examples.*

In all these cases, one or more variables describing the system obey(s) an equation of the form

$$\ddot{x}+2\gamma\dot{x}+\omega_0^2x=P\cos\omega t,$$

where $\dot{x}=dx/dt$*, etc. and the inclusion of the factor 2 is conventional. In the steady state (i.e. after the effects of any initial displacement or velocity have been damped out) the solution of the equation takes the form*

$$x(t)=A\cos(\omega t+\phi).$$

By expressing each term in the form $B\cos(\omega t+\epsilon)$ *and representing it by a vector of magnitude* B *making an angle* ϵ *with the* x*-axis, draw a closed vector diagram, at* $t=0$*, say, that is equivalent to the equation.*

(a) *Convince yourself that whatever the value of* ω (>0) ϕ *must be negative* $(-\pi<\phi\le 0)$ *and that*

$$\phi=\tan^{-1}\left(\frac{-2\gamma\omega}{\omega_0^2-\omega^2}\right).$$

(b) *Obtain an expression for* A *in terms of* P, ω_0 *and* ω.

Substituting $x(t)=A\cos(\omega t+\phi)$ into the differential equation:

$$\begin{aligned}P\cos\omega t&=\ddot{x}+2\gamma\dot{x}+\omega_0^2x,\\ P\cos\omega t&=-\omega^2A\cos(\omega t+\phi)-2\gamma\omega A\sin(\omega t+\phi)+\omega_0^2A\cos(\omega t+\phi),\\ P\cos\omega t&=\omega^2A\cos(\omega t+\phi+\pi)+2\gamma\omega A\cos(\omega t+\phi+\tfrac{1}{2}\pi)\\ &\qquad+\omega_0^2A\cos(\omega t+\phi).\end{aligned}$$

Now, set $t=0$ and represent each term as a vector with magnitude and phase as shown in figure 7.2.

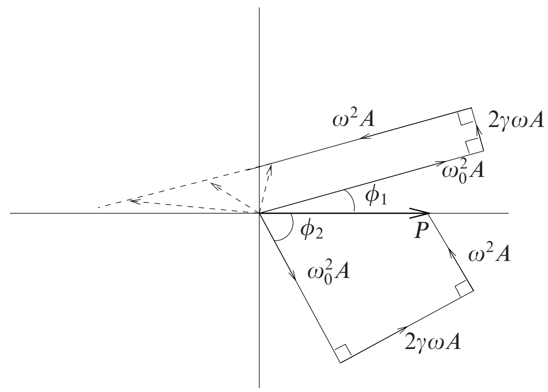

Figure 7.2 The vector diagram for the equation in exercise 7.26.

(a) For the last of these equations to be valid the three vectors representing the terms on the RHS must have a resultant equal to that representing P on the LHS, i.e. the resultant must be real and positive. As can be seen, with $\phi > 0$ (illustrated by $\phi = \phi_1$ in the figure), no matter what value ω takes, the possible resultants (broken arrows) can never equal P.

(b) However, with $\phi < 0$ (illustrated by $\phi = \phi_2$), the three vectors from the RHS can have a resultant corresponding to P. When this happens, from the geometry of the quadrilateral, it can be seen that

$$|\tan\phi_2| = \frac{2\gamma\omega A}{\omega_0^2 A - \omega^2 A} \Rightarrow \phi = \tan^{-1}\left(\frac{-2\gamma\omega}{\omega_0^2 - \omega^2}\right),$$

and, from the geometry of a right-angled triangle, that

$$P^2 = (2\gamma\omega A)^2 + (\omega_0^2 A - \omega^2 A)^2,$$

$$\Rightarrow \quad A = \frac{P}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}.$$

This is the amplitude of the response of the system when that of the sinusoidal input is P.

8

Matrices and vector spaces

8.2 *Evaluate the determinants*

$$\text{(a)} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \qquad \text{(b)} \begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix}$$

and

$$\text{(c)} \begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}.$$

(a) Using the elements and cofactors of the first row in a straightforward Laplace expansion, we have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a(bc-f) + h(fg-hc) + g(hf-gb)$$
$$= abc + 2fgh - af^2 - bg^2 - ch^2.$$

(b) At each stage we subtract a suitable multiple of the first column from each other column so as to make the first entry in each of the other columns zero; then we use a Laplace expansion with a single term. Here this reduction is carried out three times.

$$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & -2 & -11 \\ -2 & 1 & 2 & 7 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -2 & 1 \\ -3 & -2 & -11 \\ 1 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -3 & -8 & -8 \\ 1 & 4 & 6 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -8 & -8 \\ 4 & 6 \end{vmatrix} = 1 \begin{vmatrix} -8 & 0 \\ 4 & 2 \end{vmatrix} = -8|2| = -16.$$

(c) In making this reduction we (i) subtract g times the third row from the first row, (ii) subtract the second row from the fourth, (iii) use the Laplace expansion, (iv) subtract the second column from the third, and (v) use a Laplace expansion followed by direct evaluation.

$$\begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix} = \begin{vmatrix} 0 & 0 & a & 0 \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & a & 0 \\ 0 & b & b & b \\ c & e & e & b+e \\ a & 0 & f & d \end{vmatrix} = a \begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & 0 & d \end{vmatrix}$$

$$= a \begin{vmatrix} 0 & b & 0 \\ c & e & b \\ a & 0 & d \end{vmatrix} = -ab \begin{vmatrix} c & b \\ a & d \end{vmatrix}$$

$$= ab(ab - cd).$$

8.4 *Consider the matrices*

$$\text{(a)}\ \mathsf{B} = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}, \qquad \text{(b)}\ \mathsf{C} = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}.$$

Are they (i) *real,* (ii) *diagonal,* (iii) *symmetric,* (iv) *antisymmetric,* (v) *singular,* (vi) *orthogonal,* (vii) *Hermitian,* (viii) *anti-Hermitian,* (ix) *unitary,* (x) *normal*?

(a) For matrix B:

Clearly, (i)-(iii) are not true whilst (iv) is.
(v) $|\mathsf{B}| = -i(i^2 - 0) + i(i^2) = 0$ and so B is singular.
(vi) From (v) it follows that B has no inverse. In particular, its transpose cannot be its inverse, i.e. B is not orthogonal.
(vii)

$$(\mathsf{B}^*)^T = \begin{pmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix} = \mathsf{B},$$

i.e. B is Hermitian.
(viii) In view of (vii), B cannot be anti-Hermitian.
(ix) As in (vi), B cannot be unitary.
(x) Since B is Hermitian, it commutes with its Hermitian conjugate (itself) and is therefore normal.

(b) For matrix C:

C is clearly real, i.e. satisfies (i), and, equally clearly, satisfies none of (ii)-(iv).
(v)

$$|\mathsf{C}| = \left(\frac{1}{\sqrt{8}}\right)^3 \begin{vmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{vmatrix} = \left(\frac{1}{\sqrt{8}}\right)^3 \begin{vmatrix} \sqrt{3} & -\sqrt{2} & 0 \\ 1 & \sqrt{6} & 0 \\ 2 & 0 & 4 \end{vmatrix}$$
$$= \frac{1}{\sqrt{32}}(\sqrt{18} + \sqrt{2})$$
$$= \tfrac{1}{4}(3 + 1) = 1 \neq 0.$$

Thus C is not singular.
(vi) Consider $\mathsf{C}^T\mathsf{C}$, which is given by

$$\frac{1}{8}\begin{pmatrix} \sqrt{3} & 1 & 2 \\ -\sqrt{2} & \sqrt{6} & 0 \\ -\sqrt{3} & -1 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathsf{I}_3.$$

Thus C is orthogonal.
(vii) & (viii) In view of (i), (iii) and (iv), C cannot be either Hermitian or anti-Hermitian.
(ix) In view of (i) and (vi), C is unitary.
(x) In view of (i) and (vi), $\mathsf{C}^\dagger\mathsf{C} = \mathsf{C}^T\mathsf{C} = \mathsf{I} = \mathsf{C}\mathsf{C}^T = \mathsf{C}\mathsf{C}^\dagger$. Hence C is normal.

8.6 *This exercise considers a crystal whose unit cell has base vectors that are not necessarily mutually orthogonal.*

(a) *The basis vectors of the unit cell of a crystal, with the origin O at one corner, are denoted by* $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. *The matrix* G *has elements* G_{ij}, *where* $G_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ *and* H_{ij} *are the elements of the matrix* $\mathsf{H} \equiv \mathsf{G}^{-1}$. *Show that the vectors* $\mathbf{f}_i = \sum_j H_{ij}\,\mathbf{e}_j$ *are the reciprocal vectors and that* $H_{ij} = \mathbf{f}_i \cdot \mathbf{f}_j$.

(b) *If the vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are given by*

$$\mathbf{u} = \sum_i u_i \mathbf{e}_i, \qquad \mathbf{v} = \sum_i v_i \mathbf{f}_i,$$

obtain expressions for $|\mathbf{u}|$, $|\mathbf{v}|$, *and* $\mathbf{u} \cdot \mathbf{v}$.

(c) *If the basis vectors are each of length a and the angle between each pair is* $\pi/3$, *write down* G *and hence obtain* H.

(d) *Calculate* (i) *the length of the normal from O onto the plane containing the points* $p^{-1}\mathbf{e}_1$, $q^{-1}\mathbf{e}_2$, $r^{-1}\mathbf{e}_3$, *and* (ii) *the angle between this normal and* $\mathbf{e}_1$.

(a) With $\mathbf{f}_i$ defined by $\mathbf{f}_i = \sum_j H_{ij}\,\mathbf{e}_j$, consider

$$\mathbf{f}_i \cdot \mathbf{e}_k = \sum_j H_{ij}\,\mathbf{e}_j \cdot \mathbf{e}_k = \sum_j \left(G^{-1}\right)_{ij} G_{jk} = \left(G^{-1}G\right)_{ik} = \delta_{ik}.$$

Thus the $\mathbf{f}_i$ are the reciprocal vectors of the cell's base vectors.

Now consider

$$\begin{aligned}\mathbf{f}_i \cdot \mathbf{f}_j &= \sum_k H_{ik}\mathbf{e}_k \sum_m H_{jm}\mathbf{e}_m = \sum_{k,m} H_{ik} H_{jm} G_{km} \\ &= \sum_m H_{jm} \sum_k H_{ik} G_{km} = \sum_m H_{jm}\delta_{im} = H_{ji} = H_{ij}.\end{aligned}$$

(b) With $\mathbf{u} = \sum_i u_i\,\mathbf{e}_i$,

$$|\mathbf{u}|^2 = \sum_i u_i\,\mathbf{e}_i \sum_j u_j\,\mathbf{e}_j = \sum_{i,j} u_i\,G_{ij}\,u_j \Rightarrow |\mathbf{u}| = \left(\sum_{i,j} u_i\,G_{ij}\,u_j\right)^{1/2}.$$

Similarly,

$$|\mathbf{v}| = \left(\sum_{i,j} v_i\,H_{ij}\,v_j\right)^{1/2}.$$

For the scalar product of $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} \cdot \mathbf{v} = \sum_i u_i\,\mathbf{e}_i \sum_j v_j\,\mathbf{f}_j = \sum_{i,j} u_i\,v_j\,\delta_{ij} = \sum_i u_i\,v_i.$$

(c) For $i = j$, $\mathbf{e}_i \cdot \mathbf{e}_j = a^2$ whilst, for $i \neq j$, $\mathbf{e}_i \cdot \mathbf{e}_j = a^2 \cos(\pi/3) = \frac{1}{2}a^2$. Thus

$$\mathsf{G} = \tfrac{1}{2}a^2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The matrix $2\mathsf{G}/a^2$ has determinant 4 and all of its co-factors are either 3 or ± 1. The matrix H, computed using this data, is found to be

$$\mathsf{H} \equiv \mathsf{G}^{-1} = \frac{1}{2a^2} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

(d)(i) The normal to the plane is in the direction

$$(q^{-1}\mathbf{e}_2 - p^{-1}\mathbf{e}_1) \times (r^{-1}\mathbf{e}_3 - p^{-1}\mathbf{e}_1) \propto \left[(qr)^{-1}\mathbf{f}_1 + (pr)^{-1}\mathbf{f}_2 + (qp)^{-1}\mathbf{f}_3\right]$$

i.e. in the direction $\mathbf{f} = p\,\mathbf{f}_1 + q\,\mathbf{f}_2 + r\,\mathbf{f}_3$.

A unit vector in this direction is

$$\hat{\mathbf{n}} = \frac{p\,\mathbf{f}_1 + q\,\mathbf{f}_2 + r\,\mathbf{f}_3}{(\mathbf{f} \cdot \mathbf{f})^{1/2}}$$

and the distance from the origin to the plane is the scalar product of this unit vector and the position vector of any one of the three points (necessarily, they all give the same answer). Using $p^{-1}\mathbf{e}_1$ and denoting (p, q, r) by v_i, we have the distance d as

$$\begin{aligned} d = \hat{\mathbf{n}} \cdot p^{-1}\mathbf{e}_1 &= \frac{p^{-1}p + 0 + 0}{(\mathbf{f} \cdot \mathbf{f})^{1/2}} \\ &= \frac{1}{(\sum_i v_i \mathbf{f}_i \sum_j v_j \mathbf{f}_j)^{1/2}} \\ &= \frac{1}{(\sum_{i,j} v_i H_{ij} v_j)^{1/2}} = \frac{1}{M}, \end{aligned}$$

where $M^2 = (2a^2)^{-1}[3(p^2 + q^2 + r^2) - 2(qr + rp + pq)]$.

(d)(ii) The angle θ between $\hat{\mathbf{n}}$ and $\mathbf{e}_1$ is given by

$$\theta = \cos^{-1} \frac{\hat{\mathbf{n}} \cdot \mathbf{e}_1}{|\mathbf{e}_1|} = \cos^{-1} \frac{p\,d}{a} = \cos^{-1} \frac{p}{aM}.$$

8.8 *A and B are real non-zero 3×3 matrices and satisfy the equation*

$$(\mathsf{AB})^{\mathrm{T}} + \mathsf{B}^{-1}\mathsf{A} = 0.$$

(a) *Prove that if B is orthogonal then A is antisymmetric.*
(b) *Without assuming that B is orthogonal, prove that A is singular.*

We have that $(\mathsf{AB})^{\mathrm{T}} = -\mathsf{B}^{-1}\mathsf{A}$.

(a) Given $\mathsf{B}^{\mathrm{T}}\mathsf{B} = \mathsf{I}$ (i.e. B is orthogonal),

$$\mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} = -\mathsf{B}^{-1}\mathsf{A} \quad \Rightarrow \quad \mathsf{BB}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} = -\mathsf{BB}^{-1}\mathsf{A} \quad \Rightarrow \quad \mathsf{A}^{\mathrm{T}} = -\mathsf{A},$$

i.e. A is antisymmetric.

(b) Since B^{-1} is defined, $|\mathsf{B}| \neq 0$.

$$\begin{aligned}
\mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} &= -\mathsf{B}^{-1}\mathsf{A} \\
\mathsf{BB}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} &= -\mathsf{A} \\
|\mathsf{B}|\,|\mathsf{B}^{\mathrm{T}}|\,|\mathsf{A}^{\mathrm{T}}| &= |-\mathsf{A}| \\
|\mathsf{B}|^2\,|\mathsf{A}| &= (-1)^3|\mathsf{A}|, \quad \text{since } |\mathsf{B}^{\mathrm{T}}| = |\mathsf{B}|.
\end{aligned}$$

In the last line the factor $(-1)^3$ arises because A is a 3×3 matrix. The two sides of the last equation have opposite (contradictory) signs unless $|\mathsf{A}| = 0$, i.e. unless A is singular.

8.10 *The four matrices S_x, S_y, S_z and I are defined by*

$$\mathsf{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\mathsf{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $i^2 = -1$. Show that $\mathsf{S}_x^2 = \mathsf{I}$ and $\mathsf{S}_x\mathsf{S}_y = i\mathsf{S}_z$, and obtain similar results by permutting x, y and z. Given that $\mathbf{v}$ is a vector with Cartesian components (v_x, v_y, v_z), the matrix $\mathsf{S}(\mathbf{v})$ is defined as

$$\mathsf{S}(\mathbf{v}) = v_x\mathsf{S}_x + v_y\mathsf{S}_y + v_z\mathsf{S}_z.$$

Prove that, for general non-zero vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathsf{S}(\mathbf{a})\mathsf{S}(\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\,\mathsf{I} + i\,\mathsf{S}(\mathbf{a}\times\mathbf{b}).$$

Without further calculation, deduce that $\mathsf{S}(\mathbf{a})$ and $\mathsf{S}(\mathbf{b})$ commute if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

As we have only the definitions to work with, these basic products must be found by explicit matrix multiplication:

$$\mathsf{S}_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathsf{I}.$$

$$\mathsf{S}_x\mathsf{S}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\mathsf{S}_z.$$

$$\mathsf{S}_y\mathsf{S}_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\mathsf{S}_x.$$

Similarly $\mathsf{S}_y^2 = \mathsf{S}_z^2 = \mathsf{I}$ and $\mathsf{S}_z\mathsf{S}_x = i\mathsf{S}_y$. We also note that

$$\mathsf{S}_y\mathsf{S}_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\mathsf{S}_z,$$

i.e. that S_x and S_y anticommute. This applies to any pair of the matrices (excluding I of course).

We first note that if $\mathbf{0}$ is the zero vector then $\mathsf{S}(\mathbf{0}) = \mathsf{O}$, the zero matrix; conversely, if $\mathsf{S}(\mathbf{v}) = \mathsf{O}$ then $\mathbf{v} = \mathbf{0}$. Now consider the product of the two matrices $\mathsf{S}(\mathbf{a})$ and $\mathsf{S}(\mathbf{b})$.

$$\begin{aligned}
\mathsf{S}(\mathbf{a})\mathsf{S}(\mathbf{b}) &= (a_x\mathsf{S}_x + a_y\mathsf{S}_y + a_z\mathsf{S}_z)(b_x\mathsf{S}_x + b_y\mathsf{S}_y + b_z\mathsf{S}_z) \\
&= (a_xb_x + a_yb_y + a_zb_z)\mathsf{I} + a_xb_y(i\mathsf{S}_z) + a_yb_x(-i\mathsf{S}_z) + \cdots \\
&= (a_xb_x + a_yb_y + a_zb_z)\mathsf{I} + i(\mathbf{a}\times\mathbf{b})_z\mathsf{S}_z + \cdots \\
&= (\mathbf{a}\cdot\mathbf{b})\mathsf{I} + i\mathsf{S}(\mathbf{a}\times\mathbf{b}), \qquad \text{as stated in the question.}
\end{aligned}$$

Interchanging $\mathbf{a}$ and $\mathbf{b}$ gives $\mathsf{S}(\mathbf{b})\mathsf{S}(\mathbf{a}) = (\mathbf{b}\cdot\mathbf{a})\mathsf{I} + i\mathsf{S}(\mathbf{b}\times\mathbf{a})$. It then follows that

$$\mathsf{S}(\mathbf{a})\mathsf{S}(\mathbf{b}) - \mathsf{S}(\mathbf{b})\mathsf{S}(\mathbf{a}) = 2i\mathsf{S}(\mathbf{a}\times\mathbf{b}).$$

The matrix on the RHS is the zero matrix if and only if $\mathbf{a}\times\mathbf{b} = \mathbf{0}$, i.e. $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

8.12 *Given a matrix*

$$\mathsf{A} = \begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where α and β are non-zero complex numbers, find its eigenvalues and eigenvectors. Find the respective conditions for (a) *the eigenvalues to be real and* (b) *the eigenvectors to be orthogonal. Show that the conditions are jointly satisfied if and only if* A *is Hermitian.*

The eigenvalues λ of A are the roots of

$$\begin{vmatrix} 1-\lambda & \alpha & 0 \\ \beta & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0,$$

i.e. the values of λ that satisfy

$$(1-\lambda)\left[(1-\lambda)^2 - \alpha\beta\right] = 0.$$

This means that either $\lambda = 1$ or that

$$\lambda^2 - 2\lambda + 1 - \alpha\beta = 0,$$
$$\Rightarrow \quad \lambda = 1 \pm (\alpha\beta)^{1/2}.$$

For $\lambda = 1$ the corresponding eigenvector is obviously $\mathsf{e}^1 = (0,\ 0,\ 1)^{\mathrm{T}}$.

For $\lambda = 1 \pm (\alpha\beta)^{1/2}$, we have for eigenvector $(x,\ y,\ z)^{\mathrm{T}}$ that

$$\mp(\alpha\beta)^{1/2}x + \alpha y = 0,$$
$$\mp(\alpha\beta)^{1/2}z = 0.$$

Thus $\mathsf{e}^{2,3} = \left(\sqrt{\alpha},\ \pm\sqrt{\beta},\ 0\right)^{\mathrm{T}}$.

(a) For all the eigenvalues to be real, we need the product $\alpha\beta$ to be real and positive.

(b) For the eigenvectors to be mutually orthogonal we need (recall that α and β can be complex)

$$0 = (\mathsf{e}^2)^{\dagger} \cdot \mathsf{e}^3 = (\alpha^*)^{1/2}\alpha^{1/2} - (\beta^*)^{1/2}\beta^{1/2} \quad \Rightarrow \quad |\alpha| = |\beta|.$$

The orthogonality of e^3 to the other two is trivially obvious.

(i) If both conditions are satisfied and we write $\alpha = ce^{i\theta}$, then the first condition requires that the argument of β is $-\theta$ whilst the second requires its magnitude to be c. Thus $\beta = ce^{-i\theta}$ and $\beta = \alpha^*$, making A Hermitian.

(ii) If A is Hermitian, $\beta = \alpha^*$ and so $\alpha\beta$ is real and positive. The eigenvalues are then $1,\ 1 + |\alpha|$ and $1 - |\alpha|$, i.e. all real.

The corresponding eigenvectors have the forms $(0,\ 0,\ 1)^{\mathrm{T}}$, $\left(\sqrt{\alpha},\ \sqrt{\alpha^*},\ 0\right)^{\mathrm{T}}$ and $\left(\sqrt{\alpha},\ -\sqrt{\alpha^*},\ 0\right)^{\mathrm{T}}$, and are clearly mutually orthogonal.

Thus, the two conditions are jointly satisfied if and only if A is Hermitian.

> ***8.14*** *If a unitary matrix* U *is written as* $\mathsf{A} + i\mathsf{B}$*, where* A *and* B *are Hermitian with non-degenerate eigenvalues, show the following:*
>
> (a) A *and* B *commute*;
> (b) $\mathsf{A}^2 + \mathsf{B}^2 = \mathsf{I}$;
> (c) *The eigenvectors of* A *are also eigenvectors of* B;
> (d) *The eigenvalues of* U *have unit modulus (as is necessary for any unitary matrix).*

Given that $\mathsf{U} = \mathsf{A} + i\mathsf{B}$, with $\mathsf{A}^\dagger = \mathsf{A}$, $\mathsf{B}^\dagger = \mathsf{B}$ and $\mathsf{U}^\dagger\mathsf{U} = \mathsf{I}$, consider

$$\begin{aligned}
\mathsf{I} = \mathsf{U}^\dagger\mathsf{U} &= (\mathsf{A}^\dagger - i\mathsf{B}^\dagger)(\mathsf{A} + i\mathsf{B}) \\
&= (\mathsf{A} - i\mathsf{B})(\mathsf{A} + i\mathsf{B}) \\
&= \mathsf{A}^2 + \mathsf{B}^2 + i(\mathsf{AB} - \mathsf{BA}). \\
\text{and} \qquad \mathsf{I} = \mathsf{U}\mathsf{U}^\dagger &= (\mathsf{A} + i\mathsf{B})(\mathsf{A}^\dagger - i\mathsf{B}^\dagger) \\
&= (\mathsf{A} + i\mathsf{B})(\mathsf{A} - i\mathsf{B}) \\
&= \mathsf{A}^2 + \mathsf{B}^2 + i(\mathsf{BA} - \mathsf{AB}).
\end{aligned}$$

Comparison of the two results implies

(a) $\mathsf{BA} - \mathsf{AB} = \mathsf{O}$, i.e. A and B commute, and, consequently, (b) $\mathsf{A}^2 + \mathsf{B}^2 = \mathsf{I}$.

(c) Let x be an eigenvector of A with eigenvalue λ, i.e. $\mathsf{Ax} = \lambda\mathsf{x}$. Then

$$\mathsf{ABx} = \mathsf{BAx} = \mathsf{B}\lambda\mathsf{x} = \lambda\mathsf{Bx},$$

where we have used result (a) to justify the first equality.
Now, the above result shows that $\mathsf{y} = \mathsf{Bx}$ is an eigenvector of A with eigenvalue λ.
But the eigenvalues of A are non-degenerate and so y must be a multiple of x, i.e. $\mathsf{Bx} = \mu\mathsf{x}$ for some μ.
However, this is the statement that x is an eigenvector of B (as well as of A). Hence each eigenvector of A is also an eigenvector of B.

(d) Let x be an eigenvector of U with (complex) eigenvalue λ. We then have

$$\begin{aligned}
(\mathsf{A} + i\mathsf{B})\mathsf{x} &= \lambda\mathsf{x}, \quad \text{take the hermitian conjugate,} \\
\mathsf{x}^\dagger(\mathsf{A}^\dagger - i\mathsf{B}^\dagger) &= \lambda^*\mathsf{x}^\dagger, \\
\mathsf{x}^\dagger(\mathsf{A}^2 + \mathsf{B}^2)\mathsf{x} &= \mathsf{x}^\dagger|\lambda|^2\mathsf{x}, \\
\mathsf{x}^\dagger\ \mathsf{I}\ \mathsf{x} &= \mathsf{x}^\dagger|\lambda|^2\mathsf{x}, \\
|\mathsf{x}|^2 &= |\lambda|^2|\mathsf{x}|^2.
\end{aligned}$$

To obtain the third equation we multiplied the two LHS and the two RHS of the first two equations together. Finally, since x is a non-zero vector it follows from the last equation that $|\lambda| = 1$.

8.16 *Find the eigenvalues and a set of eigenvectors of the matrix*

$$\begin{pmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{pmatrix}.$$

Verify that its eigenvectors are mutually orthogonal.

The eigenvalues must be the roots of

$$\begin{vmatrix} 1-\lambda & 3 & -1 \\ 3 & 4-\lambda & -2 \\ -1 & -2 & 2-\lambda \end{vmatrix} = 0.$$

Evaluating the determinant gives

$$\begin{aligned} (1-\lambda)(\lambda^2 - 6\lambda + 4) + 3(-4 + 3\lambda) - 1(-2-\lambda) &= 0, \\ (1-\lambda)(\lambda^2 - 6\lambda + 4) - 10 + 10\lambda &= 0, \\ (1-\lambda)(\lambda^2 - 6\lambda - 6) &= 0. \end{aligned}$$

Thus $\lambda = 1$ or $\lambda = 3 \pm \sqrt{15}$.

Writing an eigenvector as $\mathsf{e} = (x,\ y,\ z)^{\mathrm{T}}$:

For $\lambda = 1$, $0x + 3y - z = 0$ and $3x + 3y - 2z = 0$ which imply

$$\mathsf{e}^1 = (1,\ 1,\ 3)^{\mathrm{T}}.$$

For $\lambda = 3 \pm \sqrt{15}$,

$$\begin{aligned} (-2 \mp \sqrt{15})x + 3y - z &= 0, \\ 3x + (1 \mp \sqrt{15})y - 2z &= 0. \end{aligned}$$

We now eliminate z and obtain

$$(-7 \mp 2\sqrt{15})x + (5 \pm \sqrt{15})y = 0.$$

Taking $\mathsf{e} = \left(5 \pm \sqrt{15},\ 7 \pm 2\sqrt{15},\ z\right)^{\mathrm{T}}$, the first equation gives

$$\begin{aligned} z &= (-2 \mp \sqrt{15})(5 \pm \sqrt{15}) + 3(7 \pm 2\sqrt{15}) \\ &= -4 \mp \sqrt{15}. \end{aligned}$$

Thus the three eigenvectors are

$$\mathsf{e}^1 = (1,\ 1,\ 3)^{\mathrm{T}},$$
$$\mathsf{e}^2 = \left(5+\sqrt{15},\ 7+2\sqrt{15},\ -4-\sqrt{15}\right)^{\mathrm{T}},$$
$$\mathsf{e}^3 = \left(5-\sqrt{15},\ 7-2\sqrt{15},\ -4+\sqrt{15}\right)^{\mathrm{T}}.$$

Their mutual orthogonality is established by considering the following scalar products.

$$\mathsf{e}^1\cdot\mathsf{e}^2 = (5+7-12)+(1+2-3)\sqrt{15} = 0,$$
$$\mathsf{e}^1\cdot\mathsf{e}^3 = (5+7-12)+(-1-2+3)\sqrt{15} = 0,$$
$$\mathsf{e}^2\cdot\mathsf{e}^3 = (25-15)+(49-60)+(16-15) = 0,$$

i.e. they are mutually orthogonal. We note that, formally, the first factor in each scalar product should be the hermitian conjugate of the eigenvector; here this makes no difference as all components are real.

8.18 *Use the results of the first worked example in section 8.14 to evaluate, without repeated matrix multiplication, the expression* $\mathsf{A}^6\mathsf{x}$*, where* $\mathsf{x} = (2 \quad 4 \quad -1)^{\mathrm{T}}$ *and* A *is the matrix given in the example.*

A set of three (un-normalised, but that does not matter here) independent eigenvectors of A, and their corresponding eigenvalues, are, as given in section 8.14,

$$\mathsf{x}^1 = (1,\ 1,\ 0)^{\mathrm{T}} \quad \text{for} \quad \lambda = 2,$$
$$\mathsf{x}^2 = (1,\ -1,\ 1)^{\mathrm{T}} \quad \text{for} \quad \lambda = 3,$$
$$\mathsf{x}^3 = (1,\ -1,\ -2)^{\mathrm{T}} \quad \text{for} \quad \lambda = 2.$$

We first express the given vector x in terms of the eigenvectors as

$$\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 3\mathsf{x}^1 - \mathsf{x}^2.$$

We now use the fact that, for an eigenvector, $\mathsf{A}^n\mathsf{x} = \lambda^n\mathsf{x}$. This gives

$$\begin{aligned}\mathsf{A}^6\mathsf{x} = 3\ 2^6\mathsf{x}^1 - 3^6\mathsf{x}^2 &= \left(3\ 2^6(1) - 3^6(1),\ 3\ 2^6(1) - 3^6(-1),\ 3\ 2^6(0) - 3^6(1)\right)^{\mathrm{T}} \\ &= (-537,\ 921,\ -729)^{\mathrm{T}}.\end{aligned}$$

8.20 *Demonstrate that the matrix*

$$\mathsf{A} = \begin{pmatrix} 2 & 0 & 0 \\ -6 & 4 & 4 \\ 3 & -1 & 0 \end{pmatrix},$$

is defective, i.e. does not have three linearly independent eigenvectors, by showing the following:

(a) *its eigenvalues are degenerate and, in fact, all equal;*
(b) *any eigenvector has the form* $(\mu \quad (3\mu - 2\nu) \quad \nu)^{\mathrm{T}}$;
(c) *if two pairs of values,* μ_1, ν_1 *and* μ_2, ν_2, *define two independent eigenvectors* v_1 *and* v_2 *then* any *third similarly defined eigenvector* v_3 *can be written as a linear combination of* v_1 *and* v_2, *i.e.*

$$\mathsf{v}_3 = a\mathsf{v}_1 + b\mathsf{v}_2$$

where

$$a = \frac{\mu_3\nu_2 - \mu_2\nu_3}{\mu_1\nu_2 - \mu_2\nu_1} \quad \text{and} \quad b = \frac{\mu_1\nu_3 - \mu_3\nu_1}{\mu_1\nu_2 - \mu_2\nu_1}.$$

Illustrate (c) *using the example* $(\mu_1, \nu_1) = (1, 1)$, $(\mu_2, \nu_2) = (1, 2)$ *and* $(\mu_3, \nu_3) = (0, 1)$.

Show further that any matrix of the form

$$\begin{pmatrix} 2 & 0 & 0 \\ 6n - 6 & 4 - 2n & 4 - 4n \\ 3 - 3n & n - 1 & 2n \end{pmatrix}$$

is defective, with the same eigenvalues and eigenvectors as A.

(a) The eigenvalues of A are given by

$$\begin{vmatrix} 2 - \lambda & 0 & 0 \\ -6 & 4 - \lambda & 4 \\ 3 & -1 & -\lambda \end{vmatrix} = (2 - \lambda)(-4\lambda + \lambda^2 + 4) = (2 - \lambda)^3 = 0.$$

Thus A has three equal eigenvalues $\lambda = 2$.

(b) Using this value for λ, an eigenvector $(x,\ y,\ z)^{\mathrm{T}}$ must satisfy

$$\begin{aligned} 0x + 0y + 0z &= 0, \\ -6x + 2y + 4z &= 0, \\ 3x - y - 2z &= 0, \end{aligned}$$

leading to the conclusion that $\mathsf{v} = (\mu,\ 3\mu - 2\nu,\ \nu)^{\mathrm{T}}$, with μ and ν arbitrary, will be an eigenvector. Clearly any two components of a vector of this form can be chosen arbitrarily, but the third one is then determined.

(c) Given two eigenvectors

$$\mathsf{v}_1 = (\mu_1,\ 3\mu_1 - 2\nu_1,\ \nu_1)^{\mathrm{T}} \quad \text{and} \quad \mathsf{v}_2 = (\mu_2,\ 3\mu_2 - 2\nu_2,\ \nu_2)^{\mathrm{T}},$$

any third vector v_3 of the same form with parameters μ_3 and ν_3 can be written as a linear combination of v_1 and v_2, as is shown by the following argument. Define the vector v and the numbers a and b by

$$\mathsf{v} = a\mathsf{v}_1 + b\mathsf{v}_2 \equiv \frac{\mu_3\nu_2 - \mu_2\nu_3}{\mu_1\nu_2 - \mu_2\nu_1}\mathsf{v}_1 + \frac{\mu_1\nu_3 - \mu_3\nu_1}{\mu_1\nu_2 - \mu_2\nu_1}\mathsf{v}_2.$$

Now consider the first component, say, of the vector on the RHS

$$\frac{\mu_3\nu_2 - \mu_2\nu_3}{\mu_1\nu_2 - \mu_2\nu_1}\mu_1 + \frac{\mu_1\nu_3 - \mu_3\nu_1}{\mu_1\nu_2 - \mu_2\nu_1}\mu_2 = \frac{\mu_1\mu_3\nu_2 - \mu_2\mu_3\nu_1}{\mu_1\nu_2 - \mu_2\nu_1} = \mu_3.$$

Similarly, the second component is $3\mu_3 - 2\nu_3$ and the third one ν_3. In other words, $\mathsf{v} = \mathsf{v}_3$ and v_3 has been expressed explicitly as a linear combination of v_1 and v_2. This establishes that A does *not* have three linearly independent eigenvectors, i.e. it is defective.

(d) With

$$(\mu_1, \nu_1) = (1, 1) \text{ and } \mathsf{v}_1 = (1,\ 1,\ 1)^{\mathrm{T}},$$
$$(\mu_2, \nu_2) = (1, 2) \text{ and } \mathsf{v}_2 = (1,\ -1,\ 2)^{\mathrm{T}},$$
$$(\mu_3, \nu_3) = (0, 1) \text{ and } \mathsf{v}_3 = (0,\ -2,\ 1)^{\mathrm{T}},$$

$$a = \frac{(0 \times 2) - (1 \times 1)}{(1 \times 2) - (1 \times 1)} = -1 \quad \text{and} \quad b = \frac{(1 \times 1) - (0 \times 1)}{(1 \times 2) - (1 \times 1)} = 1.$$

Thus

$$\mathsf{v} = -1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \mathsf{v}_3,$$

as expected.

The eigenvalues of this more general matrix, $\mathsf{A}(n)$ say, are given by

$$\begin{aligned} 0 &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 6n-6 & 4-2n-\lambda & 4-4n \\ 3-3n & n-1 & 2n-\lambda \end{vmatrix} \\ &= (2-\lambda)[(4-2n-\lambda)(2n-\lambda) - (n-1)(4-4n)] \\ &= (2-\lambda)(8n - 4\lambda - 4n^2 + 2n\lambda - 2n\lambda + \lambda^2 - 4n + 4 + 4n^2 - 4n) \\ &= (2-\lambda)(2-\lambda)^2. \end{aligned}$$

This shows that the eigenvalues of $\mathsf{A}(n)$ are the same as those of A.

The equations to be satisfied by the components of an eigenvector of $\mathsf{A}(n)$, $\mathsf{v} = (x,\ y,\ z)^{\mathrm{T}}$, are

$$\begin{aligned} 0x + 0y + 0z &= 0, \\ (6n - 6)x + (2 - 2n)y + (4 - 4n)z &= 0, \\ (3 - 3n)x + (n - 1)y + (2n - 2)z &= 0. \end{aligned}$$

When the common factor $(n-1)$ has been cancelled from the second and third of these, the equations remaining are identical to those satisfied by the components of the eigenvectors of A. The eigenvectors will therefore be identical to those of A; it also follows that $\mathsf{A}(n)$ is defective.

> ***8.22*** *Use the stationary properties of quadratic forms to determine the maximum and minimum values taken by the expression*
>
> $$Q = 5x^2 + 4y^2 + 4z^2 + 2xz + 2xy$$
>
> *on the unit sphere, $x^2 + y^2 + z^2 = 1$. For what values of x, y and z do they occur?*

Since all vectors on the unit sphere have unit modulus the maximum and minimum values of Q will be equal to the largest and smallest of the eigenvalues of the associated symmetric matrix. These we find by considering

$$\begin{aligned} \begin{vmatrix} 5-\lambda & 1 & 1 \\ 1 & 4-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} &= 0, \\ (5-\lambda)(16 - 8\lambda + \lambda^2) - 4 + \lambda - 4 + \lambda &= 0, \\ \lambda^3 - 13\lambda^2 + 54\lambda - 72 &= 0. \end{aligned}$$

There is no concise automatic way to solve this cubic equation, but by inspection it is clear that if $\lambda = 3$ the top row of the determinant is equal to the sum of the other two, implying that $\lambda = 3$ is one root. The polynomial equation is now easily factorised as

$$(\lambda - 3)(\lambda - 4)(\lambda - 6) = 0,$$

showing that the maximum and minimum values of Q are 6 and 3.

The corresponding values of $(x,\ y,\ z)^{\mathrm{T}}$ are given by the associated (normalised) eigenvectors:

For $\lambda = 6$ (the maximum),

$$-x + y + z = 0,\ x - 2y = 0,\ x - 2z = 0 \Rightarrow \mathsf{x}_{\max} = \pm(6)^{-1/2}\,(2,\ 1,\ 1)^{\mathrm{T}}.$$

For $\lambda = 3$ (the minimum),

$$2x + y + z = 0, \; x + y = 0, \; x + z = 0 \Rightarrow \mathsf{x}_{\min} = \pm(3)^{-1/2}\,(1,\; -1,\; -1)^{\mathrm{T}}.$$

8.24 *Find the lengths of the semi-axes of the ellipse*

$$73x^2 + 72xy + 52y^2 = 100,$$

and determine its orientation.

This is a quadric surface with no z-dependence (an elliptical cylinder) and if its semi-axes are a and b then a^{-2} and b^{-2} are given by the eigenvalues of the associated matrix after the RHS has been made unity. The eigenvalues therefore satisfy

$$\begin{aligned} 0 &= \begin{vmatrix} 0.73 - \lambda & 0.36 \\ 0.36 & 0.52 - \lambda \end{vmatrix} \\ &= 0.3796 - 1.25\lambda + \lambda^2 - 0.1296 \\ &= \lambda^2 - 1.25\lambda + 0.25 \\ &= (\lambda - 1)(\lambda - 0.25). \end{aligned}$$

Thus $\lambda = a^{-2} = 1$, giving $a = 1$, and $\lambda = b^{-2} = 0.25$ yielding $b = 2$.

The eigenvector $(x,\; y)^{\mathrm{T}}$ corresponding to the major semi-axis ($b = 2$) has $(0.73 - 0.25)x + 0.36y = 0$, i.e. makes an angle $\tan^{-1}(-4/3)$ with the x-axis.

8.26 *Show that the quadratic surface*

$$5x^2 + 11y^2 + 5z^2 - 10yz + 2xz - 10xy = 4$$

is an ellipsoid with semi-axes of lengths 2, 1 *and* 0.5. *Find the direction of its longest axis.*

As previously, we need to solve the characteristic equation of the matrix associated with the quadric:

$$\begin{aligned} 0 &= \begin{vmatrix} 5 - \lambda & -5 & 1 \\ -5 & 11 - \lambda & -5 \\ 1 & -5 & 5 - \lambda \end{vmatrix} \\ &= (5 - \lambda)(55 - 16\lambda + \lambda^2 - 25) - 5(-5 + 25 - 5\lambda) + 1(25 - 11 + \lambda) \\ &= -\lambda^3 + 21\lambda^2 - 85\lambda + 64. \end{aligned}$$

Clearly, $\lambda = 1$ is one root of this equation, which can be written

$$-(\lambda - 1)(\lambda^2 - 20\lambda + 64) = -(\lambda - 1)(\lambda - 4)(\lambda - 16) = 0.$$

The eigenvalues are all positive and so the quadratic surface is an ellipsoid with semi-axes $(1/4)^{-1/2}$, $(4/4)^{-1/2}$ and $(16/4)^{-1/2}$, i.e. 2, 1 and 0.5. The longest axis corresponds to the smallest eigenvalue, $\lambda = 1$, and its direction $(x,\ y,\ z)^{\mathrm{T}}$ satisfies

$$(5-1)x - 5y + z = 0,$$
$$-5x + (11-1)y - 5z = 0.$$

The unit vector in this direction is $(3)^{-1/2}\,(1,\ 1,\ 1)^{\mathrm{T}}$.

8.28 *Find the eigenvalues, and sufficient of the eigenvectors, of the following matrices to be able to describe the quadratic surfaces associated with them.*

$$\text{(a)}\ \begin{pmatrix} 5 & 1 & -1 \\ 1 & 5 & 1 \\ -1 & 1 & 5 \end{pmatrix},\quad \text{(b)}\ \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},\quad \text{(c)}\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

In each case the eigenvalues and then the eigenvectors of the matrices can be found by the methods employed in the previous four exercises and the details will not be given here. The results and their interpretations are:

(a) The eigenvalues are 6, 6 and 3. Since they are all positive and two are equal, the surface is an ellipsoid with a circular cross-section perpendicular to the direction $(1, -1, 1)/\sqrt{3}$, which is the eigenvector corresponding to eigenvalue 3. If the maximimum radius of the circular cross-section is r, say, (when the section includes the origin) then the semi-axis of the ellipsoid in the direction of the axis of symmetery is of length $\sqrt{2}r$.

(b) The eigenvalues are 5, -1 and -1. Since two are negative and equal, the surface is a hyperboloid of revolution about an axis in the direction $(1, 1, 1)/\sqrt{3}$ (the direction of the eigenvector corresponding to the non-repeated eigenvalue). In tranformed coordinates the equation of the surface will take the form

$$\frac{y_1^2}{(1/\sqrt{5})^2} - \frac{y_2^2}{1^2} - \frac{y_3^2}{1^2} = a$$

from this it can be seen that the two halves of the hyperboloid are asymptotic to a cone of semi-angle $\tan^{-1}\sqrt{5}$ that passes through the origin and has the same symmetry axis as the hyperboloid.

(c) The eigenvalues are 6, 0 and 0. A zero eigenvalue formally implies an infinitely long semi-axis; in other words, the surface is a cylinder (not necessarily circular) with the correponding eigenvector as the cylinder's axis. Here, there are two such eigenvalues and we have 'infinite cylinders in two directions', i.e the notional ellipsoid has degenerated into a pair of parallel planes. They are equidistant from the origin and have their normals in the direction of the eigenvector, $(1, 2, 1)$, corresponding to the only non-zero eigenvalue. The equation of the 'surface' in transformed coordinates becomes simply

$$\frac{y_1^2}{(1/\sqrt{6})^2} + 0y_2^2 + 0y_3^2 = a,$$

which describes the two planes $y_1 = \pm\sqrt{a/6}$.

_8.30_ *Find an orthogonal transformation that takes the quadratic form*

$$Q \equiv -x_1^2 - 2x_2^2 - x_3^2 + 8x_2x_3 + 6x_1x_3 + 8x_1x_2$$

into the form

$$\mu_1 y_1^2 + \mu_2 y_2^2 - 4y_3^2,$$

and determine μ_1 and μ_2 (see section 8.17).

Expressing Q as $\mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x}$, the required transformation has the normalised eigenvectors e^i of A as its columns. So, we need to determine the eigenvalues and eigenvectors of A. Following the normal method:

$$\begin{aligned} 0 &= \begin{vmatrix} -1-\lambda & 4 & 3 \\ 4 & -2-\lambda & 4 \\ 3 & 4 & -1-\lambda \end{vmatrix} \\ &= -(1+\lambda)(\lambda^2+3\lambda-14) + 4(16+4\lambda) + 3(22+3\lambda) \\ &= -\lambda^3 - 4\lambda^2 + 36\lambda + 144 \\ &= -(\lambda+4)(\lambda^2-36) \\ &= -(\lambda+4)(\lambda-6)(\lambda+6). \end{aligned}$$

We were guided by the given answer when writing $\lambda + 4$ as a factor of the characteristic polynomial. The values of μ_1 and μ_2 are determined as 6 and -6.

We now need to find the three *normalised* eigenvectors $(x,\ y,\ z)^{\mathrm{T}}$.
For $\lambda = -4$:

$$3x + 4y + 3z = 0, \quad 4x + 2y + 4z = 0, \quad \Rightarrow \mathsf{e}^3 = \frac{1}{\sqrt{2}}(1,\ 0,\ -1)^{\mathrm{T}}.$$

For $\lambda = 6$:

$$-7x + 4y + 3z = 0, \quad 4x - 8y + 4z = 0, \quad \Rightarrow \mathsf{e}^1 = \frac{1}{\sqrt{3}}(1,\ 1,\ 1)^{\mathrm{T}}.$$

For $\lambda = -6$:

$$5x + 4y + 3z = 0, \quad 4x + 4y + 4z = 0, \quad \Rightarrow \mathsf{e}^2 = \frac{1}{\sqrt{6}}\,(1,\ -2,\ 1)^{\mathrm{T}}.$$

Thus, the required new coordinates are:

$$y_1 = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3),\ y_2 = \frac{1}{\sqrt{6}}(x_1 - 2x_2 + x_3),\ y_3 = \frac{1}{\sqrt{2}}(x_1 - x_3).$$

The labelling of the y_i is, of course, arbitrary.

8.32 *Do the following sets of equations have non-zero solutions? If so, find them.*

(a) $3x + 2y + z = 0, \qquad x - 3y + 2z = 0, \qquad 2x + y + 3z = 0.$
(b) $2x = b(y + z), \qquad x = 2a(y - z), \qquad x = (6a - b)y - (6a + b)z.$

(a) For the equations, written in the form

$$\mathsf{Ax} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

to have a non-zero solution we must have $|\,\mathsf{A}\,| = 0$. But

$$|\,\mathsf{A}\,| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 3(-11) + 2(1) + 1(7) = -24 \neq 0,$$

and so the equations have no non-trivial solutions.

(b) Rearranged in standard form, the equations read

$$\mathsf{Ax} = \begin{pmatrix} 2 & -b & -b \\ 1 & -2a & 2a \\ 1 & b - 6a & 6a + b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Either by direct calculation or by observing that the sum of the first and third rows is equal to three time the second row, we conclude that $|\,\mathsf{A}\,| = 0$ and that a non-trivial solution is possible.

Arbitrarily taking $x = 1$, we require that

$$\begin{aligned} 2 - by - bz &= 0, \\ 1 - 2ay + 2az &= 0, \\ 4a + b - 2aby - 2aby &= 0, \\ \Rightarrow \quad y &= \frac{4a+b}{4ab}, \\ \Rightarrow \quad z &= \frac{2}{b} - y = \frac{4a-b}{4ab}. \end{aligned}$$

Thus the solution is any multiple of $(4ab,\ 4a+b,\ 4a-b)^{\mathrm{T}}$.

8.34 *Solve the following simultaneous equations for x_1, x_2 and x_3, using matrix methods:*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1, \\ 3x_1 + 4x_2 + 5x_3 &= 2, \\ x_1 + 3x_2 + 4x_3 &= 3. \end{aligned}$$

We need to invert the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 3 & 4 \end{pmatrix},$$

whose determinant is $1(1) + 2(-7) + 3(5) = 2$. This is non-zero and so A has an inverse. The matrix of cofactors is

$$\mathsf{C} = \begin{pmatrix} 1 & -7 & 5 \\ 1 & 1 & -1 \\ -2 & 4 & -2 \end{pmatrix},$$

from which it follows that $\mathsf{A}^{-1} = (1/|\mathsf{A}|)\mathsf{C}^{\mathrm{T}}$ is given by

$$\mathsf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -2 \\ -7 & 1 & 4 \\ 5 & -1 & -2 \end{pmatrix}.$$

Finally, rewriting the given equation $\mathsf{Ax} = \mathsf{y}$ as $\mathsf{x} = \mathsf{A}^{-1}\mathsf{y}$, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -2 \\ -7 & 1 & 4 \\ 5 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 \\ 7 \\ -3 \end{pmatrix}.$$

Thus $x_1 = -\frac{3}{2}$, $x_2 = \frac{7}{2}$ and $x_3 = -\frac{3}{2}$.

8.36 *Find the condition(s) on α such that the simultaneous equations*

$$x_1 + \alpha x_2 = 1,$$
$$x_1 - x_2 + 3x_3 = -1,$$
$$2x_1 - 2x_2 + \alpha x_3 = -2$$

have (a) *exactly one solution,* (b) *no solutions, or* (c) *an infinite number of solutions; give all solutions where they exist.*

As usual, and in the normal notation, we start by examining

$$|\,\mathsf{A}\,| = 1(6-\alpha) + \alpha(6-\alpha) = (1+\alpha)(6-\alpha).$$

(a) For exactly one solution we need $|\mathsf{A}| \neq 0$, i.e. $\alpha \neq -1$ and $\alpha \neq 6$. Then

$$\mathsf{A}^{-1} = \frac{1}{(1+\alpha)(6-\alpha)} \begin{pmatrix} 6-\alpha & -\alpha^2 & 3\alpha \\ 6-\alpha & \alpha & -3 \\ 0 & 2+2\alpha & -1-\alpha \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \frac{1}{(1+\alpha)(6-\alpha)} \begin{pmatrix} 6-\alpha & -\alpha^2 & 3\alpha \\ 6-\alpha & \alpha & -3 \\ 0 & 2+2\alpha & -1-\alpha \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \\ &= \frac{1}{(1+\alpha)(6-\alpha)} \begin{pmatrix} 6-\alpha+\alpha^2-6\alpha \\ 6-\alpha-\alpha+6 \\ -2-2\alpha+2+2\alpha \end{pmatrix} \\ &= \left(\frac{1-\alpha}{1+\alpha}, \frac{2}{1+\alpha}, 0 \right)^{\mathrm{T}}. \end{aligned}$$

(b) and (c). For no or infinitely many solutions the matrix must have rank 2 or less, which requires that either $\alpha = -1$ or $\alpha = 6$.

With $\alpha = -1$ the equations become

$$x_1 - x_2 = 1,$$
$$x_1 - x_2 + 3x_3 = -1,$$
$$2x_1 - 2x_2 - x_3 = -2.$$

Substituting from the first equation for $x_1 - x_2$ leaves two equations for x_3 which are contradictory; this is case (b) of no solution.

With $\alpha = 6$ the equations become

$$x_1 + 6x_2 = 1,$$
$$x_1 - x_2 + 3x_3 = -1,$$
$$2x_1 - 2x_2 + 6x_3 = -2.$$

The last two equations are multiples of each other, but not of the first. Therefore there are infinitely many solutions containing one free parameter. Taking this as $x_2 = \beta$, the general solution is

$$(x_1,\ x_2,\ x_3)^{\mathrm{T}} = \left(1 - 6\beta,\ \beta,\ \tfrac{1}{3}(7\beta - 2)\right)^{\mathrm{T}}.$$

This solution is that for case (c) and corresponds to figure 8.1(a) in the main text; case (b) corresponds to figure 8.1(b).

8.38 *Make an LU decomposition of the matrix*

$$\mathsf{A} = \begin{pmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{pmatrix}.$$

Hence solve $\mathsf{Ax} = \mathsf{b}$ *for* (i) $\mathsf{b} = (-4 \quad 1 \quad 8 \quad -5)^{\mathrm{T}}$, *and* (ii) $\mathsf{b} = (-10 \quad 0 \quad -3 \quad -24)^{\mathrm{T}}$. *Deduce that* $\det \mathsf{A} = -160$ *and confirm this by direct calculation.*

To avoid a lot of subscripts we will use single lower-case letters as the elements of the upper- and lower-diagonal matrices. We also make the immediately-apparent entries in U.
We need

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & 1 & 3 \\ 0 & g & h & j \\ 0 & 0 & k & l \\ 0 & 0 & 0 & m \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{pmatrix}.$$

From the 1st column of A:

$$a = \tfrac{1}{2}, \quad b = \tfrac{5}{2} \quad \text{and} \quad d = \tfrac{3}{2}.$$

Then, from the 2nd row:

$$-3a + g = 4 \Rightarrow g = \tfrac{11}{2},$$
$$a + h = -3 \Rightarrow h = -\tfrac{7}{2},$$
$$3a + j = -3 \Rightarrow j = -\tfrac{9}{2}.$$

From the 2nd column:

$$-3b + gc = 3 \Rightarrow c = \tfrac{21}{11},$$
$$-3d + ge = -6 \Rightarrow e = -\tfrac{3}{11}.$$

From the 3rd row:

$$b + ch + k = -1 \Rightarrow k = -1 + \tfrac{92}{22} = \tfrac{35}{11},$$
$$3b + cj + l = -1 \Rightarrow l = -1 + \tfrac{24}{22} = \tfrac{1}{11}.$$

From the 3rd column:

$$d + he + fk = -3 \Rightarrow f = (\tfrac{-120}{20})/(\tfrac{35}{11}) = -\tfrac{12}{7}.$$

Finally, from the 4th row:

$$3d + je + fl + m = 1 \Rightarrow m = 1 - \tfrac{9}{2} - \tfrac{27}{22} + \tfrac{12}{77} = -\tfrac{32}{7}.$$

(i) We first solve $\mathsf{Ly} = \mathsf{b}$ as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{5}{2} & \frac{21}{11} & 1 & 0 \\ \frac{3}{2} & -\frac{3}{11} & -\frac{12}{7} & 1 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 8 \\ -5 \end{bmatrix}.$$

That $y_1 = -4$ and $y_2 = 3$ are immediately apparent. The third row gives $y_3 = 8 + 10 - \frac{63}{11} = \frac{135}{11}$, whilst the fourth yields $y_4 = -5 + 6 + \frac{9}{11} + \frac{12}{7}\frac{135}{11} = \frac{160}{7}$. The solution vector x is now deduced from $\mathsf{Ux} = \mathsf{y}$:

$$\begin{pmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{25}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ \frac{135}{11} \\ \frac{160}{7} \end{bmatrix}.$$

That $x_4 = -5$ is obvious. The third row gives $x_3 = \frac{11}{35}(\frac{135}{11} + \frac{5}{11}) = 4$, whilst the second yields $x_2 = \frac{2}{11}(3 + 14 - \frac{45}{2}) = -1$. Finally, the top row gives $x_1 = \frac{1}{2}(-4 - 3 - 4 + 15) = 2$.

(ii) This calculation proceeds just as in (i). The intermediate vector y is found to be $(-10 \quad 5 \quad \frac{137}{11} \quad \frac{96}{7})^{\mathrm{T}}$ and the solution vector $\mathsf{x} = (-1 \quad 1 \quad 4 \quad -3)^{\mathrm{T}}$.

The determinant of A is given by the product of the diagonal entries of the matrix U, i.e. $|\mathsf{A}| = 2 \times \frac{11}{2} \times \frac{35}{11} \times (-\frac{32}{7}) = -160$.

Confirming this by direct calculation:

$$\begin{vmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 7 & -5 & -3 & 6 \\ 7 & 0 & -1 & 2 \\ 9 & -15 & -3 & 10 \end{pmatrix}$$

$$= 1 \begin{vmatrix} 7 & -5 & 6 \\ 7 & 0 & 2 \\ 9 & -15 & 10 \end{vmatrix}$$

$$= 7(30) - 5(-52) + 6(-105) = -160.$$

At the first step, an appropriate multiple of the 3rd column was subtracted from each of the other columns.

> ***8.40*** *Find the equation satisfied by the squares of the singular values of the matrix associated with the following over-determined set of equations:*
>
> $$2x + 3y + z = 0$$
> $$x - y - z = 1$$
> $$2x + y = 0$$
> $$2y + z = -2.$$
>
> *Show that one of the singular values is close to zero. Determine the two larger singular values by an appropriate iteration process and the smallest by indirect calculation.*

The matrix and its (Hermitian) transpose associated with the set of equations are

$$\mathsf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad \mathsf{A}^\dagger = \begin{pmatrix} 2 & 1 & 2 & 0 \\ 3 & -1 & 1 & 2 \\ 1 & -1 & 0 & 1 \end{pmatrix},$$

and their 3×3 product is

$$\mathsf{A}^\dagger\mathsf{A} = \begin{pmatrix} 9 & 7 & 1 \\ 7 & 15 & 6 \\ 1 & 6 & 3 \end{pmatrix}.$$

To determine the singular values of A we find the eigenvalues of this product:

$$
\begin{aligned}
0 &= \begin{vmatrix} 9-\lambda & 7 & 1 \\ 7 & 15-\lambda & 6 \\ 1 & 6 & 3-\lambda \end{vmatrix} \\
&= (9-\lambda)(9-18\lambda+\lambda^2) + 7(7\lambda-15) + 1(\lambda+27) \\
&= -(\lambda^3 - 27\lambda^2 + 121\lambda - 3).
\end{aligned}
$$

This is the equation satisfied by the squares of the singular values of A.

Using the properties of the three roots λ_i of the cubic equation, we conclude that, since their sum $\sum_i \lambda_i = 27$ whilst their product is only $\prod_i \lambda_i = 3$, at least one of the roots must be close to zero.

Using either the rearrangement iteration method,

$$x_{n+1} = \left(27x_n^2 - 121x_n + 3\right)^{1/3},$$

or the Newton-Raphson method,

$$x_{n+1} = x_n - \frac{x_n^3 - 27x_n^2 + 121x_n - 3}{3x_n^2 - 54x_n + 121},$$

the two larger roots are found to be 21.33521 and 5.639852. The third root can be found most accurately as $3/(21.33521 \times 5.639852) = 0.024938$.

The corresponding singular values are the square roots of these eigenvalues, namely, 4.6190, 2.3748 and 0.1579.

8.42 *Find the SVD form of the matrix*

$$\mathsf{A} = \begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}.$$

Use it to determine the best solution x *of the equation* $\mathsf{Ax} = \mathsf{b}$ *when* (i) $\mathsf{b} = (6 \quad -39 \quad 15 \quad 18)^{\mathrm{T}}$, (ii) $\mathsf{b} = (9 \quad -42 \quad 15 \quad 15)^{\mathrm{T}}$, *showing that* (i) *has an exact solution, but that the best solution to* (ii) *has a residual of* $\sqrt{18}$.

We start by computing

$$\mathsf{A}^\dagger\mathsf{A} = \begin{pmatrix} 22 & 1 & 19 & -6 \\ 28 & -2 & -2 & 12 \\ -22 & -19 & -1 & 6 \end{pmatrix} \begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 882 & 504 & -558 \\ 504 & 936 & -504 \\ -558 & -504 & 882 \end{pmatrix}.$$

And then find its eigenvalues:

$$|\mathsf{A}^\dagger\mathsf{A} - \lambda| = \begin{vmatrix} 882-\lambda & 504 & -558 \\ 504 & 936-\lambda & -504 \\ -558 & -504 & 882-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 324-\lambda & 0 & 324-\lambda \\ 504 & 936-\lambda & -504 \\ -558 & -504 & 882-\lambda \end{vmatrix}$$

$$= (324-\lambda)\left(\begin{vmatrix} 936-\lambda & -504 \\ -504 & 882-\lambda \end{vmatrix} + \begin{vmatrix} 504 & 936-\lambda \\ -558 & -504 \end{vmatrix} \right)$$

$$= (324-\lambda)(\lambda^2 - 1818\lambda + 571536 - 558\lambda + 268272)$$

$$= (324-\lambda)(\lambda^2 - 2376\lambda + 839808)$$

$$= (324-\lambda)(\lambda - 432)(\lambda - 1944).$$

This shows that the singular values ($\sqrt{\lambda}$) are $18\sqrt{6}$, $12\sqrt{3}$ and 18. We have, as usual, taken the singular values to be positive; this choice is reflected in the signs of the terms in the matrix U calculated later.

The corresponding normalised eigenvectors $(x_1, x_2, x_3)^{\mathrm{T}}$ satisfy:

$$-1062x_1 + 504x_2 - 558x_3 = 0,$$
$$504x_1 - 1008x_2 - 504x_3 = 0. \Rightarrow \mathsf{v}^1 = \frac{1}{\sqrt{3}}(1,\ 1,\ -1)^{\mathrm{T}}.$$
$$450x_1 + 504x_2 - 558x_3 = 0,$$
$$504x_1 + 504x_2 - 504x_3 = 0. \Rightarrow \mathsf{v}^2 = \frac{1}{\sqrt{6}}(1,\ -2,\ -1)^{\mathrm{T}}.$$
$$558x_1 + 504x_2 - 558x_3 = 0,$$
$$504x_1 + 504x_2 - 504x_3 = 0. \Rightarrow \mathsf{v}^3 = \frac{1}{\sqrt{2}}(1,\ 0,\ 1)^{\mathrm{T}}.$$

The next step is to calculate the (normalised) column vectors u^i from $(s_i)^{-1}\mathsf{A}\mathsf{v}^i =$

u^i:

$$\mathsf{u}^1 = \frac{1}{18\sqrt{6}}\frac{1}{\sqrt{3}}\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\mathsf{u}^2 = \frac{1}{12\sqrt{3}}\frac{1}{\sqrt{6}}\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ 2 \\ 2 \\ -3 \end{bmatrix}.$$

$$\mathsf{u}^3 = \frac{1}{18}\frac{1}{\sqrt{2}}\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Although we will not need its components for the present exercise, we now find the fourth base vector (to make U a unitary matrix). It has to be orthogonal to the three vectors just found; simple simultaneous equations show that, when normalised, it is $\mathsf{u}^4 = (1/\sqrt{18})(-1 \quad 2 \quad 2 \quad 3)^{\mathrm{T}}$.

Thus, finally, we are able to write $\mathsf{A} = \mathsf{USV}^\dagger$ explicitly as

$$\frac{1}{N}\begin{pmatrix} 4 & -1 & 0 & -1 \\ 1 & 2 & -3 & 2 \\ 1 & 2 & 3 & 2 \\ 0 & -3 & 0 & 3 \end{pmatrix}\begin{pmatrix} 18\sqrt{6} & 0 & 0 \\ 0 & 12\sqrt{3} & 0 \\ 0 & 0 & 18 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -2 & -1 \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix},$$

where $N = \sqrt{18} \times \sqrt{6}$.

The best solution to $\mathsf{Ax} = \mathsf{b}$ is given by $\mathsf{x} = \mathsf{V}\bar{\mathsf{S}}\mathsf{U}^\dagger\mathsf{b}$. We therefore compute $\mathsf{R} = \mathsf{V}\bar{\mathsf{S}}\mathsf{U}^\dagger$ as (with N defined as previously)

$$\frac{1}{N}\begin{pmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ -\sqrt{2} & -1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} \frac{1}{18\sqrt{6}} & 0 & 0 & 0 \\ 0 & \frac{1}{12\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{18} & 0 \end{pmatrix}\begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 2 & -3 \\ 0 & -3 & 3 & 0 \\ -1 & 2 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{N}\begin{pmatrix} \frac{1}{18\sqrt{3}} & \frac{1}{12\sqrt{3}} & \frac{\sqrt{3}}{18} & 0 \\ \frac{1}{18\sqrt{3}} & -\frac{1}{6\sqrt{3}} & 0 & 0 \\ -\frac{1}{18\sqrt{3}} & -\frac{1}{12\sqrt{3}} & \frac{\sqrt{3}}{18} & 0 \end{pmatrix}\begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 2 & -3 \\ 0 & -3 & 3 & 0 \\ -1 & 2 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{108}}\frac{1}{36\sqrt{3}}\begin{pmatrix} 5 & -10 & 26 & -9 \\ 14 & -10 & -10 & 18 \\ -5 & -26 & 10 & 9 \end{pmatrix}.$$

(i) With $\mathsf{b} = (6 \quad -39 \quad 15 \quad 18)^{\mathrm{T}}$ the best solution is

$$\mathsf{x} = \frac{1}{\sqrt{108}}\frac{1}{36\sqrt{3}}\begin{pmatrix} 5 & -10 & 26 & -9 \\ 14 & -10 & -10 & 18 \\ -5 & -26 & 10 & 9 \end{pmatrix}\begin{bmatrix} 6 \\ -39 \\ 15 \\ 18 \end{bmatrix}$$

$$x_1 = \tfrac{1}{648}(30 + 390 + 390 - 162) = 1,$$
$$x_2 = \tfrac{1}{648}(84 + 390 - 150 + 324) = 1,$$
$$x_3 = \tfrac{1}{648}(-30 + 1014 + 150 + 162) = 2.$$

Thus, the best solution is $(1,\ 1,\ 2)^{\mathrm{T}}$ and the residual vector given by

$$\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{bmatrix} 6 \\ -39 \\ 15 \\ 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The residual vector is the zero vector and the best solution is an exact solution.

(ii) With $\mathsf{b} = (9 \quad -42 \quad 15 \quad 15)^{\mathrm{T}}$ the best solution is

$$\mathsf{x} = \frac{1}{\sqrt{108}}\frac{1}{36\sqrt{3}}\begin{pmatrix} 5 & -10 & 26 & -9 \\ 14 & -10 & -10 & 18 \\ -5 & -26 & 10 & 9 \end{pmatrix}\begin{bmatrix} 9 \\ -42 \\ 15 \\ 15 \end{bmatrix} = \frac{1}{36}\begin{pmatrix} 40 \\ 37 \\ 74 \end{pmatrix}.$$

With $\frac{1}{36}(40,\ 37,\ 74)^{\mathrm{T}}$ as the best solution, the residual vector is

$$\frac{1}{36}\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 40 \\ 37 \\ 74 \end{pmatrix} - \begin{bmatrix} 9 \\ -42 \\ 15 \\ 15 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

We conclude that the solution is not exact and that the residual (equal to the modulus of the residual vector) is $[\,(-1)^2 + 2^2 + 2^2 + 3^2\,]^{1/2} = \sqrt{18}$.

9

Normal modes

9.2 *A double pendulum, smoothly pivoted at A, consists of two light rigid rods, AB and BC, each of length l, which are smoothly jointed at B and carry masses m and αm at B and C respectively. The pendulum makes small oscillations in one plane under gravity; at time t, AB and BC make angles θ(t) and φ(t) respectively with the downward vertical. Find quadratic expressions for the kinetic and potential energies of the system and hence show that the normal modes have angular frequencies given by*

$$\omega^2 = \frac{g}{l}\left[1 + \alpha \pm \sqrt{\alpha(1+\alpha)}\right].$$

For α = 1/3, show that in one of the normal modes the mid-point of BC does not move during the motion.

For small oscillations, the sideways displacements and consequent velocities of the masses are

$$x_1 = l\theta \quad \text{and} \quad \dot{x}_1 = l\dot{\theta}$$
$$x_2 = l\theta + l\phi \quad \text{and} \quad \dot{x}_2 = l\dot{\theta} + l\dot{\phi}$$

To first order in small quantities (i.e. ignoring any vertical components of velocity) the total kinetic energy of the system is therefore

$$\begin{aligned}\text{KE} &= \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}\alpha m\dot{x}_2^2 \\ &= \tfrac{1}{2}ml^2[\dot{\theta}^2 + \alpha(\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2)],\end{aligned}$$

and the kinetic energy matrix

$$\mathsf{T} = \tfrac{1}{2}ml^2\begin{pmatrix} 1+\alpha & \alpha \\ \alpha & \alpha \end{pmatrix}.$$

Remembering that the raising of the lower mass receives a contribution from that of the upper mass and working to second order in the displacements, the potential energy is

$$\begin{aligned}\text{PE} &= mgl(1-\cos\theta) + \alpha mgl[(1-\cos\theta) + (1-\cos\phi)] \\ &\approx \tfrac{1}{2}mgl\theta^2 + \tfrac{1}{2}\alpha mgl(\theta^2+\phi^2).\end{aligned}$$

The potential energy matrix is therefore

$$\mathsf{V} = \tfrac{1}{2}mgl\begin{pmatrix} 1+\alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

The normal frequencies, determined by $|-\omega^2\mathsf{T} + \mathsf{V}| = 0$, are given by

$$\tfrac{1}{2}m\left| -l^2\omega^2\begin{pmatrix} 1+\alpha & \alpha \\ \alpha & \alpha \end{pmatrix} + gl\begin{pmatrix} 1+\alpha & 0 \\ 0 & \alpha \end{pmatrix}\right| = 0.$$

Writing $\omega^2 l/g$ as λ, this requirement is

$$\begin{vmatrix} (1+\alpha) - \lambda(1+\alpha) & -\lambda\alpha \\ -\lambda\alpha & \alpha - \lambda\alpha \end{vmatrix} = 0,$$

i.e. $\quad (1+\alpha)(1-\lambda)\alpha(1-\lambda) - \lambda^2\alpha^2 = \lambda^2 - 2(1+\alpha)\lambda + (1+\alpha) = 0$

The angular frequencies of the two normal modes are given by the roots of this quadratic equation as

$$\omega^2 = \frac{g\lambda}{l} = \frac{g}{l}[(1+\alpha) \pm \sqrt{\alpha(1+\alpha)}]. \qquad (*)$$

For $\alpha = \frac{1}{3}$ the two values of ω^2 become $\dfrac{2g}{l}$ and $\dfrac{2g}{3l}$, and the components of the solution vector must satisfy (using the second line of the matrix-vector equation)

$$\begin{aligned} -2\frac{1}{3}\theta + \left(\frac{1}{3} - \frac{2}{3}\right)\phi = 0 \Rightarrow \phi = -2\theta \quad \text{when} \quad \lambda = 2, \\ -\frac{2}{3}\frac{1}{3}\theta + \left(\frac{1}{3} - \frac{2}{9}\right)\phi = 0 \Rightarrow \phi = 2\theta \quad \text{when} \quad \lambda = \frac{2}{3}. \end{aligned}$$

For the mid-point of BC, $x = l\theta + \frac{1}{2}l\phi$. In the higher frequency mode, $\phi = -2\theta$ and $\dot{x} = l\dot{\theta} + \frac{1}{2}l(-2\dot{\theta}) = 0$, i.e. the mid-point does not move.

Note It is of some interest to check that (∗) gives the correct limits for small and large α. It obviously leads (correctly) to $\omega^2 = g/l$ (repeated) as $\alpha \to 0$. For $\alpha \to \infty$ one solution has an (unphysical) infinite frequency; the other has

$$\begin{aligned}\frac{l\omega^2}{g} &= \lim_{\alpha\to\infty} 1 + \alpha - \sqrt{\alpha(1+\alpha)} = \lim_{\alpha\to\infty} 1 + \alpha - \alpha\left(1 + \frac{1}{\alpha}\right)^{1/2} \\ &= \lim_{\alpha\to\infty} 1 + \alpha - \alpha\left(1 + \frac{1}{2\alpha} + \cdots\right) = \frac{1}{2},\end{aligned}$$

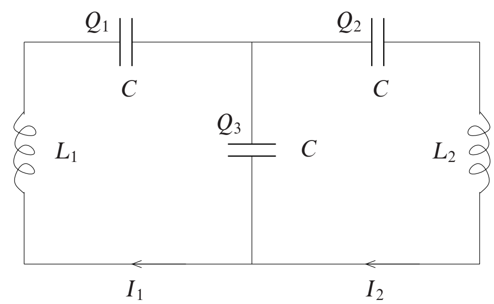

Figure 9.1 The circuit and notation for example 9.4.

i.e. the correct value for a simple pendulum of length $2l$.

9.4 *Consider the circuit consisting of three equal capacitors and two different inductors shown in figure 9.1. For charges Q_i on the capacitors and currents I_i through the components, write down Kirchhoff's law for the total voltage change around each of two complete circuit loops. Note that, to within an unimportant constant, the conservation of current implies that $Q_3 = Q_1 - Q_2$ and hence express the loop equations in the form given in (9.7), namely*

$$\mathsf{A}\ddot{\mathsf{Q}} + \mathsf{B}\mathsf{Q} = 0.$$

Use this to show that the normal frequencies of the circuit are given by

$$\omega^2 = \frac{1}{CL_1L_2}\left[L_1 + L_2 \pm (L_1^2 + L_2^2 - L_1L_2)^{1/2}\right].$$

Obtain the same matrices and result by finding the total energy stored in the various capacitors (typically $Q^2/(2C)$) and in the inductors (typically $LI^2/2$).

For the special case $L_1 = L_2 = L$ determine the relevant eigenvectors and so describe the patterns of current flow in the circuit.

We apply Kirchhoff's law to a loop taken round the left-hand part of the circuit and to one taken round the whole circuit (one round the right-hand part does not give any further independent information as there are only two currents needed to specify the situtation).

$$L_1\dot{I}_1 + \frac{Q_1}{C} + \frac{Q_3}{C} = 0,$$
$$L_1\dot{I}_1 + \frac{Q_1}{C} + \frac{Q_2}{C} + L_2\dot{I}_2 = 0,$$

with $I_1 = \dot{Q}_1$, $I_2 = \dot{Q}_2$ and $I_1 - I_2 = \dot{Q}_3 = \dot{Q}_1 - \dot{Q}_2$.

Writing everything in terms of Q_1, Q_2 and their time derivatives,

$$L_1\ddot{Q}_1 + \frac{Q_1}{C} + \frac{Q_1 - Q_2}{C} = 0,$$
$$L_1\ddot{Q}_1 + \frac{Q_1}{C} + \frac{Q_2}{C} + L_2\ddot{Q}_2 = 0.$$

In matrix and vector form, $\mathsf{A}\ddot{\mathsf{Q}} + \mathsf{B}\mathsf{Q} = \mathsf{0}$, these equations read

$$\begin{pmatrix} L_1 & 0 \\ L_1 & L_2 \end{pmatrix}\begin{pmatrix} \ddot{Q}_1 \\ \ddot{Q}_2 \end{pmatrix} + \begin{pmatrix} 2C^{-1} & -C^{-1} \\ C^{-1} & C^{-1} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To find the normal frequencies, we now have to solve $|\mathsf{B} - \omega^2\mathsf{A}| = 0$. After mutiplying through by C, this reads

$$\begin{vmatrix} 2 - \omega^2 L_1 C & -1 \\ 1 - \omega^2 L_1 C & 1 - \omega^2 L_2 C \end{vmatrix} = 0,$$
$$2 - (L_1 + 2L_2)C\omega^2 + L_1 L_2 C^2 \omega^4 + 1 - L_1 C\omega^2 = 0,$$
$$L_1 L_2 C^2 \omega^4 - 2(L_1 + L_2)C\omega^2 + 3 = 0.$$

Hence the normal frequencies are

$$\omega^2 = \frac{(L_1 + L_2)C \pm \sqrt{(L_1 + L_2)^2 C^2 - 3L_1 L_2 C^2}}{L_1 L_2 C^2}$$
$$= \frac{1}{CL_1L_2}\left[L_1 + L_2 \pm (L_1^2 + L_2^2 - L_1 L_2)^{1/2}\right].$$

We now repeat this derivation, working in terms of stored energy, rather than the equations of motion.

The total 'kinetic energy' is the energy stored in the magnetic fields of the inductors (typically $\frac{1}{2}LI^2$). This is

$$T = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2.$$

The 'potential energy' term is the energy stored in the capacitors (typically $\frac{1}{2}C^{-1}Q^2$). This is

$$V = \tfrac{1}{2}C^{-1}[\,Q_1^2 + Q_2^2 + (Q_1 - Q_2)^2\,].$$

The charateristic equation determining (the squares of) the normal mode frequencies is therefore

$$\begin{vmatrix} 2C^{-1} - \omega^2 L_1 & -C^{-1} \\ -C^{-1} & 2C^{-1} - \omega^2 L_2 \end{vmatrix} = 0,$$
$$4C^{-2} - 2(L_1 + L_2)C^{-1}\omega^2 + L_1 L_2 \omega^4 - C^{-2} = 0.$$

After multiplication by C^2, this is the same equation as that obtained previously and has the same roots for ω^2.

If $L_1 = L_2 = L$ then one mode has $\omega^2 = (LC)^{-1}$ and the eigenvector is given by

$$(2-1)Q_1 - Q_2 = 0 \quad \Rightarrow \quad Q_1 = Q_2.$$

Under these circumstances $Q_3 = 0$ and no current flows through the central capacitor.

The other mode has $\omega^2 = 3(LC)^{-1}$; for this mode,

$$(2-3)Q_1 - Q_2 = 0 \quad \Rightarrow \quad Q_1 = -Q_2.$$

In this case, equal currents I (one clockwise, one anticlockwise) flow in the two loops and the current through the central capacitor is $2I$.

9.6 The simultaneous reduction to diagonal form of two real symmetric quadratic forms.

Consider the two real symmetric quadratic forms $\mathsf{u}^{\mathrm{T}}\mathsf{A}\mathsf{u}$ *and* $\mathsf{u}^{\mathrm{T}}\mathsf{B}\mathsf{u}$, *where* u^{T} *stands for the row matrix* $(x \quad y \quad z)$, *and denote by* u^n *those column matrices that satisfy*

$$\mathsf{B}\mathsf{u}^n = \lambda_n \mathsf{A}\mathsf{u}^n \qquad (*),$$

in which n *is a label and the* λ_n *are real, non-zero and all different.*

(a) *By multiplying* $(*)$ *on the left by* $(\mathsf{u}^m)^{\mathrm{T}}$ *and the transpose of the corresponding equation for* u^m *on the right by* u^n, *show that* $(\mathsf{u}^m)^{\mathrm{T}}\mathsf{A}\mathsf{u}^n = 0$ *for* $n \neq m$.
(b) *By noting that* $\mathsf{A}\mathsf{u}^n = (\lambda_n)^{-1}\mathsf{B}\mathsf{u}^n$, *deduce that* $(\mathsf{u}^m)^{\mathrm{T}}\mathsf{B}\mathsf{u}^n = 0$ *for* $m \neq n$.
(c) *It can be shown that the* u^n *are linearly independent; the next step is to construct a matrix* P *whose columns are the vectors* u^n.
(d) *Make a change of variables* $\mathsf{u} = \mathsf{P}\mathsf{v}$ *such that* $\mathsf{u}^{\mathrm{T}}\mathsf{A}\mathsf{u}$ *becomes* $\mathsf{v}^{\mathrm{T}}\mathsf{C}\mathsf{v}$, *and* $\mathsf{u}^{\mathrm{T}}\mathsf{B}\mathsf{u}$ *becomes* $\mathsf{v}^{\mathrm{T}}\mathsf{D}\mathsf{v}$. *Show that* C *and* D *are diagonal by showing that* $c_{ij} = 0$ *if* $i \neq j$ *and similarly for* d_{ij}.

Thus $\mathsf{u} = \mathsf{P}\mathsf{v}$ *or* $\mathsf{v} = \mathsf{P}^{-1}\mathsf{u}$ *reduces both quadratics to diagonal form.*

To summarise, the method is as follows:

(a) *find the* λ_n *that allow* $(*)$ *a non-zero solution, by solving* $|\mathsf{B} - \lambda\mathsf{A}| = 0$;
(b) *for each* λ_n *construct* u^n;
(c) *construct the non-singular matrix* P *whose columns are the vectors* u^n;
(d) *make the change of variable* $\mathsf{u} = \mathsf{P}\mathsf{v}$.

We are given that $\mathsf{A}^{\mathrm{T}} = \mathsf{A}$, $\mathsf{B}^{\mathrm{T}} = \mathsf{B}$ and $\lambda_m \neq \lambda_n$ if $m \neq n$.

(a) From (∗) and its transpose, with n replaced by m in the latter,

$$\begin{aligned} \mathsf{B}\mathsf{u}^n = \lambda_n \mathsf{A}\mathsf{u}^n \quad &\text{and } (\mathsf{u}^m)^\mathrm{T}\mathsf{B}^\mathrm{T} = \lambda_m (\mathsf{u}^m)^\mathrm{T}\mathsf{A}^\mathrm{T}, \\ (\mathsf{u}^m)^\mathrm{T}\mathsf{B}\mathsf{u}^n = \lambda_n (\mathsf{u}^m)^\mathrm{T}\mathsf{A}\mathsf{u}^n \quad &\text{and } (\mathsf{u}^m)^\mathrm{T}\mathsf{B}\mathsf{u}^n = \lambda_m (\mathsf{u}^m)^\mathrm{T}\mathsf{A}\mathsf{u}^n, \\ \Rightarrow \quad (\lambda_n - \lambda_m)(\mathsf{u}^m)^\mathrm{T}\mathsf{A}\mathsf{u}^n &= 0, \\ \Rightarrow \quad (\mathsf{u}^m)^\mathrm{T}\mathsf{A}\mathsf{u}^n &= 0 \quad \text{if} \quad m \neq n. \end{aligned}$$

(b) We next rearrange (∗) to read $\mathsf{A}\mathsf{u}^n = (\lambda_n)^{-1}\mathsf{B}\mathsf{u}^n$. This equation is of the same form as (∗) but with the roles of A and B interchanged. It therefore follows that, since $(\lambda_m)^{-1} \neq (\lambda_n)^{-1}$ for $m \neq n$, $(\mathsf{u}^m)^\mathrm{T}\mathsf{B}\mathsf{u}^n = 0$ if $m \neq n$.

(c) We now change variables from u to v where the two variables are connected by $\mathsf{u} = \mathsf{P}\mathsf{v}$; here P is the matrix whose columns are the u^n. Thus

$$\mathsf{P} = (\mathsf{u}^1 \quad \mathsf{u}^2 \quad \mathsf{u}^3), \quad \text{i.e.} \quad P_{ij} = (\mathsf{u}^j)_i \quad \text{and} \quad P^\mathrm{T}_{ij} = (\mathsf{u}^i)_j.$$

Then,

$$Q_1 = \mathsf{u}^\mathrm{T}\mathsf{A}\mathsf{u} = (\mathsf{P}\mathsf{v})^\mathrm{T}\mathsf{A}(\mathsf{P}\mathsf{v}) = \mathsf{v}^\mathrm{T}\mathsf{P}^\mathrm{T}\mathsf{A}\mathsf{P}\mathsf{v} = \mathsf{v}^\mathrm{T}\mathsf{C}\mathsf{v},$$

where the elements of C are c_{ij} given by

$$\begin{aligned} c_{ij} &= (\mathsf{P}^\mathrm{T}\mathsf{A}\mathsf{P})_{ij} \\ &= P^\mathrm{T}_{ik} A_{kl} P_{lj} \\ &= (\mathsf{u}^i)_k A_{kl} (\mathsf{u}^j)_l \\ &= (\mathsf{u}^i)^\mathrm{T}\mathsf{A}(\mathsf{u}^j) \\ &= 0 \quad \text{if} \quad i \neq j. \end{aligned}$$

Similarly,

$$Q_2 = \mathsf{u}^\mathrm{T}\mathsf{B}\mathsf{u} = \mathsf{v}^\mathrm{T}\mathsf{D}\mathsf{v},$$

with $d_{ij} = 0$ if $i \neq j$.

Thus, the transformation $\mathsf{u} = \mathsf{P}\mathsf{v}$ (or $\mathsf{v} = \mathsf{P}^{-1}\mathsf{u}$) reduces both Q_1 and Q_2 to diagonal form. We note that, in general, P is not an orthogonal matrix, even if the vectors u^n are normalised.

9.8 *(It is recommended that the reader does not attempt this question until exercise 9.6 has been studied.)*

Find a real linear transformation that simultaneously reduces the quadratic forms

$$3x^2 + 5y^2 + 5z^2 + 2yz + 6zx - 2xy,$$

$$5x^2 + 12y^2 + 8yz + 4zx$$

to diagonal form.

With the two quadratic forms

$$Q_1 = \mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x} = 3x^2 + 5y^2 + 5z^2 + 2yz + 6zx - 2xy,$$
$$Q_2 = \mathsf{x}^{\mathrm{T}}\mathsf{B}\mathsf{x} = 5x^2 + 12y^2 + 8yz + 4zx,$$

we must find the vectors that satisfy $\mathsf{B}\mathsf{u} = \lambda\mathsf{A}\mathsf{u}$. To do this, we evaluate

$$\begin{aligned}
&|\mathsf{B} - \lambda\mathsf{A}| \\
&= \begin{vmatrix} 5-3\lambda & \lambda & 2-3\lambda \\ \lambda & 12-5\lambda & 4-\lambda \\ 2-3\lambda & 4-\lambda & -5\lambda \end{vmatrix} \\
&= \begin{vmatrix} 5 & 36-14\lambda & 14-6\lambda \\ \lambda & 12-5\lambda & 4-\lambda \\ 2 & 40-16\lambda & 12-8\lambda \end{vmatrix} \\
&= \begin{vmatrix} 5 & -6+4\lambda & 14-6\lambda \\ \lambda & -2\lambda & 4-\lambda \\ 2 & 4+8\lambda & 12-8\lambda \end{vmatrix} \\
&= 5(24\lambda^2 - 52\lambda - 16) + \lambda(-16\lambda^2 - 8\lambda + 128) + 2(-16\lambda^2 + 50\lambda - 24) \\
&= -16\lambda^3 + 80\lambda^2 - 32\lambda - 128.
\end{aligned}$$

Setting this expression equal to zero gives the cubic equation satisfied by acceptable values of λ:

$$\begin{aligned}
\lambda^3 - 5\lambda^2 + 2\lambda + 8 &= 0, \\
(\lambda + 1)(\lambda - 2)(\lambda - 4) &= 0, \\
\Rightarrow \quad \lambda &= -1 \text{ or } 2 \text{ or } 4.
\end{aligned}$$

The three required vectors u^i must have components that satisfy:

For $\lambda = -1$

$$\begin{aligned}
8x - y + 5z &= 0, \\
-x + 17y + 5z &= 0, \quad \Rightarrow \quad \mathsf{u}^1 = (2,\ 1,\ -3)^{\mathrm{T}}.
\end{aligned}$$

For $\lambda = 2$

$$\begin{aligned}
-x + 2y - 4z &= 0, \\
2x + 2y + 2z &= 0, \quad \Rightarrow \quad \mathsf{u}^2 = (-2,\ 1,\ 1)^{\mathrm{T}}.
\end{aligned}$$

For $\lambda = 4$

$$\begin{aligned}
-7x + 4y - 10z &= 0, \\
4x - 8y &= 0, \quad \Rightarrow \quad \mathsf{u}^3 = (2,\ 1,\ -1)^{\mathrm{T}}.
\end{aligned}$$

The final step is to form the transformation matrix P, using these three vectors as its columns:

$$\mathsf{P} = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ -3 & 1 & -1 \end{pmatrix}$$

and read off from its rows the required transformation

$$\begin{aligned} x &= 2\xi - 2\eta + 2\chi, \\ y &= \xi + \eta + \chi, \\ z &= -3\xi + \eta - \chi. \end{aligned}$$

[They are not needed for the question as set, but the transformed expressions are $16\xi^2 + 16\eta^2 + 4\chi^2$ and $-16\xi^2 + 32\eta^2 + 16\chi^2$; as expected, they contain no cross terms. Note that corresponding coefficients are in the ratio given by the associated eigenvalue. Explicitly: $-16/16 = -1$ for ξ; $32/16 = 2$ for η; $16/4 = 4$ for χ.]

9.10 *Use the Rayleigh–Ritz method to estimate the lowest oscillation frequency of a heavy chain of N links, each of length a $(= L/N)$, which hangs freely from one end. Consider simple calculable configurations such as all links but one vertical, or all links collinear, etc.*

Intuitively, having all links collinear should give a good estimate of the lowest oscillation frequency of the chain. However, the example discussed in the text, of a rod on the end of a string, suggests that in the true lowest-frequency mode the lower links will tend to be at a larger inclination to the vertical than are the upper ones.

With θ_i as the (small) angle the ith link makes with the vertical, the potential energy of the ith link is

$$mg \sum_{j=1}^{i-1} a(1 - \cos\theta_j) + mg\frac{a}{2}(1 - \cos\theta_i) \approx \frac{mga}{2}\left(\sum_{j=1}^{i-1} \theta_j^2 + \tfrac{1}{2}\theta_i^2\right).$$

The lateral velocity of the same link is

$$\sum_{j=1}^{i-1} a\dot{\theta}_j + \tfrac{1}{2}a\dot{\theta}_i.$$

The link's kinetic energy is therefore

$$\frac{1}{2}m\left(\sum_{j=1}^{i-1} a\dot{\theta}_j + \tfrac{1}{2}a\dot{\theta}_i\right)^2.$$

It also has some rotational kinetic energy about its own centre of mass but this is small compared to the two contributions considered above and can be ignored in an estimate such as this.

The two quadratic forms $\mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x}$ and $\mathsf{x}^{\mathrm{T}}\mathsf{B}\mathsf{x}$ are, respectively, the total kinetic energy divided by ω^2 and the total potential energy. We now evaluate them with the trial configuration

$$\mathsf{x} = (\theta_1,\ \theta_2,\ \ldots\ ,\ \theta_N)^{\mathrm{T}} = \alpha(1,\ 1,\ \ldots\ ,\ 1)^{\mathrm{T}}.$$

The contribution of the ith link to the potential energy is

$$\frac{mga}{2}\left(\sum_{j=1}^{i-1}\theta_j^2 + \tfrac{1}{2}\theta_i^2\right) = \frac{mga}{2}(i-\tfrac{1}{2})\alpha^2,$$

and its contribution to $\mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x}/\omega^2$ is

$$\frac{1}{2}m\left(\sum_{j=1}^{i-1}a\dot{\theta}_j + \tfrac{1}{2}a\dot{\theta}_i\right)^2 = \frac{ma^2}{2}(i-\tfrac{1}{2})^2\alpha^2.$$

For the whole chain:

$$\begin{aligned} V &= \tfrac{1}{2}mga\alpha^2\textstyle\sum_{i=1}^{N}(i-\tfrac{1}{2}) \\ &= \tfrac{1}{2}mga\alpha^2[\,\tfrac{1}{2}N(N+1) - \tfrac{1}{2}N\,] = \tfrac{1}{4}mga\alpha^2N^2. \\ \frac{T}{\omega^2} &= \tfrac{1}{2}ma^2\alpha^2\textstyle\sum_{i=1}^{N}(i-\tfrac{1}{2})^2 \\ &= \tfrac{1}{2}ma^2\alpha^2[\,\tfrac{1}{6}N(N+1)(2N+1) - \tfrac{1}{2}N(N+1) + \tfrac{1}{4}N\,] \\ &\approx \tfrac{1}{6}ma^2\alpha^2N^3 \quad \text{for large } N. \end{aligned}$$

Thus the estimate is

$$\lambda \approx \frac{\tfrac{1}{4}mga\alpha^2N^2}{\tfrac{1}{6}ma^2\alpha^2N^3} = \frac{3g}{2Na} = \frac{3g}{2L}.$$

10

Vector calculus

***10.2** At time $t = 0$, the vectors $\mathbf{E}$ and $\mathbf{B}$ are given by $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{B} = \mathbf{B}_0$, where the fixed unit vectors $\mathbf{E}_0$ and $\mathbf{B}_0$ are orthogonal. The equations of motion are*

$$\frac{d\mathbf{E}}{dt} = \mathbf{E}_0 + \mathbf{B} \times \mathbf{E}_0,$$
$$\frac{d\mathbf{B}}{dt} = \mathbf{B}_0 + \mathbf{E} \times \mathbf{B}_0.$$

Find $\mathbf{E}$ and $\mathbf{B}$ at a general time t, showing that after a long time the directions of $\mathbf{E}$ and $\mathbf{B}$ have almost interchanged.

Use a coordinate system in which $\mathbf{E}_0$ has components $(1, 0, 0)$ and $\mathbf{B}_0$ has components $(0, 1, 0)$. Then

$$\mathbf{B} \times \mathbf{E}_0 = (0, B_z, -B_y) \quad \text{and} \quad \mathbf{E} \times \mathbf{B}_0 = (-E_z, 0, E_x)$$

and, on equating components in the equations of motion,

$$\frac{dE_x}{dt} = 1, \qquad \frac{dE_y}{dt} = B_z, \qquad \frac{dE_z}{dt} = -B_y,$$
$$\frac{dB_x}{dt} = -E_z, \qquad \frac{dB_y}{dt} = 1, \qquad \frac{dB_z}{dt} = E_x.$$

Recalling that $E_x(0) = 1 = B_y(0)$, we see that the first and fifth of these equations integrate to

$$E_x = t + 1 \qquad \text{and} \qquad B_y = t + 1.$$

We note that all other components have zero values at $t = 0$, leading to:

$$\frac{dB_z}{dt} = E_x = t + 1 \Rightarrow B_z = \tfrac{1}{2}t^2 + t,$$
$$\frac{dE_z}{dt} = -B_y = -t - 1 \Rightarrow E_z = -\tfrac{1}{2}t^2 - t,$$
$$\frac{dE_y}{dt} = B_z = \tfrac{1}{2}t^2 + t \Rightarrow E_y = \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2,$$
$$\frac{dB_x}{dt} = -E_z = \tfrac{1}{2}t^2 + t \Rightarrow B_x = \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2.$$

Thus,

$$\mathbf{E}(t) = (t + 1, \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2, -\tfrac{1}{2}t^2 - t) \quad \text{and} \quad \mathbf{B}(t) = (\tfrac{1}{6}t^3 + \tfrac{1}{2}t^2, t + 1, \tfrac{1}{2}t^2 + t).$$

So, after a long time, when the terms cubic in t dominate, $\mathbf{E}$ is almost along the y-direction and $\mathbf{B}$ is almost along the x-direction, i.e. the directions of $\mathbf{E}$ and $\mathbf{B}$ have almost interchanged.

10.4 *Use vector methods to find the maximum angle to the horizontal at which a stone may be thrown so as to ensure that it is always moving away from the thrower.*

The equation of motion of the stone is

$$\ddot{\mathbf{r}} = \mathbf{g} \quad \text{with} \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0 \quad \text{and} \quad \mathbf{r}(0) = \mathbf{0}.$$

Integrating the equation with the given boundary conditions yields

$$\dot{\mathbf{r}} = \mathbf{g}t + \mathbf{v}_0 \quad \text{and} \quad \mathbf{r} = \tfrac{1}{2}\mathbf{g}t^2 + \mathbf{v}_0 t.$$

The requirement that the stone is always moving away from the thrower can be expressed as $\dot{\mathbf{r}} \cdot \mathbf{r} > 0$ for all t, i.e. that $\dot{\mathbf{r}} \cdot \mathbf{r} = 0$ has no real roots for $t > 0$:

$$\tfrac{1}{2}g^2t^3 + \tfrac{3}{2}\mathbf{v}_0 \cdot \mathbf{g}t^2 + v_0^2 t > 0$$

for all t, which requires that

$$4 \times \tfrac{1}{2}g^2 \times v_0^2 > (\tfrac{3}{2})^2(\mathbf{v}_0 \cdot \mathbf{g})^2,$$
$$\sqrt{\frac{8}{9}} > \frac{\mathbf{v}_0 \cdot \mathbf{g}}{v_0 g}.$$

This means that the angle between the initial trajectory and the vertical must exceed $\cos^{-1} 0.9429 = 19.5°$. The maximum permitted angle to the horizontal is therefore $70.5°$.

10.6 *Prove that for a space curve* $\mathbf{r} = \mathbf{r}(s)$*, where s is the arc length measured along the curve from a fixed point, the triple scalar product*

$$\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^3\mathbf{r}}{ds^3}$$

at any point on the curve has the value $\kappa^2\tau$*, where* κ *is the curvature and* τ *the torsion at that point.*

We start from the relationship

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$$

in which $\hat{\mathbf{b}}$, $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ are respectively the binormal, tanget and normal unit vectors at a point on the curve. This is differentiated with respect to s and use is made of the Frenet-Serret formulae. One of these is needed in the form $\kappa\hat{\mathbf{n}} = d\hat{\mathbf{t}}/ds = d^2\mathbf{r}/ds^2$ or, in terms of the radius of curvature ρ, $\hat{\mathbf{n}} = \rho\, d\hat{\mathbf{t}}/ds = \rho\, d^2\mathbf{r}/ds^2$.

$$\begin{aligned}
\hat{\mathbf{b}} &= \hat{\mathbf{t}} \times \hat{\mathbf{n}}, \\
\frac{d\hat{\mathbf{b}}}{ds} &= \left(\frac{d\hat{\mathbf{t}}}{ds} \times \hat{\mathbf{n}}\right) + \left(\hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}\right), \\
-\tau\hat{\mathbf{n}} &= (\kappa\hat{\mathbf{n}} \times \hat{\mathbf{n}}) + \left(\hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}\right), \\
-\tau\rho\frac{d^2\mathbf{r}}{ds^2} &= \mathbf{0} + \frac{d\mathbf{r}}{ds} \times \frac{d}{ds}\left(\rho\frac{d\hat{\mathbf{t}}}{ds}\right) \\
&= \frac{d\mathbf{r}}{ds} \times \frac{d}{ds}\left(\rho\frac{d^2\mathbf{r}}{ds^2}\right) \\
&= \rho\left(\frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3}\right) + \left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right)\frac{d\rho}{ds}.
\end{aligned}$$

We now take the scalar product of this vector equation with $d^2\mathbf{r}/ds^2$ (sometimes written as $\kappa\hat{\mathbf{n}}$) and obtain

$$\begin{aligned}
-\tau\rho\kappa^2(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) &= \rho\left(\frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3}\right) \cdot \frac{d^2\mathbf{r}}{ds^2} + \frac{d\rho}{ds}\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^2\mathbf{r}}{ds^2}, \\
-\tau\kappa^2 &= \left(\frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3}\right) \cdot \frac{d^2\mathbf{r}}{ds^2} + 0, \\
\tau\kappa^2 &= \left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^3\mathbf{r}}{ds^3},
\end{aligned}$$

i.e. as stated in the question.

10.8 *The shape of the curving slip road joining two motorways that cross at right angles and are at vertical heights $z = 0$ and $z = h$ can be approximated by the space curve*

$$\mathbf{r} = \frac{\sqrt{2}h}{\pi} \ln\cos\left(\frac{z\pi}{2h}\right)\mathbf{i} + \frac{\sqrt{2}h}{\pi} \ln\sin\left(\frac{z\pi}{2h}\right)\mathbf{j} + z\mathbf{k}.$$

Show that the radius of curvature ρ of the slip road is $(2h/\pi)\,\text{cosec}\,(z\pi/h)$ at height z and that the torsion $\tau = -1/\rho$. (To shorten the algebra, set $z = 2h\theta/\pi$ and use θ as the parameter.)

The slip road is given by

$$\mathbf{r} = A(\ln\cos\theta,\ \ln\sin\theta,\ \sqrt{2}\theta),$$

where $A = \sqrt{2}h/\pi$. It follows that

$$\frac{d\mathbf{r}}{d\theta} = A(-\tan\theta,\ \cot\theta,\ \sqrt{2})$$

and

$$\begin{aligned}\frac{ds}{d\theta} &= A(\tan^2\theta + \cot^2\theta + 2)^{1/2}\\ &= A(\sec^2\theta + \text{cosec}\,^2\theta)^{1/2}\\ &= \frac{A}{(\sin^2\theta\cos^2\theta)^{1/2}} = \frac{A}{\sin\theta\cos\theta}.\end{aligned}$$

Next,

$$\begin{aligned}\hat{\mathbf{t}} &= \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds}\\ &= A(-\tan\theta,\ \cot\theta,\ \sqrt{2})\frac{\sin\theta\cos\theta}{A}\\ &= (-\sin^2\theta,\ \cos^2\theta,\ \sqrt{2}\sin\theta\cos\theta).\end{aligned}$$

from which it follows that

$$\frac{1}{\rho}\hat{\mathbf{n}} = \frac{d\hat{\mathbf{t}}}{ds} = \frac{d\hat{\mathbf{t}}}{d\theta}\frac{d\theta}{ds} = \frac{\sin\theta\cos\theta}{A}(-\sin 2\theta,\ -\sin 2\theta,\ \sqrt{2}\cos 2\theta).$$

Thus,

$$\frac{1}{\rho} = \frac{\sin\theta\cos\theta}{A}(\sin^2 2\theta + \sin^2 2\theta + 2\cos^2 2\theta)^{1/2} = \frac{\sqrt{2}\sin\theta\cos\theta}{A}$$

and, as was required to be shown,

$$\rho = \sqrt{2}A\,\text{cosec}\,2\theta = \frac{2h}{\pi}\text{cosec}\,\left(\frac{\pi z}{h}\right).$$

With $\hat{\mathbf{n}}$ a unit vector in the direction of $d\hat{\mathbf{t}}/ds$, we have

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(-\sin 2\theta,\ -\sin 2\theta,\ \sqrt{2}\cos 2\theta) \quad \text{and}$$

$$\hat{\mathbf{t}} = (-\sin^2\theta,\ \cos^2\theta,\ \sqrt{2}\sin\theta\cos\theta),$$

$$\Rightarrow \quad \hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = (\cos^2\theta,\ -\sin^2\theta,\ \sqrt{2}\sin\theta\cos\theta).$$

$$\text{and} \quad \frac{d\hat{\mathbf{b}}}{ds} = \frac{d\hat{\mathbf{b}}}{d\theta}\frac{d\theta}{ds} = \frac{\sin\theta\cos\theta}{A}(-\sin 2\theta,\ -\sin 2\theta,\ \sqrt{2}\cos 2\theta)$$

$$= \frac{\sqrt{2}\sin\theta\cos\theta}{A}\hat{\mathbf{n}}.$$

From this it follows that

$$\tau = -\hat{\mathbf{n}} \cdot \frac{d\hat{\mathbf{b}}}{ds} = \frac{-\sqrt{2}\sin\theta\cos\theta}{\sqrt{2}h/\pi} = -\frac{\pi}{2h}\sin\left(\frac{z\pi}{h}\right) = -\frac{1}{\rho}.$$

10.10 *Find the areas of the given surfaces using parametric coordinates.*

(a) *Using the parameterization $x = u\cos\phi$, $y = u\sin\phi$, $z = u\cot\Omega$, find the sloping surface area of a right circular cone of semi-angle Ω whose base has radius a. Verify that it is equal to $\frac{1}{2}\times$ perimeter of the base $\times$ slope height.*

(b) *Using the same parameterization as in (a) for x and y, and an appropriate choice for z, find the surface area between the planes $z = 0$ and $z = Z$ of the paraboloid of revolution $z = \alpha(x^2 + y^2)$.*

(a) With $x = u\cos\phi$, $y = u\sin\phi$, and $z = u\cot\Omega$,

$$\frac{\partial \mathbf{r}}{\partial u} = (\cos\phi,\ \sin\phi,\ \cot\Omega),$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = (-u\sin\phi,\ u\cos\phi,\ 0),$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial \theta} = (-u\cos\phi\cot\Omega,\ -u\sin\phi\cot\Omega,\ u),$$

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| = u(1 + \cot^2\Omega)^{1/2} = u\,\mathrm{cosec}\,\Omega.$$

Thus $dS = u\,\mathrm{cosec}\,\Omega\,du\,d\phi$ and the total surface area is

$$S = \int_0^{2\pi} d\phi \int_0^a u\,\mathrm{cosec}\,\Omega\,du = \pi a^2 \mathrm{cosec}\,\Omega.$$

This can clearly be written as $\frac{1}{2} \times 2\pi a \times a\,\mathrm{cosec}\,\Omega$, i.e. $\frac{1}{2}\times$ perimeter of the base $\times$ slope height of the cone.

(b) With the given parameterization for x and y, we have

$$z = \alpha(x^2 + y^2) = \alpha u^2,$$

and so

$$\frac{d\mathbf{r}}{du} = (\cos\phi,\ \sin\phi,\ 2\alpha u),$$
$$\frac{d\mathbf{r}}{d\phi} = (-u\sin\phi,\ u\cos\phi,\ 0),$$
$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial \phi}\right| du\,d\phi = |\,(-2\alpha u^2\cos\phi,\ -2\alpha u^2\sin\phi,\ u)\,|\,du\,d\phi$$
$$= (u^2 + 4\alpha^2u^4)^{1/2}\,du\,d\phi = u(1 + 4\alpha^2u^2)^{1/2}\,du\,d\phi.$$

The total area is thus

$$S = \int_0^{2\pi} d\phi \int_0^{(Z/\alpha)^{1/2}} u(1 + 4\alpha^2u^2)^{1/2}\,du$$
$$= 2\pi\left[\frac{2}{3}\,\frac{(1 + 4\alpha^2u^2)^{3/2}}{8\alpha^2}\right]_0^{(Z/\alpha)^{1/2}}$$
$$= \frac{\pi}{6\alpha^2}\left[(1 + 4\alpha Z)^{3/2} - 1\right].$$

This is only the curved surface area; if the plane end of the paraboloid is also counted, an additional $\pi Z/\alpha$ must be included.

10.12 *For the function*

$$z(x, y) = (x^2 - y^2)e^{-x^2-y^2},$$

find the location(s) at which the steepest gradient occurs. What are the magnitude and direction of that gradient? The algebra involved is easier if plane polar coordinates are used.

The function is antisymmetric under the interchange of x and y and so we need to consider explicitly only $x > 0$.

With $x = \rho\cos\phi$ and $y = \rho\sin\phi$ we can write $z(x, y) = f(\rho, \phi)$ as

$$f(\rho, \phi) = \rho^2(\cos^2\phi - \sin^2\phi)e^{-\rho^2} = \rho^2\cos 2\phi e^{-\rho^2}.$$

From this it follows that

$$\nabla f = \left(2(\rho - \rho^3)e^{-\rho^2}\cos 2\phi, -2\rho^2e^{-\rho^2}\sin 2\phi\right),$$
$$|\nabla f|^2 = 4\rho^2e^{-2\rho^2}[\,(1 - \rho^2)^2\cos^2 2\phi + \rho^2\sin^2 2\phi\,]$$
$$= 4se^{-2s}[\,(1 - s)^2\cos^2 2\phi + s\sin^2 2\phi\,],$$

where we have set $\rho^2 = s$ for brevity.

For the line of steepest gradient,

(i) $\partial|\nabla f|^2/\partial\phi = 0$ giving

$$-4(1-s)^2\cos 2\phi\sin 2\phi + 4s\sin 2\phi\cos 2\phi = 0,$$

i.e. $\sin 4\phi = 0$ and $\phi = n\pi/4$ *or* $s = (1-s)^2 \Rightarrow s = (3 \pm \sqrt{5})/2$.

(ii) $\partial|\nabla f|^2/\partial\rho = 0$ or, alternatively, $\partial|\nabla f|^2/\partial s = 0$. We therefore require that

$$(1-2s)[(1-s)^2\cos^2 2\phi + s\sin^2 2\phi] + s[-2(1-s)\cos^2 2\phi + \sin^2 2\phi] = 0. \qquad (*)$$

We now need to examine the various possible combinations of conditions.

For $\sin 2\phi = 0$, $\cos^2 2\phi = 1$ and $(*)$ reduces to

$$(1-2s)(1-s)^2 - 2s(1-s) = 0 \quad \Rightarrow \quad s = 1 \text{ or } s = \frac{5 \pm \sqrt{17}}{4}.$$

The corresponding values of $|\nabla f|^2$, obtained by direct substitution, are 0, 0.156 and 0.345.

For $\cos 2\phi = 0$, $\sin^2 2\phi = 1$ and $(*)$ reduces to

$$(1-2s)s + s = 0 \quad \Rightarrow \quad s = 0 \text{ or } s = 1.$$

The corresponding values of $|\nabla f|^2$, obtained by direct substitution, are 0 and $4e^{-2} = 0.541$.

In the third case, when $s = (1-s)^2$, $|\nabla f|^2$ has no ϕ dependence and takes the form $4s^2e^{-2s}$, which has values 0.146 and 0.272 at $s = (3 \pm \sqrt{5})/2$

The largest of these seven values is 0.541, obtained when $\cos 2\phi = 0$, i.e. when $\phi = \pm\pi/4$ and $s = 1$. Thus the steepest gradient occurs on the circle $\rho = 1$ at the points $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{2}$. The gradient vector there is

$$\nabla f(\rho, \phi) = (0, -2e^{-2}\sin 2\phi)$$

and is therefore azimuthal along the lines $x \pm y = \pm\sqrt{2}$ and $x \pm y = \mp\sqrt{2}$.

10.14 *In the following exercises* **a** *is a vector field.*

(a) *Simplify*

$$\nabla \times \mathbf{a}(\nabla \cdot \mathbf{a}) \;+\; \mathbf{a} \times [\nabla \times (\nabla \times \mathbf{a})] \;+\; \mathbf{a} \times \nabla^2\mathbf{a}.$$

(b) *By explicitly writing out the terms in Cartesian coordinates prove that*

$$[\mathbf{c} \cdot (\mathbf{b} \cdot \nabla) - \mathbf{b} \cdot (\mathbf{c} \cdot \nabla)]\,\mathbf{a} = (\nabla \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c}).$$

(c) *Prove that* $\mathbf{a} \times (\nabla \times \mathbf{a}) = \nabla(\tfrac{1}{2}a^2) - (\mathbf{a} \cdot \nabla)\mathbf{a}$.

(a) Using results given in the text for $\nabla \times (\phi\mathbf{a})$ and for $\nabla \times (\nabla \times \mathbf{a})$, the first two terms can be expanded, as follows.

$$\nabla \times \mathbf{a}(\nabla \cdot \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) \times \mathbf{a} + (\nabla \cdot \mathbf{a})(\nabla \times \mathbf{a}),$$
$$\mathbf{a} \times [\nabla \times (\nabla \times \mathbf{a})] = [\mathbf{a} \times \nabla(\nabla \cdot \mathbf{a})] - [\mathbf{a} \times \nabla^2 \mathbf{a}].$$

Substituting these into the original expression gives

$$\nabla(\nabla \cdot \mathbf{a}) \times \mathbf{a} + (\nabla \cdot \mathbf{a})(\nabla \times \mathbf{a}) + \mathbf{a} \times \nabla(\nabla \cdot \mathbf{a}) - \mathbf{a} \times \nabla^2 \mathbf{a} + \mathbf{a} \times \nabla^2 \mathbf{a}.$$

Thus the original expression is equal to $(\nabla \cdot \mathbf{a})(\nabla \times \mathbf{a})$.

(b) The first term on the LHS is

$$\begin{aligned} \mathbf{c} \cdot (\mathbf{b} \cdot \nabla)\mathbf{a} = c_x b_x \frac{\partial a_x}{\partial x} + c_x b_y \frac{\partial a_x}{\partial y} + c_x b_z \frac{\partial a_x}{\partial z} \\ + c_y b_x \frac{\partial a_y}{\partial x} + c_y b_y \frac{\partial a_y}{\partial y} + c_y b_z \frac{\partial a_y}{\partial z} \\ + c_z b_x \frac{\partial a_z}{\partial x} + c_z b_y \frac{\partial a_z}{\partial y} + c_z b_z \frac{\partial a_z}{\partial z}. \end{aligned}$$

The second term has the same form, but with $\mathbf{b}$ and $\mathbf{c}$ interchanged. The difference between the two is therefore

$$\begin{aligned} &(c_x b_y - b_x c_y)\frac{\partial a_x}{\partial y} + (c_x b_z - b_x c_z)\frac{\partial a_x}{\partial z} + (c_y b_x - b_y c_x)\frac{\partial a_y}{\partial x} \\ &\quad + (c_y b_z - b_y c_z)\frac{\partial a_y}{\partial z} + (c_z b_x - b_z c_x)\frac{\partial a_z}{\partial x} + (c_z b_y - b_z c_y)\frac{\partial a_z}{\partial y}. \\ &= (b_x c_y - b_y c_x)\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) + (b_y c_z - b_z c_y)\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) \\ &\quad + (b_z c_x - b_x c_z)\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) \\ &= (\mathbf{b} \times \mathbf{c})_z (\nabla \times \mathbf{a})_z + (\mathbf{b} \times \mathbf{c})_x (\nabla \times \mathbf{a})_x + (\mathbf{b} \times \mathbf{c})_y (\nabla \times \mathbf{a})_y \\ &= (\mathbf{b} \times \mathbf{c}) \cdot (\nabla \times \mathbf{a}), \end{aligned}$$

as stated.

(c) Consider the z-component of the LHS.

$$\begin{aligned} [\mathbf{a} \times (\nabla \times \mathbf{a})]_z &= a_x (\nabla \times \mathbf{a})_y - a_y (\nabla \times \mathbf{a})_x \\ &= a_x \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) - a_y \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) \\ &= a_x \frac{\partial a_x}{\partial z} + a_y \frac{\partial a_y}{\partial z} + a_z \frac{\partial a_z}{\partial z} - a_z \frac{\partial a_z}{\partial z} - a_x \frac{\partial a_z}{\partial x} - a_y \frac{\partial a_z}{\partial y} \\ &= \frac{1}{2}\frac{\partial}{\partial z}\left(a_x^2 + a_y^2 + a_z^2\right) - (\mathbf{a} \cdot \nabla) a_z \\ &= \left[\nabla(\tfrac{1}{2}a^2) - (\mathbf{a} \cdot \nabla)\mathbf{a}\right]_z \\ &= z\text{-component of RHS.} \end{aligned}$$

In the third line $a_z(\partial a_z/\partial z)$ was both added and subtracted. The corresponding results for the x- and y-components can be proved in the same way, thus establishing the vector result.

10.16 *Verify that (10.42) is valid for each component separately when* $\mathbf{a}$ *is the Cartesian vector* $x^2y\,\mathbf{i} + xyz\,\mathbf{j} + z^2y\,\mathbf{k}$, *by showing that each side of the equation is equal to* $z\,\mathbf{i} + (2x + 2z)\,\mathbf{j} + x\,\mathbf{k}$.

With $\mathbf{a} = (x^2y,\ xyz,\ z^2y)$ we have:

For the RHS
$$\nabla \cdot \mathbf{a} = 2xy + xz + 2zy,$$
$$\nabla(\nabla \cdot \mathbf{a}) = (2y + z,\ 2x + 2z,\ x + 2y),$$
$$\nabla^2\mathbf{a} = (2y,\ 0,\ 2y),$$
$$\nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a} = (z,\ 2x + 2z,\ x).$$

For the LHS
$$\nabla \times \mathbf{a} = (z^2 - xy,\ 0,\ yz - x^2),$$
$$\nabla \times (\nabla \times \mathbf{a}) = (z,\ 2z + 2x,\ x).$$

This verifies that the vector equality is valid term-by-term for this vector field expressed in Cartesian coordinates.

10.18 *Evaluate the Laplacian of a vector field using two different coordinate systems as follows.*

(a) *For cylindrical polar coordinates* ρ, ϕ, z *evaluate the derivatives of the three unit vectors with respect to each of the coordinates, showing that only* $\partial\hat{\mathbf{e}}_\rho/\partial\phi$ *and* $\partial\hat{\mathbf{e}}_\phi/\partial\phi$ *are non-zero.*

  (i) *Hence evaluate* $\nabla^2\mathbf{a}$ *when* $\mathbf{a}$ *is the vector* $\hat{\mathbf{e}}_\rho$, *i.e. a vector of unit magnitude everywhere directed radially outwards from the* z*-axis.*

  (ii) *Note that it is trivially obvious that* $\nabla\times\mathbf{a} = \mathbf{0}$ *and hence that equation (10.41) requires that* $\nabla(\nabla \cdot \mathbf{a}) = \nabla^2\mathbf{a}$.

  (iii) *Evaluate* $\nabla(\nabla \cdot \mathbf{a})$ *and show that the latter equation holds, but that*
$$[\nabla(\nabla \cdot \mathbf{a})]_\rho \neq \nabla^2 a_\rho.$$

(b) *Rework the same problem in Cartesian coordinates (where, as it happens, the algebra is more complicated).*

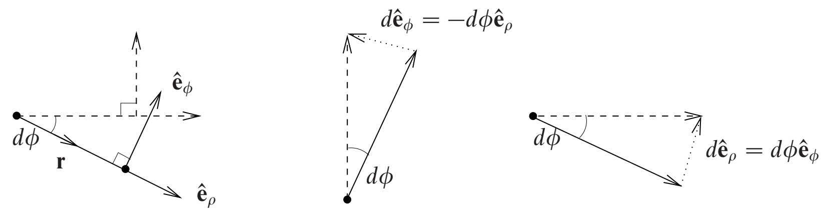

Figure 10.1 The changes in the unit base vectors calculated in exercise 10.18.

(a) It is clear that $\hat{\mathbf{e}}_z$ does not depend upon the position at which it is evaluated (recall that the vectors are determined by their magnitudes and directions, and not by their absolute positions in space); consequently all of its derivatives are zero.
Equally, $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ are unaltered if only the value of z is changed; thus $\partial\hat{\mathbf{e}}_\rho/\partial z = \partial\hat{\mathbf{e}}_\phi/\partial z = \mathbf{0}$.
In the same way, $\partial\hat{\mathbf{e}}_\rho/\partial\rho = \partial\hat{\mathbf{e}}_\phi/\partial\rho = \mathbf{0}$ are both zero.

That leaves only possible variations of $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ with ϕ to consider. Figure 10.1, a section in any plane of constant z, shows these two unit vectors and the changes in them. When ϕ is changed to $\phi + d\phi$, $\hat{\mathbf{e}}_\rho$ changes direction by $d\phi$ and its vector position has been changed by an amount $1 \times d\phi$ in the azimuthal direction parallel to $\hat{\mathbf{e}}_\phi$, i.e.

$$d\hat{\mathbf{e}}_\rho = d\phi\,\hat{\mathbf{e}}_\phi \quad \Rightarrow \quad \frac{\partial\hat{\mathbf{e}}_\rho}{\partial\phi} = \hat{\mathbf{e}}_\phi.$$

The same change, $d\phi$, also causes $\hat{\mathbf{e}}_\phi$ to change direction and alter its vector position by an amount of magnitude $d\phi$. But *this* change is along the radial direction $\hat{\mathbf{e}}_\rho$ and directed towards the polar axis, i.e.

$$d\hat{\mathbf{e}}_\phi = -d\phi\,\hat{\mathbf{e}}_\rho \quad \Rightarrow \quad \frac{\partial\hat{\mathbf{e}}_\phi}{\partial\phi} = -\hat{\mathbf{e}}_\rho.$$

(i) With $\mathbf{a} = (1,\ 0,\ 0)$,

$$\begin{aligned}
\nabla^2\mathbf{a} &= \nabla^2(1\,\hat{\mathbf{e}}_\rho) \\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\hat{\mathbf{e}}_\rho}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial}{\partial\phi}\,\frac{\partial\hat{\mathbf{e}}_\rho}{\partial\phi} + \frac{\partial}{\partial z}\,\frac{\partial\hat{\mathbf{e}}_\rho}{\partial z} \\
&= \mathbf{0} + \frac{1}{\rho^2}\frac{\partial\hat{\mathbf{e}}_\phi}{\partial\phi} + \mathbf{0} \\
&= \frac{1}{\rho^2}(-\hat{\mathbf{e}}_\rho).
\end{aligned}$$

(ii) and (iii). Using the expressions for the divergence and gradient in cylindrical polar coordinates, as given in the text, we have

$$\nabla \cdot \mathbf{a} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, 1) + 0 + 0 = \frac{1}{\rho}.$$
$$\nabla(\nabla \cdot \mathbf{a}) = -\frac{1}{\rho^2}\,\hat{\mathbf{e}}_\rho + 0\,\hat{\mathbf{e}}_\phi + 0\,\hat{\mathbf{e}}_z = -\frac{1}{\rho^2}\,\hat{\mathbf{e}}_\rho.$$

In this case, in which $\nabla \times \mathbf{a}$, and hence $\nabla \times (\nabla \times \mathbf{a})$, are trivially zero everywhere, this verifies the more general result

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}.$$

However, even in this especially simple case, it is clear that $\nabla^2 a_\rho = \nabla^2 1 = 0$ whilst the ρ-component of $\nabla(\nabla \cdot \mathbf{a})$ is equal to $-1/\rho^2$; this shows that the equality does not hold component-by-component.

(b) In Cartesian coordinates, the same vector field takes the form

$$\mathbf{a} = \left(\frac{x}{(x^2+y^2)^{1/2}},\ \frac{y}{(x^2+y^2)^{1/2}},\ 0 \right).$$

Straightforward but somewhat tedious differentiation gives the required partial derivatives of the x-component as

$$\frac{\partial a_x}{\partial x} = \frac{y^2}{(x^2+y^2)^{3/2}}, \qquad \frac{\partial a_x}{\partial y} = \frac{-xy}{(x^2+y^2)^{3/2}},$$
$$\frac{\partial^2 a_x}{\partial x^2} = \frac{-3y^2 x}{(x^2+y^2)^{5/2}}, \qquad \frac{\partial^2 a_x}{\partial x^2} = \frac{2y^2 x - x^3}{(x^2+y^2)^{5/2}}.$$

Together with the obvious $\partial^2 a_x/\partial z^2 = 0$, these results give

$$\nabla^2 a_x = \frac{-3y^2 x}{(x^2+y^2)^{5/2}} + \frac{2y^2 x - x^3}{(x^2+y^2)^{5/2}} = -\frac{x}{(x^2+y^2)^{3/2}}.$$

But,

$$\begin{aligned}
[\nabla(\nabla \cdot \mathbf{a})]_x &= \frac{\partial}{\partial x}\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} \right) \\
&= \frac{\partial}{\partial x}\left(\frac{y^2}{(x^2+y^2)^{3/2}} + \frac{x^2}{(x^2+y^2)^{3/2}} \right) \\
&= \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2)^{1/2}} \right) = -\frac{x}{(x^2+y^2)^{3/2}},
\end{aligned}$$

thus re-establishing the result. Similarly $\nabla^2 a_y = [\nabla(\nabla \cdot \mathbf{a})]_y$.

10.20 *For a description in spherical polar coordinates with axial symmetry of the flow of a very viscous fluid, the components of the velocity field* $\mathbf{u}$ *are given in terms of the* stream function ψ *by*

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = \frac{-1}{r\sin\theta}\frac{\partial \psi}{\partial r}.$$

Find an explicit expression for the differential operator E *defined by*

$$E\psi = -(r\sin\theta)(\nabla \times \mathbf{u})_\phi.$$

The stream function satisfies the equation of motion $E^2\psi = 0$ *and, for the flow of a fluid past a sphere, takes the form* $\psi(r,\theta) = f(r)\sin^2\theta$. *Show that* $f(r)$ *satisfies the (ordinary) differential equation*

$$r^4 f^{(4)} - 4r^2 f'' + 8rf' - 8f = 0.$$

Using the formulae given in the text, we have

$$\begin{aligned}(\nabla \times \mathbf{u})_\phi &= \frac{r\sin\theta}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right] \\ &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{-1}{\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right] \\ &= -\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right).\end{aligned}$$

Hence, E is the operator

$$E = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

With $\psi(r,\theta) = f(r)\sin^2\theta$,

$$\begin{aligned}E\psi &= f''\sin^2\theta + \frac{f\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{2\sin\theta\cos\theta}{\sin\theta}\right) \\ &= \left(f'' - \frac{2f}{r^2}\right)\sin^2\theta, \\ E^2\psi &= \frac{\partial^2}{\partial r^2}\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta \\ &= \sin^2\theta\left[f^{(4)} - \frac{\partial}{\partial r}\left(-\frac{4f}{r^3} + \frac{2f'}{r^2}\right)\right] - \frac{2\sin^2\theta}{r^2}\left(f'' - \frac{2f}{r^2}\right) \\ &= \sin^2\theta\left[f^{(4)} - \frac{12f}{r^4} + \frac{4f'}{r^3} - \frac{2f''}{r^2} + \frac{4f'}{r^3} - \frac{2f''}{r^2} + \frac{4f}{r^4}\right] = 0.\end{aligned}$$

Simplifying then gives

$$f^{(4)} - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0,$$

in agreement with the equation stated in the question.

10.22 *Non-orthogonal curvilinear coordinates are difficult to work with and should be avoided if at all possible, but the following example is provided to illustrate the content of section 10.10.*

In a new coordinate system for the region of space in which the Cartesian coordinate z satisfies $z \geq 0$, the position of a point $\mathbf{r}$ is given by (α_1, α_2, R), where α_1 and α_2 are respectively the cosines of the angles made by $\mathbf{r}$ with the x- and y-coordinate axes of a Cartesian system and $R = |\mathbf{r}|$. The ranges are $-1 \leq \alpha_i \leq 1$, $0 \leq R < \infty$.

(a) *Express $\mathbf{r}$ in terms of α_1, α_2, R and the unit Cartesian vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.*
(b) *Obtain expressions for the vectors $\mathbf{e}_i$ $(= \partial\mathbf{r}/\partial\alpha_1, \ldots)$ and hence show that the scale factors h_i are given by*

$$h_1 = \frac{R(1-\alpha_2^2)^{1/2}}{(1-\alpha_1^2-\alpha_2^2)^{1/2}}, \qquad h_2 = \frac{R(1-\alpha_1^2)^{1/2}}{(1-\alpha_1^2-\alpha_2^2)^{1/2}}, \qquad h_3 = 1.$$

(c) *Verify formally that the system is not an orthogonal one.*
(d) *Show that the volume element of the coordinate system is*

$$dV = \frac{R^2\, d\alpha_1\, d\alpha_2\, dR}{(1-\alpha_1^2-\alpha_2^2)^{1/2}},$$

and demonstrate that this is always less than or equal to the corresponding expression for an orthogonal curvilinear system.
(e) *Calculate the expression for $(ds)^2$ for the system, and show that it differs from that for the corresponding orthogonal system by*

$$\frac{2\alpha_1\alpha_2 R^2}{1-\alpha_1^2-\alpha_2^2}\, d\alpha_1 d\alpha_2.$$

(a) Clearly, $x = R\alpha_1$, $y = R\alpha_2$ and, since $z^2 = R^2 - x^2 - y^2$, $z = (1-\alpha_1^2-\alpha_2^2)^{1/2}R$. To save space we will write $(1-\alpha_1^2-\alpha_2^2)^{1/2}$ as β and $(1-\alpha_1^2-\alpha_2^2)^{-1/2}$ as γ. We note that $\beta\gamma = 1$ and that $\partial\beta/\partial\alpha_i = -\gamma\alpha_i$. Thus,

$$\mathbf{r} = \alpha_1 R\,\mathbf{i} + \alpha_2 R\,\mathbf{j} + \beta R\,\mathbf{k}.$$

(b) The tangent vectors are

$$\begin{aligned}
\mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial \alpha_1} = R\,\mathbf{i} + 0\,\mathbf{j} - R\gamma\alpha_1\,\mathbf{k},\\
\mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial \alpha_2} = 0\,\mathbf{i} + R\,\mathbf{j} - R\gamma\alpha_2\,\mathbf{k},\\
\mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial R} = \alpha_1\,\mathbf{i} + \alpha_2\,\mathbf{j} + \beta\,\mathbf{k},
\end{aligned}$$

and so the scale factors are

$$\begin{aligned}
h_1^2 = R^2 + R^2\gamma^2\alpha_1^2 = R^2\gamma^2(1-\alpha_2^2) &\Rightarrow h_1 = R\gamma(1-\alpha_2^2)^{1/2},\\
h_2^2 = R^2 + R^2\gamma^2\alpha_2^2 = R^2\gamma^2(1-\alpha_1^2) &\Rightarrow h_2 = R\gamma(1-\alpha_1^2)^{1/2},\\
h_3^2 = \alpha_1^2 + \alpha_2^2 + \beta^2 = 1 &\Rightarrow h_3 = 1.
\end{aligned}$$

(c) Consider the scalar products:

$$\begin{aligned}
\mathbf{e}_1\cdot\mathbf{e}_2 &= 0 + 0 + R^2\gamma^2\alpha_1\alpha_2 \neq 0,\\
\mathbf{e}_1\cdot\mathbf{e}_3 &= \alpha_1 R + 0 - \beta\gamma\alpha_1 R = 0,\\
\mathbf{e}_2\cdot\mathbf{e}_3 &= 0 + \alpha_2 R - \beta\gamma\alpha_2 R = 0.
\end{aligned}$$

These show that, whilst both $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal to $\mathbf{e}_3$, they are not orthogonal to each other, i.e the system is not an orthogonal one.

(d) The volume element is

$$\begin{aligned}
dV &= d\alpha_1\,d\alpha_2\,dR\,|(\mathbf{e}_1\times\mathbf{e}_2)\cdot\mathbf{e}_3|\\
&= d\alpha_1\,d\alpha_2\,dR\,(R^2\gamma\alpha_1,\ R^2\gamma\alpha_2,\ R^2)\cdot(\alpha_1,\ \alpha_2,\ \beta)\\
&= d\alpha_1\,d\alpha_2\,dR\,R^2(\gamma\alpha_1^2 + \gamma\alpha_2^2 + \beta)\\
&= d\alpha_1\,d\alpha_2\,dR\,\gamma R^2.
\end{aligned}$$

If the system were an orthogonal one, the elemental volume would be

$$dV_\perp = h_1 d\alpha_1\, h_2 d\alpha_2\, h_3 dR = R^2\gamma^2(1-\alpha_2^2)^{1/2}(1-\alpha_1^2)^{1/2}\,d\alpha_1\,d\alpha_2\,dR.$$

The ratio of the two is

$$\frac{dV}{dV_\perp} = \frac{1}{\gamma(1-\alpha_2^2)^{1/2}(1-\alpha_1^2)^{1/2}} = \frac{(1-\alpha_1^2-\alpha_2^2)^{1/2}}{(1-\alpha_1^2-\alpha_2^2+\alpha_1^2\alpha_2^2)^{1/2}}.$$

This is always less than or equal to unity, with equality only when α_1 and/or α_2 is equal to zero.

(e) We first note that $d\beta = -\gamma\alpha_1\, d\alpha_1 - \gamma\alpha_2\, d\alpha_2$. Then,

$$\begin{aligned}(ds)^2 &= (R\, d\alpha_1 + \alpha_1\, dR)^2 + (R\, d\alpha_2 + \alpha_2\, dR)^2 + (R\, d\beta + \beta\, dR)^2 \\ &= (d\alpha_1)^2(R^2 + R^2\gamma^2\alpha_1^2) + (d\alpha_2)^2(R^2 + R^2\gamma^2\alpha_2^2) \\ &\quad + (dR)^2(\alpha_1^2 + \alpha_2^2 + \beta^2) + (d\alpha_1\, d\alpha_2)(2R^2\gamma^2\alpha_1\alpha_2) \\ &\quad + (d\alpha_1\, dR)(2R\alpha_1 - 2R\beta\gamma\alpha_1) + (d\alpha_2\, dR)(2R\alpha_2 - 2R\beta\gamma\alpha_2) \\ &= R^2\gamma^2(1-\alpha_2^2)(d\alpha_1)^2 + R^2\gamma^2(1-\alpha_1^2)(d\alpha_2)^2 \\ &\quad + (dR)^2 + 2R^2\gamma^2\alpha_1\alpha_2\, d\alpha_1\, d\alpha_2 \\ &= h_1^2(d\alpha_1)^2 + h_2^2(d\alpha_2)^2 + h_3^2(dR)^2 + \frac{2R^2\alpha_1\alpha_2}{1-\alpha_1^2-\alpha_2^2}\, d\alpha_1\, d\alpha_2.\end{aligned}$$

This establishes the stated result.

10.24 *In a Cartesian system, A and B are the points $(0,0,-1)$ and $(0,0,1)$ respectively. In a new coordinate system a general point P is given by (u_1, u_2, u_3) with $u_1 = \frac{1}{2}(r_1 + r_2)$, $u_2 = \frac{1}{2}(r_1 - r_2)$, $u_3 = \phi$; here r_1 and r_2 are the distances AP and BP and ϕ is the angle between the plane ABP and $y = 0$.*

(a) *Express z and the perpendicular distance ρ from P to the z-axis in terms of u_1, u_2, u_3.*
(b) *Evaluate $\partial x/\partial u_i$, $\partial y/\partial u_i$, $\partial z/\partial u_i$, for $i = 1, 2, 3$.*
(c) *Find the Cartesian components of $\hat{\mathbf{u}}_j$ and hence show that the new coordinates are mutually orthogonal. Evaluate the scale factors and the infinitesimal volume element in the new coordinate system.*
(d) *Determine and sketch the forms of the surfaces $u_i =$ constant.*
(e) *Find the most general function f of u_1 only that satisfies $\nabla^2 f = 0$.*

We have the following five defining equations:

$$\begin{aligned}&\text{(i)}\;\; r_1^2 = x^2 + y^2 + (z+1)^2, \\ &\text{(ii)}\;\; r_2^2 = x^2 + y^2 + (z-1)^2, \\ &\text{(iii)}\;\; r_1 + r_2 = 2u_1, \qquad 1 \le u_1 < \infty, \\ &\text{(iv)}\;\; r_1 - r_2 = 2u_2, \qquad -1 \le u_2 \le 1, \\ &\text{(v)}\;\; \phi = u_3.\end{aligned}$$

(a) Multiplying (iii) by (iv) and subtracting (ii) from (i) gives the equality

$$4u_1u_2 = r_1^2 - r_2^2 = (z+1)^2 - (z-1)^2 = 4z \quad \Rightarrow \quad z = u_1u_2.$$

Writing $\rho^2 = x^2 + y^2$, the addition of (i) and (ii) gives

$$\begin{aligned}
2\rho^2 + 2z^2 + 2 = r_1^2 + r_2^2 &= (u_1 + u_2)^2 + (u_1 - u_2)^2 = 2u_1^2 + 2u_2^2, \\
\rho^2 &= u_1^2 + u_2^2 - u_1^2 u_2^2 - 1 \\
&= (u_1^2 - 1)(1 - u_2^2). \\
\rho\, d\rho &= (1 - u_2^2)u_1\, du_1 - (u_1^2 - 1)u_2\, du_2.
\end{aligned}$$

(b) and (c)

$$\begin{aligned}
\mathbf{r} &= \rho \cos u_3\, \mathbf{i} + \rho \sin u_3\, \mathbf{j} + u_1 u_2\, \mathbf{k}, \\
\mathbf{u}_1 &= \frac{\partial \mathbf{r}}{\partial u_1} = \frac{u_1(1 - u_2^2)}{\rho} \cos u_3\, \mathbf{i} + \frac{u_1(1 - u_2^2)}{\rho} \sin u_3\, \mathbf{j} + u_2\, \mathbf{k}, \\
\mathbf{u}_2 &= \frac{\partial \mathbf{r}}{\partial u_2} = -\frac{u_2(u_1^2 - 1)}{\rho} \cos u_3\, \mathbf{i} - \frac{u_2(u_1^2 - 1)}{\rho} \sin u_3\, \mathbf{j} + u_1\, \mathbf{k}, \\
\mathbf{u}_3 &= \frac{\partial \mathbf{r}}{\partial u_3} = -\rho \sin u_3\, \mathbf{i} + \rho \cos u_3\, \mathbf{j}.
\end{aligned}$$

Next, consider the scalar products:

$$\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{u}_2 &= \frac{u_1 u_2 (1 - u_2^2)(u_1^2 - 1)}{\rho^2}(-\cos^2 u_3 - \sin^2 u_3) + u_1 u_2 \\
&= u_1 u_2 (1 - \cos^2 u_3 - \sin^2 u_3) = 0, \\
\mathbf{u}_1 \cdot \mathbf{u}_3 &= -(1 - u_2^2)u_1 \cos u_3 \sin u_3 + (1 - u_2^2)u_1 \sin u_3 \cos u_3 = 0, \\
\mathbf{u}_2 \cdot \mathbf{u}_3 &= (u_1^2 - 1)u_2 \cos u_3 \sin u_3 - (u_1^2 - 1)u_2 \sin u_3 \cos u_3 = 0.
\end{aligned}$$

Thus, the new coordinates form an orthogonal system. Further,

$$\begin{aligned}
h_1^2 &= \frac{(1 - u_2^2)^2 u_1^2}{\rho^2}(\cos^2 u_3 + \sin^2 u_3) + u_2^2 = \frac{u_1^2 - u_2^2}{u_1^2 - 1}, \\
h_2^2 &= \frac{(u_1^2 - 1)^2 u_2^2}{\rho^2}(\cos^2 u_3 + \sin^2 u_3) + u_1^2 = \frac{u_1^2 - u_2^2}{1 - u_2^2}, \\
h_3^2 &= \rho^2(\sin^2 u_3 + \cos^2 u_3) = \rho^2,
\end{aligned}$$

giving the scale factors as

$$h_1 = \sqrt{\frac{u_1^2 - u_2^2}{u_1^2 - 1}}, \qquad h_2 = \sqrt{\frac{u_1^2 - u_2^2}{1 - u_2^2}}, \qquad h_3 = \sqrt{(u_1^2 - 1)(1 - u_2^2)}.$$

The volume element is

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3 = |u_1^2 - u_2^2|\, du_1\, du_2\, du_3.$$

(d) Since one definition of an ellipsoid is the locus of a point the sum of whose distances from two fixed points is a constant, the surfaces $u_1 = \frac{1}{2}(r_1 + r_2) = c$ must be ellipsoids, *all* with foci at $(0, 0, \pm 1)$. The range of c is $1 \le c < \infty$, with $c = 1$ corresponding to the line AB.

Similarly, $u_2 = \frac{1}{2}(r_1 - r_2) = c$, with $-1 \leq c \leq 1$, is a set confocal hyperboloids. The extreme values for c of $+1$ and -1 correspond to the parts of the z-axis $1 \leq z < \infty$ and $-1 \geq z > -\infty$, respectively.

The surfaces $u_3 =$ constant are clearly half-planes containing the z-axis.

(e) If $f = f(u_1)$ is a solution of $\nabla^2 f = 0$, then Laplace's equation reduces to

$$\begin{aligned} 0 &= \frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) \\ &= \frac{\partial}{\partial u_1}\left(\frac{\sqrt{u_1^2 - 1}\sqrt{u_1^2 - 1}\sqrt{1 - u_2^2}}{\sqrt{1 - u_2^2}}\frac{\partial f}{\partial u_1}\right). \end{aligned}$$

Integrating this and simplifying the factor containing square roots, now gives

$$\frac{\partial f}{\partial u_1} = \frac{k}{u_1^2 - 1} = \frac{k}{2(u_1 - 1)} - \frac{k}{2(u_1 + 1)}$$

which on further integration gives the most general function of u_1 that satisfies Laplace's equation as

$$f(u_1) = A \ln \frac{u_1 - 1}{u_1 + 1} + B,$$

where A and B are arbitrary constants.

11

Line, surface and volume integrals

11.2 *The vector field* $\mathbf{Q}$ *is defined as*

$$\mathbf{Q} = \left[3x^2(y+z) + y^3 + z^3\right]\mathbf{i} + \left[3y^2(z+x) + z^3 + x^3\right]\mathbf{j} + \left[3z^2(x+y) + x^3 + y^3\right]\mathbf{k}.$$

Show that $\mathbf{Q}$ *is a conservative field, construct its potential function and hence evaluate the integral* $J = \int \mathbf{Q}\cdot d\mathbf{r}$ *along any line connecting the point* A *at* $(1, -1, 1)$ *to* B *at* $(2, 1, 2)$.

To test whether $\mathbf{Q}$ is conservative we consider the components of $\nabla \times \mathbf{Q}$, which are

$$(\nabla \times \mathbf{Q})_x = \frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z} = (3z^2 + 3y^2) - (3y^2 + 3z^2) = 0,$$

$$(\nabla \times \mathbf{Q})_y = \frac{\partial Q_x}{\partial z} - \frac{\partial Q_z}{\partial x} = (3x^2 + 3z^2) - (3z^2 + 3x^2) = 0,$$

$$(\nabla \times \mathbf{Q})_z = \frac{\partial Q_y}{\partial x} - \frac{\partial Q_x}{\partial y} = (3y^2 + 3x^2) - (3x^2 + 3y^2) = 0.$$

Hence, $\nabla \times \mathbf{Q} = \mathbf{0}$ which implies that $\mathbf{Q}$ is indeed a conservative field.

Let its potential function be $\phi(x, y, z) = f(x, y, z) + g(y, z) + h(z)$. Then, from the x-component of $\mathbf{Q}$,

$$\frac{\partial f}{\partial x} = 3x^2(y+z) + y^3 + z^3 \quad \Rightarrow \quad f(x, y, z) = x^3(y+z) + x(y^3 + z^3).$$

From its y-component,

$$x^3 + 3y^2x + \frac{\partial g}{\partial y} = 3y^2(z+x) + z^3 + x^3 \quad \Rightarrow \quad g(y, z) = y^3z + z^3y.$$

And finally, from its z-component,

$$x^3 + 3z^2x + y^3 + 3z^2y + \frac{\partial h}{\partial z} = 3z^2(x+y) + x^3 + y^3 \quad \Rightarrow \quad h(z) = c.$$

Thus,

$$\begin{aligned} \phi(x,y,z) &= x^3(y+z) + (y^3+z^3)x + y^3z + z^3y + c \\ &= yz(y^2+z^2) + zx(z^2+x^2) + xy(x^2+y^2) + c. \end{aligned}$$

Because the field is conservative, J is independent of the path taken and equal to $\phi(2,1,2) - \phi(1,-1,1) = (52+c) - (-2+c) = 54$.

11.4 *By making an appropriate choice for the functions $P(x,y)$ and $Q(x,y)$ that appear in Green's theorem in a plane, show that the integral of $x-y$ over the upper half of the unit circle centred on the origin has the value $-\frac{2}{3}$. Show the same result by direct integration in Cartesian coordinates.*

To obtain the integral of $x-y$ over the bounded region we must choose $Q(x,y)$ such that $\partial Q/\partial x$ is x, and $P(x,y)$ such that $\partial P/\partial y$ is y. Clearly $Q(x,y) = \frac{1}{2}x^2$ and $P(x,y) = \frac{1}{2}y^2$ will do. Green's theorem then reads

$$\frac{1}{2}\oint_C \left(y^2\,dx + x^2\,dy\right) = \int\int_R (x-y)\,dx\,dy.$$

We now evaluate the line integral on the LHS using $x = \cos\theta$ and $y = \sin\theta$ on the semi-circular part of the contour and ordinary integration with $y = 0$ on the straight-line portion joining $(-1,0)$ to $(1,0)$. Clearly, the latter contributes nothing, as both $y = 0$ and $dy = 0$.

With this parameterisation, the integral is

$$\begin{aligned} I &= \int_0^\pi \sin^2\theta(-\sin\theta\,d\theta) + \cos^2\theta(\cos\theta\,d\theta) \\ &= -\int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta + \int_0^\pi (1-\sin^2\theta)\cos\theta\,d\theta \\ &= \left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^\pi + \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^\pi \\ &= -2 + \frac{2}{3} + 0 - 0 = -\frac{4}{3}. \end{aligned}$$

The integral of $x-y$ is therefore one-half of this, i.e. $-\frac{2}{3}$.

As a double integral in Cartesian coordinates, we have

$$\begin{aligned}\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} (x-y)\,dy &= \int_{-1}^{1} \left[x\sqrt{1-x^2} - \frac{1}{2}(1-x^2) \right] dx \\ &= \left[-\frac{1}{3}(1-x^2)^{3/2} \right]_{-1}^{1} - \frac{1}{2}\left[x - \frac{x^3}{3} \right]_{-1}^{1} \\ &= 0 - 0 - 1 + \tfrac{1}{3} = -\tfrac{2}{3}.\end{aligned}$$

11.6 *By using parameterisations of the form $x = a\cos^n\theta$ and $y = a\sin^n\theta$ for suitable values of n, find the area bounded by the curves*

$$x^{2/5} + y^{2/5} = a^{2/5} \quad \textit{and} \quad x^{2/3} + y^{2/3} = a^{2/3}.$$

Consider first $x^{2/5} + y^{2/5} = a^{2/5}$, which is clearly parameterised by $x = a\cos^5\theta$ and $y = a\sin^5\theta$. As shown in the worked example in section 11.3, the area of a region R enclosed by a simple closed curve C is given by $A = \frac{1}{2}\oint_C (x\,dy - y\,dx) = \oint_C x\,dy = -\oint_C y\,dx$. Applying this to the present case,

$$\begin{aligned}A_1 &= \frac{1}{2}\oint (x\,dy - y\,dx) \\ &= \frac{1}{2}\int_0^{2\pi} [\,5a^2\cos^5\theta\sin^4\theta\cos\theta - 5a^2\sin^5\theta\cos^4\theta(-\sin\theta)\,]\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 5a^2\cos^4\theta\sin^4\theta\,d\theta.\end{aligned}$$

In the same way, the area bounded by $x^{2/3} + y^{2/3} = a^{2/3}$ will be given by

$$A_2 = \frac{1}{2}\int_0^{2\pi} 3a^2\cos^2\theta\sin^2\theta\,d\theta.$$

Integrals of this sort were considered in exercise 2.42 where it was shown that

$$J(m,n) = \int_0^{\pi/2} \cos^m\theta\sin^n\theta\,d\theta = \frac{m-1}{m+n}J(m-2,n) = \frac{n-1}{m+n}J(m,n-2),$$

with $J(0,0) = \pi/2$. Hence,

$$\begin{aligned}A_1 &= \frac{5a^2}{2}\,4\,J(4,4) = 10a^2\frac{3}{8}J(2,4) \\ &= \frac{15a^2}{4}\frac{1}{6}J(0,4) = \frac{5a^2}{8}\frac{3}{4}J(0,2) \\ &= \frac{15a^2}{32}\frac{1}{2}J(0,0) = \frac{15\pi a^2}{128}.\end{aligned}$$

In the same way,

$$A_2 = \frac{3a^2}{2}\, 4\, J(2,2) = 6a^2 \frac{1}{4} J(0,2)$$
$$= \frac{3a^2}{2}\frac{1}{2} J(0,0) = \frac{3\pi a^2}{8}.$$

The area in the first quadrant enclosed between the two curves is therefore

$$\pi a^2 \left(\frac{3}{8} - \frac{15}{128}\right) = \frac{33\pi a^2}{128}.$$

11.8 *Criticise the following 'proof' that* $\pi = 0$.

(a) *Apply Green's theorem in a plane to the two functions* $P(x, y) = \tan^{-1}(y/x)$ *and* $Q(x, y) = \tan^{-1}(x/y)$, *taking the region* R *to be the unit circle centred on the origin.*
(b) *The RHS of the equality so produced is*

$$\int\int_R \frac{y - x}{x^2 + y^2}\, dx\, dy$$

which, either by symmetry considerations or by changing to plane polar co-ordinates, can be shown to have zero value.
(c) *In the LHS of the equality set* $x = \cos\theta$ *and* $y = \sin\theta$, *yielding* $P(\theta) = \theta$ *and* $Q(\theta) = \pi/2 - \theta$. *The line integral becomes*

$$\int_0^{2\pi} \left[\left(\frac{\pi}{2} - \theta\right)\cos\theta - \theta\sin\theta\right] d\theta,$$

which has value 2π.
(d) *Thus* $2\pi = 0$ *and the stated result follows.*

All of the mathematical steps are as indicated with, in part (b),

$$\frac{\partial P}{\partial y} = \frac{x}{x^2 + y^2} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{y}{x^2 + y^2},$$

and, in part (c),

$$P = \tan^{-1}\frac{\sin\theta}{\cos\theta} = \theta \quad \text{and} \quad Q = \tan^{-1}\frac{\cos\theta}{\sin\theta} = \tan^{-1}\cot\theta = \frac{\pi}{2} - \theta.$$

The non-zero contribution to the integral on the LHS comes from the integral of $\theta\,(-\sin\theta\, d\theta)$. Thus the false result does not arise from an algebraic or integration error.

However, the functions $P(x,y) = \tan^{-1}(y/x)$ and $Q(x,y) = \tan^{-1}(x/y)$, are not continuous (let alone differentiable!) at the origin. As this point is enclosed by the contour, the conditions for Green's theorem to apply are not met and the 'proof' is false.

11.10 *Find the vector area* $\mathbf{S}$ *of the part of the curved surface of the hyperboloid of revolution*

$$\frac{x^2}{a^2} - \frac{y^2+z^2}{b^2} = 1$$

that lies in the region $z \geq 0$ *and* $a \leq x \leq \lambda a$.

The curved surface in question, together with the semicircular intersection of the hyperboloid with the plane $x = \lambda a$ and its hyperbolic intersection with the plane $z = 0$, make up a closed surface. Since the vector area of a closed surface vanishes, the vector area of the curved surface can be found by subtracting the vector areas of the other two plane surfaces from $\mathbf{0}$. Thus, $\mathbf{S} = -S_1\,\mathbf{i} + S_2\,\mathbf{k}$ where S_1 is the area of the semicircle and S_2 that of the hyperbolic intersection.

For S_1, $x = \lambda a$ and

$$\frac{\lambda^2 a^2}{a^2} - 1 = \frac{y^2+z^2}{b^2},$$

i.e the radius of the semicircular intersection is $b\sqrt{\lambda^2-1}$ and the corresponding area is $S_1 = \frac{1}{2}\pi b^2(\lambda^2-1)$.

For S_2, $z = 0$ and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and so, making the substitution $x = a\cosh\theta$ and writing $\cosh^{-1}\lambda$ as μ, we obtain

$$\begin{aligned}
S_2 &= \int_a^{\lambda a} 2b\sqrt{\frac{x^2}{a^2} - 1}\,dx \\
&= \int_0^{\mu} 2b\sinh\theta\,(a\sinh\theta\,d\theta) \\
&= \int_0^{\mu} ab(\cosh 2\theta - 1)\,d\theta \\
&= ab\left[\frac{\sinh 2\theta}{2} - \theta\right]_0^{\mu} \\
&= ab(\lambda\sqrt{\lambda^2-1} - \cosh^{-1}\lambda).
\end{aligned}$$

In summary, $\mathbf{S} = -\frac{1}{2}\pi b^2(\lambda^2-1)\,\mathbf{i} + ab(\lambda\sqrt{\lambda^2-1} - \cosh^{-1}\lambda)\,\mathbf{k}$.

11.12 *Show that the expression below is equal to the solid angle subtended by a rectangular aperture of sides 2a and 2b at a point a distance c from the aperture along the normal to its centre:*

$$\Omega = 4\int_0^b \frac{ac}{(y^2+c^2)(y^2+c^2+a^2)^{1/2}}\,dy.$$

By setting $y = (a^2+c^2)^{1/2}\tan\phi$, *change this integral into the form*

$$\int_0^{\phi_1} \frac{4ac\cos\phi}{c^2+a^2\sin^2\phi}\,d\phi,$$

where $\tan\phi_1 = b/(a^2+c^2)^{1/2}$, *and hence show that*

$$\Omega = 4\tan^{-1}\left[\frac{ab}{c(a^2+b^2+c^2)^{1/2}}\right].$$

The general expression for the solid angle subtended at the origin is

$$\Omega = \int_S \frac{\mathbf{r}\cdot d\mathbf{S}}{r^3}.$$

In the present case, taking the plane's normal along the z-axis, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + c\,\mathbf{k}$ and $d\mathbf{S} = dx\,dy\,\mathbf{k}$. Therefore

$$\Omega = 4\int_0^a\int_0^b \frac{c\,dx\,dy}{(c^2+x^2+y^2)^{3/2}}.$$

If we write $c^2+y^2=p^2$ and $x = p\tan\theta$ with $\tan^{-1}(a/p)=\mu$, then this becomes

$$\begin{aligned}\Omega &= 4\int_0^b dy\int_0^{\mu}\frac{cp\sec^2\theta\,d\theta}{(p^2\sec^2\theta)^{3/2}}\\ &= 4\int_0^b dy\int_0^{\mu}\frac{c\cos\theta}{p^2}\,d\theta\\ &= 4\int_0^b \frac{c\sin\mu}{p^2}\,dy.\end{aligned}$$

Now,

$$\sin\mu = \frac{a}{(a^2+p^2)^{1/2}} = \frac{a}{(a^2+c^2+y^2)^{1/2}},$$

and so

$$\Omega = 4\int_0^b \frac{ac}{(y^2+c^2)(y^2+c^2+a^2)^{1/2}}\,dy,$$

as given in the question.

Next, as suggested, set $y = (a^2+c^2)^{1/2}\tan\phi$ and define ϕ_1 by $b = (a^2+c^2)^{1/2}\tan\phi_1$. Then,

$$\begin{aligned}\Omega &= 4ac\int_0^{\phi_1}\frac{(a^2+c^2)^{1/2}\sec^2\phi\,d\phi}{[\,c^2+(a^2+c^2)\tan^2\phi\,](a^2+c^2)^{1/2}(\sec^2\phi)^{1/2}}\\ &= 4ac\int_0^{\phi_1}\frac{\sec\phi\,d\phi}{c^2+(a^2+c^2)\tan^2\phi}\\ &= 4ac\int_0^{\phi_1}\frac{\cos\phi\,d\phi}{a^2\sin^2\phi+c^2}\\ &= \frac{4ac}{a^2}\int_0^{\phi_1}\frac{\cos\phi\,d\phi}{\sin^2\phi+\dfrac{c^2}{a^2}}\\ &= \frac{4c}{a}\left[\frac{a}{c}\tan^{-1}\frac{a\sin\phi}{c}\right]_0^{\phi_1}\\ &= 4\tan^{-1}\frac{a\sin\phi_1}{c} = 4\tan^{-1}\frac{ab}{c(a^2+c^2+b^2)^{1/2}}.\end{aligned}$$

This establishes the explicit expression for the solid angle subtended by the rectangle.

11.14 *A vector field* **a** *is given by* $(z^2+2xy)\,\mathbf{i}+(x^2+2yz)\,\mathbf{j}+(y^2+2zx)\,\mathbf{k}$. *Show that* **a** *is conservative and that the line integral* $\int \mathbf{a}\cdot d\mathbf{r}$ *along any line joining* $(1,1,1)$ *and* $(1,2,2)$ *has the value* 11.

We show that the field is conservative by showing that it is possible to construct a suitable potential function, as follows.

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= z^2+2xy \quad\Rightarrow\quad \phi(x,y,z) = xz^2+x^2y+f(y,z),\\ \frac{\partial\phi}{\partial y} &= x^2+2yz = x^2+\frac{\partial f}{\partial y} \quad\Rightarrow\quad f(y,z) = y^2z+g(z),\\ \frac{\partial\phi}{\partial z} &= y^2+2zx = 2xz+y^2+\frac{\partial g}{\partial z} \quad\Rightarrow\quad g(z)=c.\end{aligned}$$

Thus

$$\phi(x,y,z) = xz^2+x^2y+y^2z+c$$

is a suitable potential function.

It follows that the line integral of **a** along *any* line joining $(1,1,1)$ to $(1,2,2)$ has the value $\phi(1,2,2)-\phi(1,1,1) = (14+c)-(3+c) = 11$.

11.16 *One of Maxwell's electromagnetic equations states that all magnetic fields* **B** *are solenoidal (i.e.* $\nabla \cdot \mathbf{B} = 0$*). Determine whether each of the following vectors could represent a real magnetic field; where it could, try to find a suitable vector potential* **A***, i.e. such that* $\mathbf{B} = \nabla \times \mathbf{A}$*. (Hint: seek a vector potential that is parallel to* $\nabla \times \mathbf{B}$*.):*

(a) $\dfrac{B_0 b}{r^3}[(x-y)z\,\mathbf{i} + (x-y)z\,\mathbf{j} + (x^2-y^2)\,\mathbf{k}]$ *in Cartesians with* $r^2 = x^2 + y^2 + z^2$*;*

(b) $\dfrac{B_0 b^3}{r^3}[\cos\theta\ \cos\phi\,\hat{\mathbf{e}}_r - \sin\theta\ \cos\phi\,\hat{\mathbf{e}}_\theta + \sin 2\theta\ \sin\phi\,\hat{\mathbf{e}}_\phi]$ *in spherical polars;*

(c) $B_0 b^2\left[\dfrac{z\rho}{(b^2+z^2)^2}\,\hat{\mathbf{e}}_\rho + \dfrac{1}{b^2+z^2}\,\hat{\mathbf{e}}_z\right]$ *in cylindrical polars.*

(a) We calculate $\nabla \cdot \mathbf{B}$ in Cartesian coordinates.

$$\frac{\nabla \cdot \mathbf{B}}{B_0 b} = -\frac{3(x-y)zx}{r^5} + \frac{z}{r^3} - \frac{3(x-y)zy}{r^5} - \frac{z}{r^3} - \frac{3(x^2-y^2)z}{r^5}$$
$$= -\frac{6(x^2-y^2)z}{r^5} \neq 0 \quad \Rightarrow \quad \mathbf{B} \text{ cannot be a real field.}$$

(b) Working in spherical polar coordinates:

$$\nabla \cdot \mathbf{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}$$
$$= B_0 b^3\left[-\frac{\cos\theta\cos\phi}{r^4} - \frac{\cos\phi\,2\sin\theta\cos\theta}{r^4\sin\theta} + \frac{\sin 2\theta\cos\phi}{r^4\sin\theta}\right]$$
$$= -\frac{B_0 b^3\cos\theta\cos\phi}{r^4} \neq 0 \quad \Rightarrow \quad \mathbf{B} \text{ cannot be a real field.}$$

(c) **B** has no ϕ-dependence and so

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z}$$
$$\frac{\nabla \cdot \mathbf{B}}{B_0 b^2} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho\, z\, \rho}{(b^2+z^2)^2}\right) + \frac{\partial}{\partial z}\left(\frac{1}{b^2+z^2}\right)$$
$$= \frac{2z}{(b^2+z^2)^2} - \frac{2z}{(b^2+z^2)^2} = 0$$
$$\Rightarrow \mathbf{B} \text{ could be a real magnetic field.}$$

Following the hint (and with no ϕ component or ϕ-dependence in **B**),

$$\frac{\nabla \times \mathbf{B}}{B_0 b^2} = \left(\frac{1}{\rho}(0-0),\ \frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial\rho},\ \frac{1}{\rho}(0-0)\right)$$
$$= \left(0,\ \frac{\rho}{(b^2+z^2)^2} - \frac{4\rho z^2}{(b^2+z^2)^3} - 0,\ 0\right).$$

$\nabla \times \mathbf{B}$ has only a ϕ-component and so take $\mathbf{A} = (0, A_\phi, 0)$. We then require

$$\begin{aligned}
\frac{B_0 b^2 z\rho}{(b^2+z^2)^2} &= (\nabla \times \mathbf{A})_\rho = -\frac{\partial A_\phi}{\partial z} \\
\Rightarrow \quad A_\phi &= \frac{B_0 b^2 \rho}{2(b^2+z^2)} + g(\rho), \\
\frac{B_0 b^2}{b^2+z^2} &= (\nabla \times \mathbf{A})_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) \\
\Rightarrow \quad \rho A_\phi &= \frac{B_0 b^2 \rho^2}{2(b^2+z^2)} + f(z).
\end{aligned}$$

These two equations, taken together, imply that $\mathbf{A} = (0, A_\phi, 0)$, with

$$A_\phi = \frac{B_0 b^2 \rho}{2(b^2+z^2)}$$

a suitable component. To this $\mathbf{A}$ could be added *any* vector field that is the gradient of a scalar.

> ***11.18*** *A vector field* $\mathbf{a} = f(r)\mathbf{r}$ *is spherically symmetric and everywhere directed away from the origin. Show that* $\mathbf{a}$ *is irrotational but that it is also solenoidal only if* $f(r)$ *is of the form* Ar^{-3}.

In spherical polar coordinates, $\mathbf{a} = f(r)\mathbf{r} = rf(r)\hat{\mathbf{e}}_r$, and

$$\begin{aligned}
\nabla \times \mathbf{a} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ rf(r) & 0 & 0 \end{vmatrix} \\
&= 0\hat{\mathbf{e}}_r + \frac{1}{r\sin\theta}\frac{\partial(rf)}{\partial\phi}\hat{\mathbf{e}}_\theta - \frac{1}{r}\frac{\partial(rf)}{\partial\theta}\hat{\mathbf{e}}_\phi \\
&= \mathbf{0}.
\end{aligned}$$

Hence $\mathbf{a}$ is irrotational. For it also to be solenoidal requires that

$$0 = \nabla \cdot \mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\, rf(r)\right] + 0 + 0 \;\Rightarrow\; r^3 f(r) = A \;\Rightarrow\; f(r) = \frac{A}{r^3}.$$

11.20 *Obtain an expression for the value ϕ_P at a point P of a scalar function ϕ that satisfies $\nabla^2\phi = 0$ in terms of its value and normal derivative on a surface S that encloses it, by proceeding as follows.*

(a) *In Green's second theorem take ψ at any particular point Q as $1/r$, where r is the distance of Q from P. Show that $\nabla^2\psi = 0$ except at $r = 0$.*
(b) *Apply the result to the doubly connected region bounded by S and a small sphere Σ of radius δ centred on P.*
(c) *Apply the divergence theorem to show that the surface integral over Σ involving $1/\delta$ vanishes, and prove that the term involving $1/\delta^2$ has the value $4\pi\phi_P$.*
(d) *Conclude that*

$$\phi_P = -\frac{1}{4\pi}\int_S \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial \phi}{\partial n}\, dS.$$

This important result shows that the value at a point P of a function ϕ that satisfies $\nabla^2\phi = 0$ everywhere within a closed surface S that encloses P may be expressed entirely *in terms of its value and normal derivative on S. This matter is taken up more generally in connection with Green's functions in chapter 21 and in connection with functions of a complex variable in section 24.10.*

Green's theorems apply to any suitably differentiable pair of functions, but here we apply them to a function ϕ that satisfies $\nabla^2\phi = 0$ and ψ, which has a value at any point Q equal to the reciprocal of its distance r from a fixed point P.

(a) Using spherical polar coordinates centred on P, we have

$$\nabla^2\psi = \nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) = \frac{1}{r^2}\frac{\partial(-1)}{\partial r} = 0.$$

Thus $\nabla^2\psi = 0$ except at $r = 0$ where the function is not differentiable.

(b) When these results are put into Green's second theorem applied to the doubly connected region bounded by S and a small sphere Σ of radius δ centred on P, the integrand in the volume integral vanishes, leading to

$$\int_S \frac{1}{r}\frac{\partial\phi}{\partial n}\, dS + \int_\Sigma \frac{1}{r}\frac{\partial\phi}{\partial n}\, dS = \int_S \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \int_\Sigma \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS.$$

(c) For the term on the LHS taken over the sphere Σ, the factor r^{-1} is a constant and equal to δ^{-1} and, by the divergence theorem, the surface integral of $\partial\phi/\partial n$ is equal to the volume integral of $\nabla^2\phi$. But this is zero and so the term vanishes.

For the term on the RHS taken over the sphere Σ,

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = -\left(-\frac{1}{r^2}\right)_{r=\delta} = \frac{1}{\delta^2}.$$

The additional minus sign arises because $\hat{\mathbf{n}}$ is the outward normal to the space and this is in the direction of decreasing r. The surface area is $4\pi\delta^2$ and so the value of the integral is $4\pi\phi_P$ in the limit of $\delta \to 0$.

(d) Taking all terms involving integrals over S to one side of the equation, it can be rearranged as

$$\phi_P = -\frac{1}{4\pi}\int_S \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\,dS + \frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial\phi}{\partial n}\,dS,$$

thus establishing the stated result.

11.22 *A rigid body of volume V and surface S rotates with angular velocity $\boldsymbol{\omega}$. Show that*

$$\boldsymbol{\omega} = -\frac{1}{2V}\oint_S \mathbf{u}\times d\mathbf{S},$$

where $\mathbf{u}(\mathbf{x})$ is the velocity of the point $\mathbf{x}$ on the surface S.

From result (11.22), which is proved in exercise 11.24, we have in general that

$$\int_V (\nabla\times\mathbf{b})\,dV = \oint_S d\mathbf{S}\times\mathbf{b}.$$

For the current application we set $\mathbf{b}$ equal to $\mathbf{u}(\mathbf{x}) = \boldsymbol{\omega}\times\mathbf{x}$, giving

$$\begin{aligned}\oint_S d\mathbf{S}\times\mathbf{u} &= \int_V \nabla\times(\boldsymbol{\omega}\times\mathbf{x})\,dV\\ &= \int_V [\,\boldsymbol{\omega}(\nabla\cdot\mathbf{x}) - \mathbf{x}(\nabla\cdot\boldsymbol{\omega}) + (\mathbf{x}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{x}\,]\,dV\\ &= \int_V [\,3\boldsymbol{\omega} - \mathbf{0} - \mathbf{0} - (\omega_x\,\mathbf{i} + \omega_y\,\mathbf{j} + \omega_z\,\mathbf{k})\,]\,dV\\ &= 3\boldsymbol{\omega}V - \boldsymbol{\omega}V.\end{aligned}$$

To obtain the second line we used the standard identity for $\nabla\times(\mathbf{a}\times\mathbf{b})$ (see table 10.1). Thus,

$$\boldsymbol{\omega} = -\frac{1}{2V}\oint_S \mathbf{u}\times d\mathbf{S}.$$

11.24 *Prove equation (11.22) and, by taking* $\mathbf{b} = zx^2\mathbf{i} + zy^2\mathbf{j} + (x^2 - y^2)\mathbf{k}$, *show that the two integrals*

$$I = \int x^2\, dV \quad \textit{and} \quad J = \int \cos^2\theta \sin^3\theta \cos^2\phi\, d\theta\, d\phi,$$

both taken over the unit sphere, must have the same value. Evaluate both directly to show that the common value is $4\pi/15$.

We have to prove that

$$\int_V (\nabla \times \mathbf{b})\, dV = \oint_S d\mathbf{S} \times \mathbf{b}.$$

Let $\mathbf{a} = \mathbf{b} \times \mathbf{c}$, where $\mathbf{c}$ is an arbitrary but fixed vector. Then, from the divergence theorem

$$\begin{aligned} \int_V \nabla \cdot \mathbf{a} &= \oint_S \mathbf{a} \cdot d\mathbf{S}, \\ \int_V \nabla \cdot (\mathbf{b} \times \mathbf{c}) &= \oint_S (\mathbf{b} \times \mathbf{c}) \cdot d\mathbf{S}, \\ \int_V [\,\mathbf{c} \cdot (\nabla \times \mathbf{b}) - \mathbf{b} \cdot (\nabla \times \mathbf{c})\,]\, dV &= \oint_S (d\mathbf{S} \times \mathbf{b}) \cdot \mathbf{c}. \end{aligned}$$

To obtain this last line we have used a result from table 10.1 and the cyclic property of a triple scalar product. But $\nabla \times \mathbf{c} = \mathbf{0}$ and so

$$\mathbf{c} \cdot \int_V (\nabla \times \mathbf{b})\, dV = \mathbf{c} \cdot \oint_S d\mathbf{S} \times \mathbf{b},$$

and, since $\mathbf{c}$ is also arbitrary, it follows that

$$\int_V (\nabla \times \mathbf{b})\, dV = \oint_S d\mathbf{S} \times \mathbf{b}.$$

With $\mathbf{b} = zx^2\,\mathbf{i} + zy^2\,\mathbf{j} + (x^2 - y^2)\,\mathbf{k}$,

$$\nabla \times \mathbf{b} = (-2y - y^2)\,\mathbf{i} + (x^2 - 2x)\,\mathbf{j}.$$

Clearly, on (anti-) symmetry grounds, $\int x\, dV = \int y\, dV = 0$ for integrals over the unit sphere and so $\int (\nabla \times \mathbf{b})\, dV$ has the form $(-I,\ I,\ 0)$ where $I = \int x^2\, dV = \int y^2\, dV$.

On the surface of the unit sphere, where $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$ and $z = \cos\theta$,

$$\begin{aligned} d\mathbf{S} &= \sin\theta(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k})\, d\theta\, d\phi, \\ \mathbf{b} &= \cos\theta\sin^2\theta\cos^2\phi\,\mathbf{i} + \cos\theta\sin^2\theta\sin^2\phi\,\mathbf{j} + \sin^2\theta\cos 2\phi\,\mathbf{k}. \end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{d\mathbf{S}\times\mathbf{b}}{d\theta\,d\phi} &= \sin\theta(\sin^3\theta\sin\phi\cos 2\phi - \cos^2\theta\sin^2\theta\sin^2\phi)\,\mathbf{i}\\ &\quad + \sin\theta(\cos^2\theta\sin^2\theta\cos^2\phi - \sin^3\theta\cos\phi\cos 2\phi)\,\mathbf{j}\\ &\quad + \sin\theta(\cos\theta\sin^3\theta\cos\phi\sin^2\phi - \sin^3\theta\cos\theta\sin\phi\cos^2\phi)\,\mathbf{k}.\end{aligned}$$

The two terms in the **k**-coordinate cancel each other when integrated over $0 \le \phi < 2\pi$. The first term in the **i**-coordinate can be written as $g(\theta)(\sin 3\phi - \sin\phi)$ and therefore integrates to zero; similarly, the second term in the **j**-coordinate does not contribute.

In summary, the integral $\oint d\mathbf{S}\times\mathbf{b}$ has the form $(-J,\ J,\ 0)$ where

$$J = \iint \cos^2\theta\sin^3\theta\sin^2\phi\,d\theta\,d\phi = \iint \cos^2\theta\sin^3\theta\cos^2\phi\,d\theta\,d\phi.$$

It follows that the integrals I and J defined in the question are equal.

It only remains to evaluate I and J. For I we have

$$\begin{aligned}I &= \int_V x^2\,dV\\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^1 r^2\sin^2\theta\cos^2\phi\, r^2\sin\theta\,dr\,d\phi\,d\theta\\ &= \int_0^{\pi}\int_0^{2\pi}\frac{1}{5}\sin^3\theta\cos^2\phi\,d\phi\,d\theta\\ &= \frac{\pi}{5}\int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta\\ &= \frac{\pi}{5}\left[2-\frac{2}{3}\right] = \frac{4\pi}{15}.\end{aligned}$$

For J the integral is

$$\begin{aligned}J &= \int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^3\theta\sin^2\phi\,d\theta\,d\phi\\ &= \pi\int_0^{\pi}(\cos^2\theta - \cos^4\theta)\sin\theta\,d\theta\\ &= \pi\left[\frac{2}{3}-\frac{2}{5}\right] = \frac{4\pi}{15},\end{aligned}$$

i.e. the same value as I.

11.26 *A vector field* $\mathbf{F}$ *is defined in cylindrical polar coordinates* ρ, θ, z *by*

$$\mathbf{F} = F_0 \left(\frac{x \cos \lambda z}{a} \mathbf{i} + \frac{y \cos \lambda z}{a} \mathbf{j} + (\sin \lambda z) \mathbf{k} \right)$$
$$\equiv \frac{F_0 \rho}{a} (\cos \lambda z) \mathbf{e}_\rho + F_0 (\sin \lambda z) \mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are the unit vectors along the Cartesian axes and* $\mathbf{e}_\rho$ *is the unit vector* $(x/\rho)\mathbf{i} + (y/\rho)\mathbf{j}$.

(a) *Calculate, as a surface integral, the flux of* $\mathbf{F}$ *through the closed surface bounded by the cylinders* $\rho = a$ *and* $\rho = 2a$ *and the planes* $z = \pm a\pi/2$.
(b) *Evaluate the same integral using the divergence theorem.*

(a) The flux through the cylindrical surfaces

$$= \int_{-a\pi/2}^{a\pi/2} \frac{F_0}{a} \cos \lambda z \, dz \left[\int_0^{2\pi} \rho \, \rho \, d\phi \right]_{\rho=a}^{\rho=2a}$$
$$= \int_{-a\pi/2}^{a\pi/2} \frac{2\pi F_0}{a} [\,(2a)^2 - (a)^2\,] \cos \lambda z \, dz$$
$$= 6\pi F_0 a \int_{-a\pi/2}^{a\pi/2} \cos \lambda z \, dz$$
$$= \frac{12\pi F_0 a}{\lambda} \sin \left(\frac{\lambda \pi a}{2} \right).$$

The flux through the planes

$$= \pi(4a^2 - a^2) \left[F_0 \sin \left(\frac{\lambda \pi a}{2} \right) - F_0 \sin \left(-\frac{\lambda \pi a}{2} \right) \right]$$
$$= 6\pi a^2 F_0 \sin \left(\frac{\lambda \pi a}{2} \right).$$

Adding these together gives

$$\text{Total flux } = 6\pi F_0 \left(a^2 + \frac{2a}{\lambda} \right) \sin \left(\frac{\lambda \pi a}{2} \right).$$

(b) Using the Cartesian form

$$\nabla \cdot \mathbf{F} = \frac{F_0}{a} (\cos \lambda z + \cos \lambda z + a\lambda \cos \lambda z),$$

which is independent of ρ. Thus the ρ and ϕ integrations are trivial and the volume integral of the divergence reduces to

$$\begin{aligned}\int_V \nabla\cdot\mathbf{F}\,dV &= \int_{-a\pi/2}^{a\pi/2} \frac{\pi F_0(4a^2-a^2)(2+a\lambda)}{a}\cos\lambda z\,dz\\ &= 3\pi F_0 a(2+a\lambda)\left[\frac{\sin\lambda z}{\lambda}\right]_{-a\pi/2}^{a\pi/2}\\ &= 6\pi F_0\left(\frac{2a}{\lambda}+a^2\right)\sin\left(\frac{\lambda\pi a}{2}\right), \quad \text{as in part (a).}\end{aligned}$$

> ***11.28*** *A vector force field* $\mathbf{F}$ *is defined in Cartesian coordinates by*
>
> $$\mathbf{F} = F_0\left[\left(\frac{y^3}{3a^3}+\frac{y}{a}e^{xy/a^2}+1\right)\mathbf{i}+\left(\frac{xy^2}{a^3}+\frac{x+y}{a}e^{xy/a^2}\right)\mathbf{j}+\frac{z}{a}e^{xy/a^2}\mathbf{k}\right].$$
>
> *Use Stokes' theorem to calculate*
>
> $$\oint_L \mathbf{F}\cdot d\mathbf{r},$$
>
> *where* L *is the perimeter of the rectangle* $ABCD$ *given by* $A=(0,a,0)$, $B=(a,a,0)$, $C=(a,3a,0)$ *and* $D=(0,3a,0)$.

The rectangle $ABCD$ lies in the plane $z=0$ and so to apply Stokes' theorem we need only the z-component of $\nabla\times\mathbf{F}$. this is given by

$$\begin{aligned}(\nabla\times\mathbf{F})_z &= F_0\left(\frac{y^2}{a^3}+\frac{1}{a}e^{xy/a^2}+\frac{(x+y)y}{a^3}e^{xy/a^2}\right)\\ &\quad - F_0\left(\frac{y^2}{a^3}+\frac{1}{a}e^{xy/a^2}+\frac{xy}{a^3}e^{xy/a^2}\right)\\ &= \frac{F_0y^2}{a^3}e^{xy/a^2}.\end{aligned}$$

So, by Stokes' theorem, the line integral has the same value as this component of curl $\mathbf{F}$ integrated over the area of the rectangle, i.e.

$$\oint_L \mathbf{F}\cdot d\mathbf{r} = \int_0^a dx\int_a^{3a} dy\,\frac{F_0y^2}{a^3}e^{xy/a^2}.$$

Since x appears in the integrand only in the form $e^{\lambda x}$, the x-integration is straightforward and is therefore carried out first to give

$$\oint_L \mathbf{F}\cdot d\mathbf{r} = \frac{F_0}{a^3}\int_a^{3a}\frac{a^2\left(e^{y/a}-1\right)y^2}{y}\,dy.$$

After simplification, this can be integrated by parts to yield the final value for the contour integral:

$$\begin{aligned}
\oint_L \mathbf{F}\cdot d\mathbf{r} &= \frac{F_0}{a}\int_a^{3a}(ye^{y/a} - y)\,dy \\
&= \frac{F_0}{a}\left\{\left[aye^{y/a}\right]_a^{3a} - \int_a^{3a} ae^{y/a}\,dy - \left[\frac{y^2}{a}\right]_a^{3a}\right\} \\
&= \frac{F_0}{a}\left(3a^2e^3 - a^2e - a^2e^3 + a^2e - \frac{9}{2}a^2 + \frac{1}{2}a^2\right) \\
&= F_0a(2e^3 - 4).
\end{aligned}$$

12

Fourier series

12.2 *Derive the Fourier coefficients b_r in a similar manner to the derivation of the a_r in section 12.2.*

As explained in the text, the method of proof is almost identical to that given in section 12.2, the only differences being multiplying the Fourier expansion (12.4) through by $\sin(2\pi px/L)$ rather than by $\cos(2\pi px/L)$ and using the first (12.1) and last (12.3) of the orthogonality relations (rather than the first and second).

The given text may be taken as a model solution once these small changes have been allowed for. There is never a b_0 term, formally because $\sin(2\pi rx/L)$ is zero for all x if $r = 0$.

12.4 *By moving the origin of t to the centre of an interval in which $f(t) = +1$, i.e. by changing to a new independent variable $t' = t - \frac{1}{4}T$, express the square-wave function in the example in section 12.2 as a cosine series. Calculate the Fourier coefficients involved* (a) *directly and* (b) *by changing the variable in result (12.8).*

With the change in origin, the function becomes an even one and only cosine terms will be needed. However, the change of origin does not affect the average value of the function, which therefore remains equal to zero. This means that the value of A_0 in the cosine series will also be zero.

(a) By direct calculation:

$$\begin{aligned}
A_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t')\cos\frac{2\pi nt'}{T}\,dt' \\
&= \frac{4}{T}\int_{0}^{T/2} f(t')\cos\frac{2\pi nt'}{T}\,dt' \\
&= \frac{4}{T}\int_{0}^{T/4} f(t')\cos\frac{2\pi nt'}{T}\,dt' - \frac{4}{T}\int_{T/4}^{T/2} f(t')\cos\frac{2\pi nt'}{T}\,dt' \\
&= 4\left[\frac{\sin(2\pi nt'/T)}{2\pi n}\right]_0^{T/4} - 4\left[\frac{\sin(2\pi nt'/T)}{2\pi n}\right]_{T/4}^{T/2} \\
&= \frac{2}{\pi n}\left[\sin\frac{n\pi}{2} - \sin 0 - \sin n\pi + \sin\frac{n\pi}{2}\right] \\
&= \frac{4}{n\pi}(-1)^{(n-1)/2} \text{ for odd } n, \text{ and } = 0 \text{ for even } n.
\end{aligned}$$

(b) By changing the variable in the result for $f(t)$ derived in the text, and writing $2\pi/T$ as ω:

$$\begin{aligned}
f(t) &= \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{\sin n\omega t}{n}, \\
g(t') &= \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{\sin n\omega(t' + \frac{1}{4}T)}{n}, \\
&= \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{1}{n}[\,\sin(n\omega t')\cos(\tfrac{1}{4}n\omega T) + \cos(n\omega t')\sin(\tfrac{1}{4}n\omega T)\,].
\end{aligned}$$

Now, for n odd, $\cos(\frac{1}{4}n\omega T) = \cos(n\pi/2) = 0$ but $\sin(\frac{1}{4}n\omega T) = \sin(n\pi/2) = (-1)^{(n-1)/2}$. Thus, only the $\cos(n\omega t')$ terms survive and

$$g(t') = \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{(-1)^{(n-1)/2}}{n}\cos(n\omega t').$$

This is, as it must be, the same result as that obtained by direct calculation.

12.6 *For the function*

$$f(x) = 1 - x, \qquad 0 \le x \le 1,$$

find (a) *the Fourier sine series and* (b) *the Fourier cosine series. Which would be better for numerical evaluation? Relate your answer to the relevant periodic continuations.*

(a) Sine series. In order to make the function both periodic and odd in x, it must be continued in the range $-1 < x \le 0$ as $f(x) = -1 - x$. The function thus has a discontinuity of 2 at $x = 0$. The Fourier coefficients are

$$\begin{aligned} b_n &= 2\,\frac{2}{2}\int_0^1 (1-x)\sin n\pi x\, dx \\ &= 2\left\{\left[-\frac{\cos n\pi x}{n\pi}\right]_0^1 + \left[\frac{x\cos n\pi x}{n\pi}\right]_0^1 - \int_0^1 \frac{\cos n\pi x}{n\pi}\, dx\right\} \\ &= 2\left(-\frac{(-1)^n - 1}{n\pi} + \frac{(-1)^n}{n\pi} - 0\right) = \frac{2}{n\pi}. \end{aligned}$$

Thus the Fourier sine series for this function is

$$1 - x = f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}.$$

(b) Cosine series. In order to make the function both periodic and even in x, it must be continued in the range $-1 < x \le 0$ as $f(x) = 1 + x$. The function then has no discontinuity at $x = 0$. The Fourier coefficients are

$$\begin{aligned} a_n &= 2\,\frac{2}{2}\int_0^1 (1-x)\cos n\pi x\, dx \\ &= 2\left\{\left[\frac{\sin n\pi x}{n\pi}\right]_0^1 - \left[\frac{x\sin n\pi x}{n\pi}\right]_0^1 + \int_0^1 \frac{\sin n\pi x}{n\pi}\, dx\right\} \\ &= 2\left(0 - 0 + \left[-\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1\right) \\ &= 2\left(-\frac{(-1)^n - 1}{n^2\pi^2}\right) \\ &= \frac{4}{n^2\pi^2} \text{ for } n \text{ odd, and } = 0 \text{ for positive even } n. \end{aligned}$$

For $n = 0$, the non-zero integral is $a_0 = 2\int_0^1 (1-x)\, dx = 1$, making the complete Fourier cosine series representation

$$1 - x = f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n\,\text{odd}}\frac{\cos n\pi x}{n^2}.$$

Because alternate terms (the positive even values of n) are missing and the series converges as n^{-2} (rather than as n^{-1}), it is clear that the cosine series is much superior for calculational purposes. This superiority is reinforced by the lack of a discontinuity in the continued function in case (b); the discontinuity in case (a) will bring additional computational difficulty as a result of the Gibbs' phenomenon.

12.8 *The function $y(x) = x\sin x$ for $0 \le x \le \pi$ is to be represented by a Fourier series of period 2π that is either even or odd. By sketching the function and considering its derivative, determine which series will have the more rapid convergence. Find the full expression for the better of these two series, showing that the convergence $\sim n^{-3}$ and that alternate terms are missing.*

As the period is to be 2π, the question is how to define $y(x)$ in the range $-\pi \le x \le 0$. The even and odd continuations would be

$$y_e(x) = x\sin x \quad \text{and} \quad y_o(x) = -x\sin x.$$

Both continuations make the function continuous at $x = 0$ and at $x = \pm\pi$. However, there is a difference in their derivatives. Both have zero derivative at $x = 0$, but, at $x = -\pi$, $y_e' = \pi$ whilst $y_o' = -\pi$. To avoid a discontinuity in the derivative, the derivative of the continuation must match that of $y(x)$ evaluated at $x = +\pi$. The value of the latter is $-\pi$, and so the odd continuation is the one to be preferred as it will give more rapid convergence and avoid problems arising from the Gibbs' phenomenon.

Thus the series is to be a sine series with

$$\begin{aligned} b_n &= \frac{2}{2\pi}\, 2\int_0^\pi x\sin x \sin nx\, dx \\ &= \frac{1}{\pi}\int_0^\pi x[\,\cos(n-1)x - \cos(n+1)x\,]\, dx. \end{aligned}$$

Now, for integer p, except when $p = 0$,

$$\int_0^\pi x\cos px\, dx = \left[\frac{x\sin px}{p}\right]_0^\pi - \int_0^\pi \frac{\sin px}{p}\, dx = 0 + \left[\frac{\cos px}{p^2}\right]_0^\pi = \frac{(-1)^p - 1}{p^2}.$$

When $p = 0$ the integral has value $\pi^2/2$.
Thus

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[\frac{(-1)^{n-1} - 1}{(n-1)^2} - \frac{(-1)^{n+1} - 1}{(n+1)^2}\right] \\ &= \frac{1}{\pi}\left[\frac{-2}{(n-1)^2} + \frac{2}{(n+1)^2}\right], \text{ for } n \text{ even}, \\ &= -\frac{8n}{\pi(n^2-1)^2}, \text{ for } n \text{ even}, \\ b_n &= 0 - 0 \text{ for } n \text{ odd, except for } n = 1 \end{aligned}$$

When $n = 1$,

$$b_n = \frac{1}{\pi}\int_0^\pi x(1 - \cos 2x)\, dx = \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0\right] = \frac{\pi}{2}.$$

Finally, collecting together the results obtained,

$$y(x) = x\sin x = \frac{\pi}{2}\sin x - \frac{8}{\pi}\sum_{n\,\text{even}} \frac{n}{(n^2-1)^2}\sin nx.$$

The series converges as $n/(n^2)^2 \sim n^{-3}$.

> ***12.10*** *By integrating term by term the Fourier series found in the previous question (exercise 12.9) and using the Fourier series for $f(x) = x$, show that $\int \exp\ x\,dx = \exp\ x + c$. Why is it not possible to show that $d(\exp\ x)/dx = \exp\ x$ by differentiating the Fourier series of $f(x) = \exp\ x$ in a similar manner?*

The series for exp x (found in exercise 12.9) is

$$\exp\ x = (\sinh 1)\left\{1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2}\left[\cos(n\pi x) - n\pi\sin(n\pi x)\right]\right\}.$$

Integrating this term by term gives

$$I = \int \exp\ x\,dx = (\sinh 1)\left\{x + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2}\left[\frac{\sin(n\pi x)}{n\pi} + \cos(n\pi x)\right]\right\}.$$

Now the function x can be expanded in $-1 \le x \le 1$ as a Fourier sine series with

$$b_n = \frac{2}{2}\int_{-1}^{1} x\sin(n\pi x)\,dx = \left[-\frac{x\cos(n\pi x)}{n\pi}\right]_{-1}^{1} + \int_{-1}^{1}\frac{\cos(n\pi x)}{n\pi}\,dx = \frac{2(-1)^{n+1}}{n\pi} + 0.$$

Thus, we may show that I has the form stated in the question as follows:

$$\begin{aligned} I &= 2(\sinh 1)\sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n}{n\pi(1+n^2\pi^2)} + \frac{(-1)^{n+1}}{n\pi}\right)\sin(n\pi x)\right. \\ &\qquad\qquad \left. + \frac{(-1)^n}{1+n^2\pi^2}\cos(n\pi x)\right] \\ &= 2(\sinh 1)\sum_{n=1}^{\infty}\left[\frac{(-1)^n(-n^2\pi^2)}{n\pi(1+n^2\pi^2)}\sin(n\pi x) + \frac{(-1)^n}{1+n^2\pi^2}\cos(n\pi x)\right] \\ &= \exp x - \sinh 1, \text{ by comparison with the original series.} \end{aligned}$$

If the original series is differentiated (with the aim of finding a series to represent the derivative of exp x) it will contain the sum

$$2(\sinh 1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n\pi)^2}{1+n^2\pi^2}\cos(n\pi x).$$

This sum does not converge since the terms do not $\to 0$ as $n \to \infty$. Consequently, no useful result is obtained.

> ***12.12*** *Find, without calculation, which terms will be present in the Fourier series for the periodic functions $f(t)$, of period T, that are given in the range $-T/2$ to $T/2$ by:*
>
> (a) $f(t) = 2$ *for* $0 \le |t| < T/4$, $f = 1$ *for* $T/4 \le |t| < T/2$;
> (b) $f(t) = \exp[-(t - T/4)^2]$;
> (c) $f(t) = -1$ *for* $-T/2 \le t < -3T/8$ *and* $3T/8 \le t < T/2, f(t) = 1$ *for* $-T/8 \le t < T/8$*; the graph of* f *is completed by two straight lines in the remaining ranges so as to form a continuous function.*

If the Fourier series for $f(t)$ is written in the form

$$f(t) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi r t}{L}\right) + b_r \sin\left(\frac{2\pi r t}{L}\right) \right],$$

then the consequences of any symmetry properties that $f(t)$ may possess can be summarised by

- if $f(t)$ is even about $t = 0$ then all $b_r = 0$,
- if $f(t)$ is odd about $t = 0$ then all $a_r = 0$,
- if $f(t)$ is even about $t = T/4$ then $a_{2r+1} = 0$ and $b_{2r} = 0$,
- if $f(t)$ is odd about $t = T/4$ then $a_{2r} = 0$ and $b_{2r+1} = 0$,
- the average value of $f(t)$ over a complete cycle is $\frac{1}{2}a_0$.

Sketching the given functions shows the following.

(a) This is a function that:
(i) is even about $t = 0 \Rightarrow$ no sine terms are present;
(ii) has a non-zero average $\Rightarrow a_0$ $(= 3)$ present;
(iii) is odd about $t = T/4$ once the average value has been subtracted $\Rightarrow$ $a_{2n} = 0$.
Thus the series contains a constant and odd-n cosine terms.

(b) The periodic version of this function does not exhibit symmetry about any value of t; there is a discontinuity of $-(e^{-T^2/16} - e^{-9T^2/16})$ at $t = T/2$. Consequently, all terms are present.

(c) This is a function that:
(i) is even about $t = 0 \Rightarrow$ no sine terms are present;
(ii) has a zero average $\Rightarrow a_0 = 0$;
(iii) is odd about $t = T/4 \Rightarrow a_{2n} = 0$.
Thus the series consists of odd-n cosine terms only.

12.14 *Show that the Fourier series for the function $y(x) = |x|$ in the range $-\pi \le x < \pi$ is*

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}.$$

By integrating this equation term by term from 0 to x, find the function g(x) whose Fourier series is

$$\frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3}.$$

Deduce the value of the sum S of the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots .$$

The function $y(x) = |x|$ is an even function and its Fourier series will therefore contain only cosine terms. They are given, for $n \ge 1$, by

$$\begin{aligned} a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx \\ &= \left\{ \left[\frac{x \sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n}\, dx \right\} \\ &= \frac{2}{\pi}\left[\frac{\cos nx}{n^2} \right]_0^{\pi} \\ &= -\frac{4}{\pi n^2} \text{ for } n \text{ odd, and } = 0 \text{ for } n \text{ even.} \end{aligned}$$

The constant term is $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x\, dx = \pi$. Thus

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}.$$

We now consider the integral of $y(x)$ from 0 to x.

(i) For $x < 0$, $\displaystyle\int_0^x |x'|\, dx' = \int_0^x (-x')\, dx' = -\frac{1}{2}x^2.$

(ii) For $x > 0$, $\displaystyle\int_0^x |x'|\, dx' = \int_0^x x'\, dx' = \frac{1}{2}x^2.$

Integrating the series gives

$$\frac{\pi x}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3}.$$

Equating these two results and isolating the series gives

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3}=\begin{cases}\frac{1}{2}x(\pi+x) & \text{for } x\le 0,\\ \frac{1}{2}x(\pi-x) & \text{for } x\ge 0.\end{cases}$$

If we set $x=\pi/2$ in this result, the sine terms have values $(-1)^m$ and we obtain

$$\frac{1}{2}\frac{\pi}{2}\left(\pi-\frac{\pi}{2}\right)=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3}=\frac{4}{\pi}S.$$

It follows that $S=\pi^3/32$.

12.16 *By finding a cosine Fourier series of period 2 for the function $f(t)$ that takes the form $f(t)=\cosh(t-1)$ in the range $0\le t\le 1$, prove that*

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2+1}=\frac{1}{e^2-1}.$$

Deduce values for the sums $\sum(n^2\pi^2+1)^{-1}$ over odd n and even n separately.

In order to obtain a cosine series we must make an even continuation $f(t)=\cosh(t+1)$ for $-1\le t\le 0$.

The constant term in the series is $a_0/2$ with

$$a_0=2\int_0^1\cosh(t-1)\,dt=[\,2\sinh(t-1)\,]_0^1=2\sinh(1).$$

The general coeeficient is

$$\begin{aligned}a_n&=2\int_0^1\cosh(t-1)\cos(n\pi t)\,dt\\&=2\left[\frac{\cosh(t-1)\sin(n\pi t)}{n\pi}\right]_0^1-2\int_0^1\frac{\sinh(t-1)\sin(n\pi t)}{n\pi}\,dt\\&=0-2\left[-\frac{\sinh(t-1)\cos(n\pi t)}{n^2\pi^2}\right]_0^1-2\int_0^1\frac{\cosh(t-1)\cos(n\pi t)}{n^2\pi^2}\,dt\end{aligned}$$

Hence,

$$a_n\left(1+\frac{1}{n^2\pi^2}\right)=-\frac{2\sinh(-1)}{n^2\pi^2}\quad\Rightarrow\quad a_n=\frac{2\sinh(1)}{n^2\pi^2+1}.$$

The Fourier expansion for $f(t)$ is thus

$$\cosh(t-1)=\sinh(1)\left(1+2\sum_{n=1}^{\infty}\frac{\cos n\pi t}{1+n^2\pi^2}\right).$$

Setting $t = 0$ gives

$$\cosh(1) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2} \qquad (*)$$

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2} = \frac{\cosh(1) - \sinh(1)}{2\sinh(1)}$$

$$= \frac{e^{-1}}{e - e^{-1}} = \frac{1}{e^2 - 1}.$$

Now, to separate the contributions to the series from the odd and the even integers, we need an extra factor of $(-1)^n$ in each term. We get this, in the form $\cos n\pi$, by setting $t = 1$ and so obtain

$$\cosh(0) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2} \qquad (**).$$

Adding $(*)$ and $(**)$ gives, with an obvious notation,

$$\frac{\cosh(1) + \cosh(0)}{\sinh(1)} = 2 + 4\,\Sigma_{\text{even}}$$

Re-arrangement and substitution of explicit expressions for the hyperbolic sinusoids give

$$\Sigma_{\text{even}} = \frac{1}{4}\,\frac{\frac{1}{2}(e + e^{-1}) + 1 - (e - e^{-1})}{\frac{1}{2}(e - e^{-1})}$$

$$= \frac{3 - e^2 + 2e}{4(e^2 - 1)} = \frac{(3 - e)(1 + e)}{4(e - 1)(e + 1)} = \frac{3 - e}{4(e - 1)}.$$

It then follows that

$$\Sigma_{\text{odd}} = \Sigma_{\text{all}} - \Sigma_{\text{even}}$$

$$= \frac{1}{e^2 - 1} - \frac{3 - e}{4(e - 1)} = \frac{4 - 2e + e^2 - 3}{4(e^2 - 1)}$$

$$= \frac{(1 - e)^2}{4(e^2 - 1)} = \frac{e - 1}{4(e + 1)}.$$

12.18 *Express the function $f(x) = x^2$ as a Fourier sine series in the range $0 < x \leq 2$ and show that it converges to zero at $x = \pm 2$.*

To ensure a sine series we take $f(x) = -x^2$ for $-2 < x \leq 0$. This means that $f(-2) = -4$ and so, since $f(2) = +4$, we expect the series to converge to the average value of $\frac{1}{2}(-4 + 4) = 0$ at $x = \pm 2$.

The coefficients in the sine series $\sum b_n \sin(n\pi x/2)$ are

$$b_n = 2\,\frac{2}{4}\int_0^2 x^2 \sin\frac{n\pi x}{2}\,dx.$$

Setting $n\pi x/2 = y$ gives $b_n = 8I_n/(n\pi)^3$ with

$$\begin{aligned}
I_n &= \int_0^{n\pi} y^2 \sin y\,dy \\
&= \left[-y^2\cos y\right]_0^{n\pi} + \int_0^{n\pi} 2y\cos y\,dy \\
&= (-1)^{n+1}n^2\pi^2 + [\,2y\sin y\,]_0^{n\pi} - \int_0^{n\pi} 2\sin y\,dy \\
&= (-1)^{n+1}n^2\pi^2 + 0 + [\,2\cos y\,]_0^{n\pi} \\
&= (-1)^{n+1}n^2\pi^2 + 2(-1)^n - 2.
\end{aligned}$$

Thus,

$$\begin{aligned}
b_n &= \frac{(-1)^{n+1}8}{n\pi} - \frac{32}{n^3\pi^3} \text{ for } n \text{ odd}, \\
&= \frac{(-1)^{n+1}8}{n\pi} \text{ for } n \text{ even}.
\end{aligned}$$

For $x = \pm 2$ all terms in the series are zero and so this is the value of the expansion at these points. This is not simply as expected, but inevitable, because, for a pure Fourier sine series, the arguments of all the sine functions are bound to be of the form $n\pi$ at the end points of the period.

12.20 *Show that the Fourier series for* $|\sin\theta|$ *in the range* $-\pi \le \theta \le \pi$ *is given by*

$$|\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2m\theta}{4m^2-1}.$$

By setting $\theta = 0$ *and* $\theta = \pi/2$, *deduce values for*

$$\sum_{m=1}^{\infty}\frac{1}{4m^2-1} \quad \text{and} \quad \sum_{m=1}^{\infty}\frac{1}{16m^2-1}.$$

This is an even function about $\theta = 0$ with cosine coefficient a_0 given by

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi}|\sin\theta|\,d\theta = \frac{1}{\pi}\,2\,[\,-\cos\theta\,]_0^{\pi} = \frac{4}{\pi}.$$

For $n \geq 1$, we have

$$\begin{aligned}
a_n &= \frac{2}{2\pi}\, 2 \int_0^{\pi} \sin\theta \cos n\theta \, d\theta \\
&= \frac{2}{\pi} \int_0^{\pi} \tfrac{1}{2}[\,\sin(n+1)\theta - \sin(n-1)\theta\,]\, d\theta \\
&= \frac{1}{\pi}\left[-\frac{\cos(n+1)\theta}{n+1} + \frac{\cos(n-1)\theta}{n-1}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}-1}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right]_0^{\pi} \\
&= -\frac{4}{\pi}\frac{1}{n^2-1} \text{ if } n \text{ is even, and } = 0 \text{ if } n \text{ is odd.}
\end{aligned}$$

Hence, writing $n = 2m$,

$$|\sin\theta\,| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2m\theta}{4m^2-1}.$$

Now, setting $\theta = 0$ yields

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$

Setting $\theta = \pi/2$, instead, gives

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}.$$

Adding this to the previous result then yields

$$\begin{aligned}
1 &= \frac{4}{\pi} - \frac{4}{\pi}\, 2 \sum_{n\,\text{even}} \frac{1}{4n^2-1} \\
&= \frac{4}{\pi} - \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{1}{16m^2-1}, \\
\Rightarrow \quad \sum_{m=1}^{\infty}\frac{1}{16m^2-1} &= \frac{\pi}{8}\left(\frac{4}{\pi} - 1\right) = \frac{1}{2} - \frac{\pi}{8}.
\end{aligned}$$

12.22 *The repeating output from an electronic oscillator takes the form of a sine wave $f(t) = \sin t$ for $0 \le t \le \pi/2$; it then drops instantaneously to zero and starts again. The output is to be represented by a complex Fourier series of the form*

$$\sum_{n=-\infty}^{\infty} c_n e^{4nti}.$$

Sketch the function and find an expression for c_n. Verify that $c_{-n} = c_n^*$. Demonstrate that setting $t = 0$ and $t = \pi/2$ produces differing values for the sum

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}.$$

Determine the correct value and check it using the quoted result of exercise 12.20.

As the period of the expansion is to be $\pi/2$ the complex expansion coefficients are given by

$$\begin{aligned}\frac{\pi}{2}c_n &= \int_0^{\pi/2} \sin t\, e^{-i4nt} \\ &= \left[\frac{\sin t\, e^{-i4nt}}{-4ni}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos t\, e^{-i4nt}}{-4ni}\, dt \\ &= \frac{1}{-4ni} + \left[\frac{\cos t\, e^{-i4nt}}{4ni(-4ni)}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\sin t\, e^{-i4nt}}{4ni(-4ni)}\, dt.\end{aligned}$$

It follows that

$$\frac{\pi}{2}c_n\left(1 - \frac{1}{16n^2}\right) = \frac{i}{4n} - \frac{1}{16n^2},$$

and hence that

$$c_n = \frac{2}{\pi}\frac{4ni - 1}{16n^2 - 1}.$$

It is obvious that

$$c_{-n}^* = \frac{2}{\pi}\frac{4(-n)(-i) - 1}{16n^2 - 1} = c_n.$$

The series representation is therefore

$$\sin(t) = \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{4ni - 1}{16n^2 - 1}\, e^{i4nt}. \qquad (*)$$

Setting $t = 0$, equating real parts, noting that the $n = 0$ term has value -1 and that the $\pm n$-terms are equal for $n \neq 0$, together imply that

$$\sum_{n=-\infty}^{\infty} \frac{1}{16n^2 - 1} = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{16n^2 - 1} = \frac{1}{2}.$$

But, setting $t = \pi/2$ and then equating real parts implies that

$$\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{-1}{16n^2 - 1} = 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{16n^2 - 1} = -\frac{1}{2}\frac{\pi}{2}\left(1 - \frac{2}{\pi}\right) = \frac{1}{2} - \frac{\pi}{4}.$$

These results are clearly contradictory.

As there is a discontinuity in the function, the correct value is the mean of these two, namely $\frac{1}{2} - \frac{\pi}{8}$. This is the value obtained in solution 12.20.

12.24 *A string, anchored at $x = \pm L/2$, has a fundamental vibration frequency of $2L/c$, where c is the speed of transverse waves on the string. It is pulled aside at its centre point by a distance y_0 and released at time $t = 0$. Its subsequent motion can be described by the series*

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$

Find a general expression for a_n and show that only odd harmonics of the fundamental frequency are present in the sound generated by the released string. By applying Parseval's theorem, find the sum S of the series $\sum_0^\infty (2m+1)^{-4}$.

Since only cosine terms are present and the spatial terms have the form $\cos(n\pi x/L)$, the period of the continued function is $2L$. We take its forms beyond $x = \pm L/2$ as continuing that set by the physical string; they have the common value $-y_0$ at $x = \pm L$.

The values of the a_n are set by the initial displacement $y(x,0)$ and given by (writing π/L as k)

$$\begin{aligned}
a_n &= \frac{2}{2L}\int_{-L}^{L} y(x,0)\cos nkx\, dx \\
&= \frac{2y_0}{L}\int_0^L \left(1 - \frac{2x}{L}\right)\cos nkx\, dx \\
&= \frac{2y_0}{L}\left[\frac{\sin nkx}{nk}\right]_0^L - \frac{4y_0}{L^2}\left\{\left[\frac{x\sin nkx}{nk}\right]_0^L - \int_0^L \frac{\sin nkx}{nk}\, dx\right\} \\
&= \frac{4y_0}{L^2}\left[\frac{-\cos nkx}{n^2k^2}\right]_0^L \\
&= \frac{8y_0}{n^2\pi^2} \text{ for } n \text{ odd, and } = 0 \text{ for } n \text{ even.}
\end{aligned}$$

This shows that only the odd harmonics are present. Because of the presence of n in the denominators of several expressions in the above calculation, the case $n = 0$ needs to be considered separately; however, it is clear that the average value of the continued function is zero and so $a_0 = 0$.

In order to apply Parseval's theorem we need to evaluate both the integral over one period of the square of the magnitude of the function, and the sum of the squares of the magnitudes of its Fourier coefficients:

(i)
$$\frac{1}{2L}\int_{-L}^{L} |y(x,0)|^2\,dx = \frac{4y_0^2}{2L}\int_0^{L/2}\left(1-\frac{2x}{L}\right)^2\,dx$$
$$= \frac{2y_0^2}{L}\left[\frac{1}{3}\left(\frac{-L}{2}\right)\left(1-\frac{2x}{L}\right)^3\right]_0^{L/2}$$
$$= \frac{1}{3}y_0^2.$$

(ii)
$$\frac{1}{2}\sum_{n=1}^{\infty} a_n^2 = \frac{1}{2}\sum_{n\,\text{odd}}\frac{64y_0^2}{n^4\pi^4} = \frac{32y_0^2}{\pi^4}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}.$$

Equating these two expressions shows that

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4} = \frac{\pi^4}{96}.$$

12.26 *An odd function $f(x)$ of period 2π is to be approximated by a Fourier sine series having only m terms. The error in this approximation is measured by the square deviation*

$$E_m = \int_{-\pi}^{\pi}\left[f(x) - \sum_{n=1}^{m} b_n \sin nx\right]^2\,dx.$$

By differentiating E_m with respect to the coefficients b_n, find the values of b_n that minimise E_m.

Sketch the graph of the function $f(x)$, where

$$f(x) = \begin{cases} -x(\pi + x) & \text{for } -\pi \le x < 0, \\ x(x-\pi) & \text{for } 0 \le x < \pi. \end{cases}$$

$f(x)$ is to be approximated by the first three terms of a Fourier sine series. What coefficients minimise E_3? What is the resulting value of E_3?

We minimise E_m by differentiating it with respect to b_j and setting the partial derivative equal to zero.

$$\begin{aligned}
E_m &= \int_{-\pi}^{\pi} \left[f(x) - \sum_{n=1}^{m} b_n \sin nx \right]^2 dx, \\
\frac{\partial E_m}{\partial b_j} &= \int_{-\pi}^{\pi} 2 \left[f(x) - \sum_{n=1}^{m} b_n \sin nx \right] \sin jx \, dx, \\
0 &= 2 \int_{-\pi}^{\pi} f(x) \sin jx \, dx - 2 \sum_{n=1}^{m} b_n \frac{1}{2} 2\pi \, \delta_{jn}, \\
\Rightarrow \quad b_j &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(jx) \, dx,
\end{aligned}$$

i.e. b_j is the usual Fourier coefficient.

The function defined by

$$f(x) = \begin{cases} -x(\pi + x) & \text{for } -\pi \le x < 0, \\ x(x - \pi) & \text{for } 0 \le x < \pi, \end{cases}$$

is an odd function in x and therefore has a pure sine series Fourier expansion. The expansion coefficients are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(x - \pi) \sin nx \, dx.$$

Integrating by parts, we obtain

$$\begin{aligned}
\frac{\pi b_n}{2} &= \int_0^{\pi} x^2 \sin nx \, dx - \int_0^{\pi} \pi x \sin nx \, dx \\
&= \left[\frac{-x^2 \cos nx}{n} \right]_0^{\pi} + \int_0^{\pi} \frac{2x \cos nx}{n} \, dx \\
&\quad - \left[\frac{-\pi x \cos nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\pi \cos nx}{n} \, dx \\
&= \frac{(-1)^{n+1} \pi^2}{n} + \left[\frac{2x \sin nx}{n^2} \right]_0^{\pi} - \int_0^{\pi} \frac{2 \sin nx}{n^2} \, dx + \frac{(-1)^{n+2} \pi^2}{n} + 0 \\
&= 0 + \frac{2}{n^2} \left[\frac{\cos nx}{n} \right]_0^{\pi},
\end{aligned}$$

i.e.

$$b_n = -\frac{8}{\pi n^3} \text{ for } n \text{ odd, and } = 0 \text{ for } n \text{ even.}$$

Thus, the minimising coefficients are

$$b_1 = -\frac{8}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{8}{27\pi}.$$

As the full Fourier series reproduces $f(x)$ accurately, the error E_3 using these three calculated coefficients must be

$$\begin{aligned} E_3 &= \int_{-\pi}^{\pi} \sum_{\text{odd } n=5}^{\infty} \left(-\frac{8}{\pi n^3} \sin nx\right)^2 dx \\ &= \sum_{\text{odd } n=5}^{\infty} \frac{64}{\pi^2 n^6} \frac{1}{2} 2\pi \\ &= \frac{64}{\pi} \sum_{m=2}^{\infty} \frac{1}{(2m+1)^6}. \end{aligned}$$

13

Integral transforms

13.2 *Use the general definition and properties of Fourier transforms to show the following.*

(a) *If $f(x)$ is periodic with period a then $\tilde{f}(k) = 0$ unless $ka = 2\pi n$ for integer n.*
(b) *The Fourier transform of $tf(t)$ is $id\tilde{f}(\omega)/d\omega$.*
(c) *The Fourier transform of $f(mt + c)$ is*

$$\frac{e^{i\omega c/m}}{m}\tilde{f}\left(\frac{\omega}{m}\right).$$

(a) As f is periodic with period a,

$$f(x) = f(x - ma),$$

for any integer m. However, from the general translation property of Fourier transforms,

$$\tilde{f}(k) = \mathscr{F}\left[f(x)\right] = \mathscr{F}\left[f(x - ma)\right] = e^{-imka}\tilde{f}(k).$$

Thus

$$0 = \tilde{f}(k)(1 - e^{-imka}),$$

implying, in the particular case $m = 1$, that either $\tilde{f}(k) = 0$ or $ka = 2\pi n$ where n is an integer.

(b) This result is immediate, since differentiating under the integral sign gives

$$i\frac{d\tilde{f}(\omega)}{d\omega} = \frac{i}{\sqrt{2\pi}}\frac{\partial}{\partial\omega}\left(\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} tf(t)\, e^{-i\omega t}\, dt.$$

(c) From the definition of a Fourier transform,

$$\mathscr{F}\left[f(mt+c)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(mt+c)\,e^{-i\omega t}\,dt.$$

We make a change of integration variable by setting $mt + c = u$, with $dt = du/m$ and $-\infty < u < \infty$. This yields

$$\begin{aligned}\mathscr{F}\left[f(mt+c)\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i\omega(u-c)/m}\,\frac{du}{m}\\ &= \frac{e^{i\omega c/m}}{m}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i(\omega/m)u}\,du\\ &= \frac{e^{i\omega c/m}}{m}\,\tilde{f}\left(\frac{\omega}{m}\right),\end{aligned}$$

as stated in the question.

13.4 *Prove that the Fourier transform of the function $f(t)$ defined in the tf-plane by straight-line segments joining $(-T,0)$ to $(0,1)$ to $(T,0)$, with $f(t)=0$ outside $|t|<T$, is*

$$\tilde{f}(\omega) = \frac{T}{\sqrt{2\pi}}\text{sinc}^2\left(\frac{\omega T}{2}\right),$$

where sinc x is defined as $(\sin x)/x$.

Use the general properties of Fourier transforms to determine the transforms of the following functions, graphically defined by straight-line segments and equal to zero outside the ranges specified:

(a) $(0,0)$ *to* $(0.5,1)$ *to* $(1,0)$ *to* $(2,2)$ *to* $(3,0)$ *to* $(4.5,3)$ *to* $(6,0)$;
(b) $(-2,0)$ *to* $(-1,2)$ *to* $(1,2)$ *to* $(2,0)$;
(c) $(0,0)$ *to* $(0,1)$ *to* $(1,2)$ *to* $(1,0)$ *to* $(2,-1)$ *to* $(2,0)$.

The function $f(t)$ is given algebraically by

$$f(t) = \begin{cases} 1+\dfrac{t}{T} & \text{for } -T \le t \le 0,\\ 1-\dfrac{t}{T} & \text{for } 0 \le t \le T.\end{cases}$$

Its Fourier transform is therefore

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-T}^{0}\left(1+\frac{t}{T}\right)e^{-i\omega t}\,dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{T}\left(1-\frac{t}{T}\right)e^{-i\omega t}\,dt.$$

Setting $t = -u$ in the first integral and $t = u$ in the second yields

$$\begin{aligned}
\sqrt{2\pi}\tilde{f}(\omega) &= \int_T^0 \left(1 - \frac{u}{T}\right) e^{i\omega u}\,(-du) + \int_0^T \left(1 - \frac{u}{T}\right) e^{-i\omega u}\,du \\
&= \int_0^T \left(1 - \frac{u}{T}\right) 2\cos\omega u\,du \\
&= 2\left[\frac{\sin\omega u}{\omega}\right]_0^T - 2\left[\frac{u\sin\omega u}{\omega T}\right]_0^T + \frac{2}{T}\int_0^T \frac{\sin\omega u}{\omega}\,du \\
&= \frac{2\sin\omega T}{\omega} - 0 - \frac{2\sin\omega T}{\omega} + 0 + \frac{2}{T}\left[\frac{-\cos\omega u}{\omega^2}\right]_0^T \\
&= \frac{2}{T\omega^2}(1 - \cos\omega T) = \frac{4\sin^2(\frac{1}{2}\omega T)}{T\omega^2}, \\
\tilde{f}(\omega) &= \frac{T}{\sqrt{2\pi}}\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right),
\end{aligned}$$

where $\mathrm{sinc}(x) = \sin x/x$.

In addition to straightforward scaling, two of the other properties of Fourier transforms that are available are (i) $\mathscr{F}\left[f(t+a)\right] = e^{ia\omega}\tilde{f}(\omega)$ and (ii) $\mathscr{F}\left[f'(t)\right] = i\omega\tilde{f}(\omega)$.

(a) This function consists of three segments of the same shape as $f(x)$, but with each one scaled and shifted. The first segment is centred on $t = \frac{1}{2}$ and has $T = \frac{1}{2}$; its contribution to the transform is therefore

$$e^{-i\omega/2}\frac{1}{2\sqrt{2\pi}}\mathrm{sinc}^2\left(\frac{\omega}{4}\right).$$

The second segment is scaled by a factor of 2, is centred on $t = 2$ and has $T = 1$. The third is scaled by a factor of 3, is centred on $t = \frac{9}{2}$ and has $2T = 3$. The full function therefore has as its Fourier transform

$$\frac{1}{\sqrt{2\pi}}\left[e^{-i\omega/2}\frac{1}{2}\mathrm{sinc}^2\left(\frac{\omega}{4}\right) + e^{-i2\omega}2\mathrm{sinc}^2\left(\frac{\omega}{2}\right) + e^{-i9\omega/2}\frac{9}{2}\mathrm{sinc}^2\left(\frac{3\omega}{4}\right)\right]$$

$$= \frac{8}{\sqrt{2\pi}\omega^2}\left[e^{-i\omega/2}\sin^2\left(\frac{\omega}{4}\right) + e^{-i2\omega}\sin^2\left(\frac{\omega}{2}\right) + e^{-i9\omega/2}\sin^2\left(\frac{3\omega}{4}\right)\right]$$

(b) This function could be considered as the superposition of a 'triangle' of height 2 with $T = 2$ and two other triangles, each of unit height with $T = 1$, displaced from the first by ± 1. Alternatively, it could be considered as the difference between two 'triangles' centred on $t = 0$, one of height 4 with $T = 2$ and the other of height 2 with $T = 1$. Necessarily, both approaches give the same answer. Using the second,

$$\begin{aligned}\tilde{f}_2(\omega) &= \frac{8}{\sqrt{2\pi}}\text{sinc}^2\omega - \frac{2}{\sqrt{2\pi}}\text{sinc}^2\left(\frac{\omega}{2}\right)\\ &= \frac{8}{\sqrt{2\pi}\omega^2}\sin^2\left(\frac{\omega}{2}\right)\left[4\cos^2\left(\frac{\omega}{2}\right)-1\right]\\ &= \frac{8}{\sqrt{2\pi}\omega^2}\sin^2\left(\frac{\omega}{2}\right)(2\cos\omega+1).\end{aligned}$$

(c) This function can be viewed as the superposition of a 'triangle' with $T = 1$ centred on $t = 1$ and one cycle of a unit square-wave function, also centred on $t = 1$. But, the unit square-wave function is exactly the derivative of the triangle function, i.e. $+1$ for $0 \le t \le 1$ and -1 for $1 \le t \le 2$. If the complete function were centred on $t = 0$, its Fourier transform would be

$$\tilde{f}_3(\omega) = \mathscr{F}\left[f(t)\right] + \mathscr{F}\left[f'(t)\right] = \tilde{f}(\omega) + i\omega\tilde{f}(\omega).$$

However, it is centred on $t = 1$ and so an extra factor of $e^{-i\omega}$ has to be included to give

$$\tilde{f}_3(\omega) = \frac{(1+i\omega)\,e^{-i\omega}}{\sqrt{2\pi}}\text{sinc}^2\left(\frac{\omega}{2}\right).$$

13.6 *By differentiating the definition of the Fourier sine transform $\tilde{f}_s(\omega)$ of the function $f(t) = t^{-1/2}$ with respect to ω, and then integrating the resulting expression by parts, find an elementary differential equation satisfied by $\tilde{f}_s(\omega)$. Hence show that this function is its own Fourier sine transform, i.e. $\tilde{f}_s(\omega) = Af(\omega)$, where A is a constant. Show that it is also its own Fourier cosine transform. Assume that the limit as $x \to \infty$ of $x^{1/2}\sin\alpha x$ can be taken as zero.*

Starting from the definition

$$\tilde{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty t^{-1/2}\sin\omega t\,dt,$$

and then differentiating under the integral sign, we have

$$\begin{aligned}\frac{d\tilde{f}_s(\omega)}{d\omega} &= \sqrt{\frac{2}{\pi}}\int_0^\infty t^{-1/2}\,t\,\cos\omega t\,dt,\\ &= \sqrt{\frac{2}{\pi}}\left[\frac{t^{1/2}\sin\omega t}{\omega}\right]_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{2}t^{-1/2}\frac{\sin\omega t}{\omega}\,dt\\ &= 0 - 0 - \frac{1}{2\omega}\tilde{f}_s(\omega).\end{aligned}$$

The Fourier sine transform therefore satisfies the differential equation

$$\frac{d\tilde{f}_s(\omega)}{d\omega} + \frac{1}{2\omega}\tilde{f}_s(\omega) = 0 \quad \Rightarrow \quad \tilde{f}_s(\omega) = \frac{A}{\omega^{1/2}} = Af(\omega).$$

The Fourier cosine transform *must* behave in exactly the same way, although the constant A could be different, and the details will not be worked out here.

13.8 *Calculate the Fraunhofer spectrum produced by a diffraction grating, uniformly illuminated by light of wavelength $2\pi/k$, as follows. Consider a grating with $4N$ equal strips each of width a and alternately opaque and transparent. The aperture function is then*

$$f(y) = \begin{cases} A & \text{for } (2n+1)a \le y \le (2n+2)a, \quad -N \le n < N, \\ 0 & \text{otherwise.} \end{cases}$$

(a) *Show, for diffraction at angle θ to the normal to the grating, that the required Fourier transform can be written*

$$\tilde{f}(q) = (2\pi)^{-1/2} \sum_{r=-N}^{N-1} \exp(-2iarq) \int_a^{2a} A \exp(-iqu)\, du,$$

where $q = k \sin\theta$.

(b) *Evaluate the integral and sum to show that*

$$\tilde{f}(q) = (2\pi)^{-1/2} \exp(-iqa/2) \frac{A \sin(2qaN)}{q \cos(qa/2)},$$

and hence that the intensity distribution $I(\theta)$ in the spectrum is proportional to

$$\frac{\sin^2(2qaN)}{q^2 \cos^2(qa/2)}.$$

(c) *For large values of N, the numerator in the above expression has very closely spaced maxima and minima as a function of θ and effectively takes its mean value, $1/2$, giving a low-intensity background. Much more significant peaks in $I(\theta)$ occur when $\theta = 0$ or the cosine term in the denominator vanishes. Show that the corresponding values of $|\tilde{f}(q)|$ are*

$$\frac{2aNA}{(2\pi)^{1/2}} \quad \text{and} \quad \frac{4aNA}{(2\pi)^{1/2}(2m+1)\pi} \quad \text{with } m \text{ integral.}$$

Note that the constructive interference makes the maxima in $I(\theta) \propto N^2$, not N. Of course, observable maxima only occur for $0 \le \theta \le \pi/2$.

(a) and (b) The required Fourier transform is given by

$$\begin{aligned}
\sqrt{2\pi}\tilde{f}(q) &= \int_{-\infty}^{\infty} f(y)\, e^{-iqy}\, dy \\
&= \sum_{n=-N}^{N-1} \int_{(2n+1)a}^{(2n+2)a} A\, e^{-iqy}\, dy, \text{ set } y = 2na + u, \\
&= \sum_{n=-N}^{N-1} e^{-iq2na} \int_{a}^{2a} A\, e^{-iqu}\, du \\
&= e^{iq2Na} \frac{1 - e^{-iq4Na}}{1 - e^{-iq2a}} \frac{A(e^{-iq2a} - e^{-iqa})}{-iq} \\
&= \frac{2i\sin(2qaN)}{e^{-iqa}2i\sin(qa)} \frac{e^{-i3qa/2}2iA[-\sin(qa/2)]}{-iq} \\
&= e^{-iqa/2} \frac{\sin(2qaN)}{2\sin(qa/2)\cos(qa/2)} \frac{2A\sin(qa/2)}{q}, \\
\tilde{f}(q) &= \frac{1}{\sqrt{2\pi}} e^{-iqa/2} \frac{A\sin(2qaN)}{q\cos(qa/2)}.
\end{aligned}$$

The intensity distribution is proportional to the squared modulus of this, i.e to

$$\frac{\sin^2(2qaN)}{q^2\cos^2(qa/2)}.$$

(c) For the significant peaks:

(i) At $\theta = 0$, when $q = k\sin\theta = 0$. Using the fact that for small ϕ $\sin\phi \approx \phi$,

$$|\tilde{f}(q)| = \frac{1}{\sqrt{2\pi}} \frac{A2qaN}{q\,1} = \frac{2aNA}{\sqrt{2\pi}}.$$

(ii) When $qa/2 = (m + \frac{1}{2})\pi$ with m an integer, i.e. $ka\sin\theta = (2m+1)\pi$. The modulus of the transform becomes

$$|\tilde{f}(q)| = \frac{1}{\sqrt{2\pi}} \frac{A\sin[N(4m+2)\pi]}{a^{-1}(2m+1)\pi\cos[(m+\frac{1}{2})\pi]}.$$

This has the form $0/0$ and is indeterminate. To evaluate the ratio we set $qa = \psi$ and determine the limit of the ratio as $\psi \to (2m+1)\pi$ using l'Hôpital's rule.

$$\begin{aligned}
\sqrt{2\pi}|\tilde{f}| &= \left| \frac{Aa\sin(2\psi N)}{\psi\cos(\psi/2)} \right| \\
&= \left| \frac{2NAa\cos(2\psi N)}{\cos(\psi/2) + \frac{1}{2}\psi\sin(\psi/2)} \right| \\
&= \left| \frac{2NAa\,1}{\frac{1}{2}(2m+1)\pi(-1)^m} \right|, \\
|\tilde{f}| &= \frac{4NaA}{\sqrt{2\pi}(2m+1)\pi}.
\end{aligned}$$

13.10 *In many applications in which the frequency spectrum of an analogue signal is required, the best that can be done is to sample the signal $f(t)$ a finite number of times at fixed intervals and then use a* discrete Fourier transform F_k *to estimate discrete points on the (true) frequency spectrum $\tilde{f}(\omega)$.*

(a) *By an argument that is essentially the converse of that given in section 13.1, show that, if N samples f_n, beginning at $t = 0$ and spaced τ apart, are taken, then $\tilde{f}(2\pi k/(N\tau)) \approx F_k\tau$ where*

$$F_k = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f_n e^{-2\pi nki/N}.$$

(b) *For the function $f(t)$ defined by*

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t < 1, \\ 0 & \text{otherwise,} \end{cases}$$

from which eight samples are drawn at intervals of $\tau = 0.25$, find a formula for $|F_k|$ and evaluate it for $k = 0, 1, \ldots, 7$.

(c) *Find the exact frequency spectrum of $f(t)$ and compare the actual and estimated values of $\sqrt{2\pi}|\tilde{f}(\omega)|$ at $\omega = k\pi$ for $k = 0, 1, \ldots, 7$. Note the relatively good agreement for $k < 4$ and the lack of agreement for larger values of k.*

(a) With the exact definition of the Fourier transform of $f(t)$ (taken as zero for $t < 0$) being given by the integral

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(t)\, e^{-i\omega t}\, dt,$$

we approximate it with the sum of the areas of a series of rectangles. Each has width τ but the height of the nth is determined by the sample value f_n.

$$\tilde{f}(\omega) \approx \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f_n \tau\, e^{-i\omega n\tau}.$$

For the sample frequencies $\omega = 2\pi k/(N\tau)$ this gives the estimated spectrum values as

$$\tilde{f}\left(\frac{2\pi k}{N\tau}\right) \approx \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f_n \tau\, e^{-i2\pi nk/N} \equiv F_k \tau.$$

(b) The eight sample values are $f_n = 1$ for $n =$ 0, 1, 2 and 3, and $f_n = 0$ for $n =$ 4, 5, 6 and 7.

With $k = 0$, $\sqrt{2\pi}F_0 = 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 4.$

For a more general value of k

$$
\begin{aligned}
F_k &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{3} e^{-i2\pi nk/8} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1 - e^{-i4(k\pi/4)}}{1 - e^{-ik\pi/4}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{e^{ik\pi/8}[\,1 - (-1)^k\,]}{e^{ik\pi/8} - e^{-ik\pi/8}}.
\end{aligned}
$$

To obtain the last line we have used $e^{ik\pi} = (-1)^k$ and arranged for the final expression to be real apart from a single overall phase factor. It follows that the modulus of F_k is given by

$$
|F_k| = \frac{1}{\sqrt{2\pi}} \frac{1}{\sin k\pi/8} \qquad \text{for odd } k
$$

and is equal to zero for even k.

(c) The exact frequency spectrum of $f(t)$ is

$$
\begin{aligned}
\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_0^1 \\
&= \frac{e^{-i\omega/2}}{\sqrt{2\pi}} \frac{2\sin(\omega/2)}{\omega} \\
&= \frac{e^{-i\omega/2}}{\sqrt{2\pi}} \operatorname{sinc}\frac{\omega}{2}.
\end{aligned}
$$

Noting that $\tau = \frac{1}{4}$, giving $\omega = 2\pi k/(8 \times 0.25)$, the comparison of the magnitudes of the exact values $\sqrt{2\pi}\tilde{f}(\omega)$ and the estimated values $\sqrt{2\pi}F_k\tau$ is

$\omega = k\pi$	=	0	π	2π	3π	4π	5π	6π	7π
$\sqrt{2\pi}\lvert\tilde{f}(\omega)\rvert$	=	1	$\frac{2}{\pi}$	0	$\frac{2}{3\pi}$	0	$\frac{2}{5\pi}$	0	$\frac{2}{7\pi}$
	=	1	0.637	0	0.212	0	0.127	0	0.091
$\sqrt{2\pi}\lvert F_k\tau\rvert$	=	1	0.653	0	0.271	0	0.271	0	0.653

The lack of agreement for the higher frequencies ($k > 4$) is obvious.

13.12 *A signal obtained by sampling a function $x(t)$ at regular intervals T is passed through an electronic filter, whose response $g(t)$ to a unit δ-function input is represented in a tg-plot by straight lines joining $(0,0)$ to $(T,1/T)$ to $(2T,0)$ and is zero for all other values of t. The output of the filter is the convolution of the input, $\sum_{-\infty}^{\infty} x(t)\delta(t-nT)$, with $g(t)$.*
Using the convolution theorem, and the result given in exercise 13.4, show that the output of the filter can be written

$$y(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} sinc^2\left(\frac{\omega T}{2}\right) e^{-i\omega[(n+1)T-t]} d\omega.$$

In order to use the convolution theorem we need the Fourier transforms of both the input signal $x(t)$ and the filter response $g(t)$. The former is

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)\,dt = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} x(nT)\,e^{-in\omega T}.$$

The latter is the same as that obtained in exercise 13.4, except that it is scaled by a factor $1/T$ and centred on $t = T$, rather that $t = 0$. The required transform is therefore

$$\tilde{g}(\omega) = \frac{e^{-i\omega T}}{\sqrt{2\pi}} \text{sinc}^2\left(\frac{\omega T}{2}\right).$$

The transform of the output is therefore the product of these two transforms multiplied by $\sqrt{2\pi}$.

Using the Fourier inversion theorem, we can therefore write the output of the filter as

$$\begin{aligned} y(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\sqrt{2\pi}\,\tilde{g}(\omega)\,\tilde{x}(\omega)\right] e^{i\omega t}\,d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\sqrt{2\pi}\frac{e^{-i\omega T}}{\sqrt{2\pi}}\text{sinc}^2\left(\frac{\omega T}{2}\right) \right. \\ &\qquad\qquad \left. \times \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} x(nT)\,e^{-in\omega T}\right] e^{i\omega t}\,d\omega \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \text{sinc}^2\left(\frac{\omega T}{2}\right) e^{-i\omega[(n+1)T-t]} d\omega, \end{aligned}$$

as stated in the question.

13.14 *Prove the equality*

$$\int_0^\infty e^{-2at}\sin^2 at\,dt = \frac{1}{\pi}\int_0^\infty \frac{a^2}{4a^4+\omega^4}\,d\omega.$$

We utilise the first result of the previous exercise (13.13) in the special case where $\gamma = p = a$, $f(t) = e^{-at}\sin at$ and consequently

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{a}{(a+i\omega)^2+a^2}.$$

Applying Parseval's theorem,

$$\int_{-\infty}^\infty |f(t)|^2\,dt = \int_{-\infty}^\infty |\tilde{f}(\omega)|^2\,d\omega,$$

to this function and its transform:

$$\begin{aligned}\int_0^\infty e^{-2at}\sin^2 at\,dt &= \frac{1}{2\pi}\int_{-\infty}^\infty \left[\frac{a}{(a+i\omega)^2+a^2}\right]\left[\frac{a}{(a-i\omega)^2+a^2}\right]d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{a^2}{(a^2+\omega^2)^2+2a^2(a^2-\omega^2)+a^4}\,d\omega\\ &= \frac{1}{\pi}\int_0^\infty \frac{a^2}{4a^4+\omega^4}\,d\omega.\end{aligned}$$

13.16 *In quantum mechanics, two equal-mass particles having momenta $\mathbf{p}_j = \hbar\mathbf{k}_j$ and energies $E_j = \hbar\omega_j$ and represented by plane wavefunctions $\phi_j = \exp[i(\mathbf{k}_j\cdot\mathbf{r}_j - \omega_j t)]$, $j = 1, 2$, interact through a potential $V = V(|\mathbf{r}_1 - \mathbf{r}_2|)$. In first-order perturbation theory the probability of scattering to a state with momenta and energies $\mathbf{p}'_j, E'_j$ is determined by the modulus squared of the quantity*

$$M = \iiint \psi_f^* V \psi_i\,d\mathbf{r}_1\,d\mathbf{r}_2\,dt.$$

The initial state ψ_i is $\phi_1\phi_2$ and the final state ψ_f is $\phi'_1\phi'_2$.

(a) *By writing $\mathbf{r}_1 + \mathbf{r}_2 = 2\mathbf{R}$ and $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$ and assuming that $d\mathbf{r}_1\,d\mathbf{r}_2 = d\mathbf{R}\,d\mathbf{r}$, show that M can be written as the product of three one-dimensional integrals.*

(b) *From two of the integrals deduce energy and momentum conservation in the form of δ-functions.*

(c) *Show that M is proportional to the Fourier transform of V, i.e. $\widetilde{V}(\mathbf{k})$ where $2\hbar\mathbf{k} = (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}'_2 - \mathbf{p}'_1)$.*

Putting in explicit expressions for the wavefunctions gives

$$\begin{aligned} M &= \iiint \psi_{\rm f}^* V \psi_{\rm i} \, d\mathbf{r}_1 \, d\mathbf{r}_2 \, dt \\ &= \iiint \exp[-i(\mathbf{k}_2' \cdot \mathbf{r}_2 - \omega_2' t)] \exp[-i(\mathbf{k}_1' \cdot \mathbf{r}_1 - \omega_1' t)] V(|\mathbf{r}_1 - \mathbf{r}_2|) \\ &\qquad \times \exp[i(\mathbf{k}_2 \cdot \mathbf{r}_2 - \omega_2 t)] \exp[i(\mathbf{k}_1 \cdot \mathbf{r}_1 - \omega_1 t)] \, d\mathbf{r}_1 \, d\mathbf{r}_2 \, dt. \end{aligned}$$

(a) Writing the integrand in terms of the centre-of-mass coordinate, $\mathbf{r}$, and the coordinate of the centre of mass, $\mathbf{R}$, given by

$$\mathbf{r}_1 + \mathbf{r}_2 = 2\mathbf{R} \quad \text{and} \quad \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r},$$

with $d\mathbf{r}_1 \, d\mathbf{r}_2 = d\mathbf{R} \, d\mathbf{r}$, we can express $\mathbf{r}_1$ as $\mathbf{r}_1 = \mathbf{R} + \frac{1}{2}\mathbf{r}$ and $\mathbf{r}_2$ as $\mathbf{r}_2 = \mathbf{R} - \frac{1}{2}\mathbf{r}$.

When these substitutions are made the integral becomes

$$\begin{aligned} M = \int & \exp[i(-\mathbf{k}_2' - \mathbf{k}_1' + \mathbf{k}_2 + \mathbf{k}_1) \cdot \mathbf{R}] \, d\mathbf{R} \\ & \times \int \exp[\tfrac{1}{2} i(\mathbf{k}_2' - \mathbf{k}_1' - \mathbf{k}_2 + \mathbf{k}_1) \cdot \mathbf{r}] \, V(r) \, d\mathbf{r} \\ & \quad \times \int \exp[i(\omega_2' + \omega_1' - \omega_2 - \omega_1)t] \, dt. \end{aligned}$$

This is now the product of three 1-dimensional integrals.

(b) The first integral is, as shown in the text, a representation of the 3-dimensional δ-function and is equal to $(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_1' - \mathbf{k}_2')$. Since $\mathbf{p}_j = \hbar \mathbf{k}_j$, this is equivalent to $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$, i.e. to momentum conservation.

Similarly, the last of the three integrals produces a 1-dimensional δ-function, which, since $E_j = \hbar \omega_j$, is equivalent to energy conservation, namely $E_1' + E_2' = E_1 + E_2$.

(c) The second integral, containing $V(r)$, can be written as

$$\int V(r) \, e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r},$$

where $\mathbf{k} = \frac{1}{2}(\mathbf{k}_1' - \mathbf{k}_2' - \mathbf{k}_1 + \mathbf{k}_2)$, i.e. where $2\hbar\mathbf{k} = (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}_2' - \mathbf{p}_1')$; the integral is thus proportional to $\tilde{V}(\mathbf{k})$.

Note Since, from part (b), $(\mathbf{p}_2' - \mathbf{p}_2) = -(\mathbf{p}_1' - \mathbf{p}_1)$ and $2\hbar\mathbf{k}$ can be written as $2\hbar\mathbf{k} = (\mathbf{p}_1' - \mathbf{p}_1) - (\mathbf{p}_2' - \mathbf{p}_2)$, it follows that $\hbar\mathbf{k} = \mathbf{p}_1' - \mathbf{p}_1$. Thus the $\mathbf{k}$ appearing in $\tilde{V}(\mathbf{k})$ is the wave vector corresponding to the momentum transferred from one particle to the other.

13.18 *The equivalent duration and bandwidth, T_e and B_e, of a signal $x(t)$ are defined in terms of the latter and its Fourier transform $\tilde{x}(\omega)$:*

$$T_e = \frac{1}{x(0)} \int_{-\infty}^{\infty} x(t)\, dt,$$
$$B_e = \frac{1}{\tilde{x}(0)} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, d\omega,$$

where neither $x(0)$ nor $\tilde{x}(0)$ is zero. Show that the product $T_e B_e = 2\pi$ (this is a form of uncertainty principle), and find the equivalent bandwidth of the signal

$$x(t) = \exp(-|t|/T).$$

For this signal, determine the fraction of the total energy that lies in the frequency range $|\omega| < B_e/4$. You will need the indefinite integral with respect to x of $(a^2 + x^2)^{-2}$, which is

$$\frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3} \tan^{-1} \frac{x}{a}.$$

With $\tilde{x}(\omega)$ being the Fourier transform of $x(t)$,

$$T_e = \frac{1}{x(0)} \int_{-\infty}^{\infty} x(t)\, dt \quad \text{and} \quad B_e = \frac{1}{\tilde{x}(0)} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, d\omega,$$

we have that

$$\tilde{x}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{-i0t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, dt = \frac{1}{\sqrt{2\pi}}\, x(0) T_e.$$

Consequently,

$$x(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, e^{i\omega 0}\, d\omega = \frac{1}{\sqrt{2\pi}}\, \tilde{x}(0) B_e = \frac{1}{\sqrt{2\pi}}\, B_e \frac{1}{\sqrt{2\pi}}\, x(0) T_e.$$

It then follows that $B_e T_e = 2\pi$.

For $x(t) = \exp(-|t|/T)$, the equivalent duration is

$$T_e = \frac{1}{x(0)} \int_{-\infty}^{\infty} x(t)\, dt = 2 \int_0^{\infty} e^{-t/T}\, dt = 2 \left[-T e^{-t/T} \right]_0^{\infty} = 2T.$$

The equivalent bandwidth is therefore $B_e = 2\pi/(2T) = \pi/T$.

The energy density spectrum is proportional to $|\tilde{x}(\omega)|^2$ and the fraction of the total energy lying within $|\omega| < B_e/4$ is

$$f = \frac{\displaystyle\int_{-B_e/4}^{B_e/4} |\tilde{x}(\omega)|^2\, d\omega}{\displaystyle\int_{-\infty}^{\infty} |\tilde{x}(\omega)|^2\, d\omega}.$$

Now,

$$\begin{aligned}
\tilde{x}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{t/T}\, e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t/T}\, e^{-i\omega t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t'/T} e^{i\omega t'}\, dt' + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t/T}\, e^{-i\omega t}\, dt \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t/T} \cos(\omega t)\, dt \\
&= \frac{2}{\sqrt{2\pi}} \operatorname{Re} \left[\frac{e^{-t/T+i\omega t}}{-\frac{1}{T} + i\omega} \right]_{0}^{\infty} \\
&= \frac{2}{\sqrt{2\pi}} \frac{T^{-1}}{T^{-2}+\omega^2} = \frac{2T}{\sqrt{2\pi}} \frac{1}{1+\omega^2 T^2}.
\end{aligned}$$

The fraction f is therefore given by

$$\begin{aligned}
f &= \frac{\displaystyle\int_{-\pi/(4T)}^{\pi/(4T)} (1+\omega^2 T^2)^{-2}\, d\omega}{\displaystyle\int_{-\infty}^{\infty} (1+\omega^2 T^2)^{-2}\, d\omega} = \frac{\displaystyle 2\int_{0}^{\pi/4} (1+x^2)^{-2}\, dx}{\displaystyle 2\int_{0}^{\infty} (1+x^2)^{-2}\, dx} \\
&= \left\{ \frac{\pi}{8[1+(\pi/4)^2]} + \frac{1}{2} \tan^{-1} \frac{\pi}{4} \right\} \left\{ 0 + \frac{1}{2} \tan^{-1} \infty \right\}^{-1} \\
&= \frac{1}{2[1+(\pi/4)^2]} + \frac{2}{\pi} \tan^{-1} \frac{\pi}{4} = 0.733.
\end{aligned}$$

13.20 *Prove that the cross-correlation $C(z)$ of the Gaussian and Lorentzian distributions*

$$f(t) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right), \qquad g(t) = \left(\frac{a}{\pi}\right) \frac{1}{t^2+a^2},$$

has as its Fourier transform the function

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tau^2\omega^2}{2}\right) \exp(-a|\omega|).$$

Hence show that

$$C(z) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(\frac{a^2-z^2}{2\tau^2}\right) \cos\left(\frac{az}{\tau^2}\right).$$

We need the Fourier transforms of both $f(t)$ and $g(t)$. That for f is derived in the text as

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2\tau^2/2}.$$

That for g can be found from the result of either of the exercises 13.18 and 13.19 (or from the contour integral of a complex variable, as in chapter 24). If, in the final result of the previous exercise (13.19), we make the substitutions $\omega \to -t$, $\lambda \to a$ and $z \to \omega$, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-it\omega}}{a^2 + t^2}\, dt = \frac{\pi}{a}\, e^{-a|\omega|}.$$

From this it follows that

$$\tilde{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{a}{\pi} \frac{e^{-i\omega t}}{t^2 + a^2}\, dt = \frac{1}{\sqrt{2\pi}} e^{-a|\omega|}.$$

From the Wiener-Kinchin theorem, we can now state that the Fourier transform of the cross-correlation function is

$$\tilde{C}(\omega) = \sqrt{2\pi}\, \frac{1}{\sqrt{2\pi}} e^{-\omega^2\tau^2/2}\, \frac{1}{\sqrt{2\pi}} e^{-a|\omega|} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tau^2\omega^2}{2}\right) \exp(-a|\omega|).$$

The correlation function itself is obtained by forming the inverse tranform and evaluating it by ‘completing the square’.

$$\begin{aligned}
C(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\omega^2\tau^2/2}\, e^{-a|\omega|}\ e^{i\omega z}\, d\omega \\
&= \frac{2}{2\pi} \,\mathrm{Re} \int_0^{\infty} \exp(-\tfrac{1}{2}\omega^2\tau^2 - a\omega + i\omega z)\, d\omega \\
&= \frac{1}{\pi} \,\mathrm{Re} \int_0^{\infty} \exp\left\{-\frac{1}{2}\tau^2 \left[\omega^2 + \frac{2(a-iz)\omega}{\tau^2} + \frac{(a-iz)^2}{\tau^4}\right]\right\} \\
&\qquad \times\ \exp\left[\frac{(a-iz)^2}{2\tau^2}\right]\, d\omega \\
&= \frac{1}{\pi} \left(\frac{1}{2} \frac{\sqrt{2\pi}}{\tau}\right) \mathrm{Re}\ \left[\exp\left(\frac{a^2 - z^2}{2\tau^2}\right) \exp\left(\frac{-2aiz}{2\tau^2}\right)\right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\tau} \exp\left(\frac{a^2 - z^2}{2\tau^2}\right) \cos\left(\frac{az}{\tau^2}\right),
\end{aligned}$$

as given in the question.

13.22 *Find the functions $y(t)$ whose Laplace transforms are the following:*

(a) $1/(s^2 - s - 2)$;
(b) $2s/[(s+1)(s^2+4)]$;
(c) $e^{-(\gamma+s)t_0}/[(s+\gamma)^2 + b^2]$.

To find the original functions we must express the transforms in terms of those given in table 13.1. Partial fraction expansions (chapter 1) are needed for (a) and (b).

(a) Factorising the denominator and expressing the transform as partial fractions:

$$\bar{f}(s) = \frac{1}{s^2 - s - 2} = \frac{1}{3(s-2)} - \frac{1}{3(s+1)},$$

and from the table of Laplace transforms and the liearity of the process of taking Laplace transforms, it follows that

$$f(t) = \frac{1}{3}(e^{2t} - e^{-t}).$$

(b) The quadratic term in the denominator cannot be factorised further without involving complex roots (and, in any case, transforms containing $(s^2 + a^2)^{-2}$ appear in the table) and so we express the transform in partial fractions as

$$\begin{aligned} \frac{2s}{(s+1)(s^2+4)} &= \frac{A}{s+1} + \frac{Bs+C}{s^2+4}, \\ 2s &= s^2(A+B) + s(B+C) + (4A+C), \\ &\Rightarrow A = -\tfrac{2}{5}, \quad B = \tfrac{2}{5}, \quad C = \tfrac{8}{5}. \end{aligned}$$

Thus, we may write

$$\bar{f}(s) = -\frac{2}{5(s+1)} + \frac{2s+8}{5(s^2+4)},$$

and from the table of Laplace transforms can read off that

$$f(t) = \tfrac{2}{5}(-e^{-t} + \cos 2t + 4 \sin 2t).$$

(c) Apart from the factor $e^{-\gamma t_0}$ (which indicates a change of origin to $t = t_0$), $\bar{f}(s)$ is the product of the Laplace transforms of $\delta(t - t_0)$ and $b^{-1}e^{-\gamma t} \sin bt$. Thus, by the convolution theorem,

$$\begin{aligned} f(t) &= \frac{e^{-\gamma t_0}}{b} \int_0^t e^{-\gamma u} \sin(bu)\, H(u)\, \delta(t - u - t_0)\, du \\ &= \frac{e^{-\gamma t_0}}{b}\, e^{-\gamma(t-t_0)} \sin[\, b(t - t_0)\,]\, H(t - t_0) \\ &= \frac{1}{b}\, e^{-\gamma t} \sin[\, b(t - t_0)\,]\, H(t - t_0). \end{aligned}$$

Note that $f(t) = 0$ for $t < t_0$.

13.24 *Find the solution (the so-called* impulse response *or* Green's function*) of the equation*

$$T\frac{dx}{dt} + x = \delta(t)$$

by proceeding as follows.

(a) *Show by substitution that*

$$x(t) = A(1 - e^{-t/T})H(t)$$

is a solution, for which $x(0) = 0$, of

$$T\frac{dx}{dt} + x = AH(t), \qquad (*)$$

where $H(t)$ is the Heaviside step function.

(b) *Construct the solution when the RHS of $(*)$ is replaced by $AH(t - \tau)$ with $dx/dt = x = 0$ for $t < \tau$, and hence find the solution when the RHS is a rectangular pulse of duration τ.*

(c) *By setting $A = 1/\tau$ and taking the limit when $\tau \to 0$, show that the impulse response is $x(t) = T^{-1}e^{-t/T}$.*

(d) *Obtain the same result much more directly by taking the Laplace transform of each term in the original equation, solving the resulting algebraic equation and then using the entries in table 13.1.*

(a) For $t > 0$, consider $x(t) = A(1 - e^{-t/T})H(t)$, for which $x(0) = A(1 - 1) = 0$. Substitute it into $(*)$:

$$T\frac{A}{T}e^{-t/T}H(t) + A(1 - e^{-t/T})H(t) = AH(t),$$

which is clearly satisfied.

(b) With the RHS of $(*)$ $= AH(t - \tau)$ the solution will be $x(t) = A(1 -$

$e^{-(t-\tau)/T})H(t-\tau)$ and, because of the linearity of the equation, the solution when the RHS is a rectangular pulse of duration τ is

$$x(t) = A(1 - e^{-t/T})H(t) - A(1 - e^{-(t-\tau)/T})H(t-\tau).$$

This follows because the rectangular pulse can be thought of as the linear superposition of a positive Heaviside function and an equal and opposite negative Heaviside function, the latter being delayed by an interval τ.

(c) We now make A equal to $1/\tau$, so that the area under the pulse is unity, whatever the value of τ, and consider the limiting form of $f(t)$ as $\tau \to 0$.

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{1}{\tau}\left[(1-e^{-t/T})H(t) - (1-e^{-(t-\tau)/T})H(t-\tau)\right] \\
&= \lim_{\tau\to 0}\frac{1}{\tau}\left\{(1-e^{-t/T})H(t) - \left[\,1 - e^{-t/T}\left(1 + \frac{\tau}{T} + O(\tau^2)\right)\right] H(t-\tau)\right\} \\
&= \lim_{\tau\to 0}\left\{(1-e^{-t/T})\frac{[\,H(t) - H(t-\tau)\,]}{\tau} + \frac{e^{-t/T}}{\tau}\,\frac{\tau}{T}H(t-\tau) + O(\tau^2)\right\} \\
&= \left(1 - e^{-t/T}\right)_{t=0} + \frac{e^{-t/T}}{T}H(t) \\
&= \frac{e^{-t/T}}{T}H(t).
\end{aligned}$$

This is the impulse response.

(d) Laplace transforming the original equation and incorporating the initial value $x(0) = 0$ gives

$$Ts\bar{x} - 0 + \bar{x} = 1.$$

From this it follows that

$$\begin{aligned}
\bar{x} &= \frac{1}{1+sT} = \frac{1}{T(T^{-1}+s)}, \\
\Rightarrow \quad x(t) &= \frac{e^{-t/T}}{T}H(t),
\end{aligned}$$

in agreement with the result in part (c).

13.26 *By writing $f(x)$ as an integral involving the δ-function $\delta(\xi - x)$ and taking the Laplace transforms of both sides, show that the transform of the solution of the equation*

$$\frac{d^4 y}{dx^4} - y = f(x)$$

for which y and its first three derivatives vanish at $x = 0$ can be written as

$$\bar{y}(s) = \int_0^\infty f(\xi) \frac{e^{-s\xi}}{s^4 - 1}\, d\xi.$$

Use the properties of Laplace transforms and the entries in table 13.1 to show that

$$y(x) = \frac{1}{2} \int_0^x f(\xi) \left[\sinh(x - \xi) - \sin(x - \xi)\right]\, d\xi.$$

We first need to write $f(x)$ as an integral involving the δ-function $\delta(\xi - x)$ so that the only x-dependence of the RHS is on functions for which we know the explicit Laplace transform; we do not know that of $f(x)$. Thus, we take the transform of the equation in the form

$$\frac{d^4 y}{dx^4} - y = \int_{-\infty}^{\infty} H(\xi)\, f(\xi)\, \delta(\xi - x)\, d\xi.$$

Since y and its first three derivatives vanish at $x = 0$ the transform of $d^4 y/dx^4$ does not contain any terms involving s^3, s^2, s or a constant. The transform of $\delta(x - x_0)$ is e^{-sx_0}, and so the Laplace transform of the original equation reads

$$s^4 \bar{y} - \bar{y} = \int_0^\infty f(\xi) e^{-s\xi}\, d\xi.$$

This can be rearranged to express $\bar{y}$ explicitly as

$$\bar{y}(s) = \int_0^\infty f(\xi) \frac{e^{-s\xi}}{s^4 - 1}\, d\xi,$$

which is the form stated in the question.

To find the form of $y(x)$, we begin by rewriting the integrand using partial fractions. The denominator could be written as the product of four linear factors, but, with one eye on the form of the quoted solution (and the other on table 13.1), we write it as the product of two quadratic functions leading to the partial fractions representation:

$$\bar{y}(s) = \frac{1}{2} \int_0^\infty f(\xi) \left(\frac{e^{-s\xi}}{s^2 - 1} - \frac{e^{-s\xi}}{s^2 + 1} \right) d\xi.$$

Now, using the table and recognising the implication of the factor $e^{-s\xi}$ so far as the arguments of the inverted functions are concerned:

$$y(x) = \frac{1}{2}\int_0^\infty f(\xi)[\,\sinh(x-\xi) - \sin(x-\xi)\,]\,H(x-\xi)\,d\xi$$
$$= \frac{1}{2}\int_0^x f(\xi)[\,\sinh(x-\xi) - \sin(x-\xi)\,]\,d\xi,$$

i.e. as stated in the question.

13.28 *Show that the Laplace transform of $f(t-a)H(t-a)$, where $a \geq 0$, is $e^{-as}\bar{f}(s)$ and that, if $g(t)$ is a periodic function of period T, $\bar{g}(s)$ can be written as*

$$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}g(t)\,dt.$$

(a) *Sketch the periodic function defined in $0 \leq t \leq T$ by*

$$g(t) = \begin{cases} 2t/T & 0 \leq t < T/2, \\ 2(1-t/T) & T/2 \leq t \leq T, \end{cases}$$

and, using the previous result, find its Laplace transform.

(b) *Show, by sketching it, that*

$$\frac{2}{T}[tH(t) + 2\sum_{n=1}^{\infty}(-1)^n(t - \tfrac{1}{2}nT)H(t - \tfrac{1}{2}nT)]$$

is another representation of $g(t)$ and hence derive the relationship

$$\tanh x = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}.$$

From the definition of a Laplace transform,

$$\mathscr{L}\,[f(t-a)H(t-a)] = \int_0^\infty f(t-a)H(t-a)\,e^{-st}\,dt.$$

We start by changing the integration variable to $u = t - a$, with a corresponding change in the integration limits:

$$\mathscr{L}\,[f(t-a)H(t-a)] = \int_{-a}^\infty f(u)H(u)e^{-su}e^{-sa}\,du$$
$$= \int_0^\infty f(u)e^{-su}e^{-sa}\,du$$
$$= e^{-sa}\bar{f}(s).$$

With $g(t)$ periodic, $g(t) = g(t - T)$ and the Laplace transform can be written as

$$\int_0^\infty H(t)g(t)\,e^{-st}\,dt = \int_0^T H(t)g(t)\,e^{-st}\,dt + \int_T^\infty H(t)g(t)\,e^{-st}\,dt.$$

However, it follows from the properties of the Heavisde function that

$$\int_T^\infty H(t)\cdots dt = \int_0^\infty H(t-T)\cdots dt$$

and so, using the previous result, we can rewrite the above equation as

$$\begin{aligned}\bar{g}(s) = \int_0^\infty g(t)\,e^{-st}\,dt &= \int_0^T g(t)\,e^{-st}\,dt + \int_0^\infty H(t-T)g(t-T)\,e^{-st}\,dt \\ &= \int_0^T g(t)\,e^{-st}\,dt + e^{-sT}\bar{g}(s) \\ \Rightarrow \quad \bar{g}(s) &= \frac{1}{1-e^{-sT}}\int_0^T g(t)\,e^{-st}\,dt.\end{aligned}$$

(a) The graph of $g(t)$ consists of a continuously repeating pattern of isosceles triangles, each of unit height and base width T. Any one of these triangles has the same shape as the function $g_a(x)$ found in Exercise 13.27, except that a has been replaced by $T/2$ and the height of the triangle is unity rather than a. Its Laplace transform is therefore

$$\frac{1}{(T/2)}\frac{1}{s^2}\left(1-e^{-sT/2}\right)^2.$$

From our earlier result it now follows that the Laplace transform of $g(t)$ is

$$\begin{aligned}\bar{g}(s) &= \frac{2}{Ts^2}\frac{\left(1-e^{-sT/2}\right)^2}{1-e^{-sT}} \\ &= \frac{2}{Ts^2}\frac{1-e^{-sT/2}}{1+e^{-sT/2}} \\ &= \frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right).\end{aligned}$$

(b) The contributions to

$$f(t) = \sum_{n=0}^\infty f_n(t) = \frac{2}{T}[tH(t) + 2\sum_{n=1}^\infty (-1)^n(t - \tfrac{1}{2}nT)H(t - \tfrac{1}{2}nT)]$$

are shown in the sketch (figure 13.1).

For $0 \le t \le T/2$, only f_0 contributes; it is identical to $g(t)$ with $f(T/2) = 1$.

For $T/2 \le t \le T$, f_0 and f_1 contribute with *net* slope $-2/T$ and $f_1 = -f_0$ at $t = T$, making $f(T) = 0$.

For $T \le t \le 3T/2$, $f_2(t) + f_1(t) = f_1(T)$, since the two terms contribute equal and opposite changes as t varies; the change in $f(t)$ is entirely due to that in f_0.

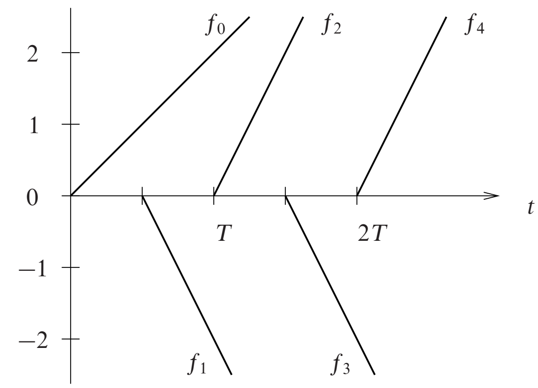

Figure 13.1 The contributions to the function considered in exercise 13.28. Their analytic forms in terms of the Heaviside function are $f_0 = 2tH(t)/T$, with slope $2/T$; $f_n = 4(-1)^n(t - \frac{1}{2}nT)H(t - \frac{1}{2}nT)/T$, with slope $4/T$.

For $3T/2 \leq t \leq 2T$, the change in $f(t)$ is due to those in f_0 and f_3, i.e. has a net slope $-2/T$, making $f(2T) = 0$.

This sequence is then repeated in successive blocks of length $2T$.

Hence $f(t)$ is an alternative representation of $g(t)$ with

$$\bar{f}(s) = \frac{2}{T}\left[\mathscr{L}\,[tH(t)] + 2\sum_{n=1}^{\infty}(-1)^n e^{-nTs/2}\mathscr{L}\,[tH(t)]\right],$$

where we have used the result from part (a). But,

$$\mathscr{L}\,[tH(t)] = \int_0^{\infty} t\,e^{-st}\,dt = \left[\frac{t\,e^{-st}}{-s}\right]_0^{\infty} - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty} = \frac{1}{s^2}.$$

Thus,

$$\frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right) = \frac{2}{Ts^2}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-nTs/2}\right].$$

Finally, setting $sT = 4x$ gives

$$\tanh x = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}.$$

14

First-order ordinary differential equations

In this chapter unspecified symbols appearing in solutions are arbitrary constants. Some of the constants may have specific relationships to earlier ones in the same solution, but this will not be indicated unless it has particular significance in the final answer.

> ***14.2*** *Solve the following equations by separation of the variables:*
>
> (a) $y' - xy^3 = 0$;
> (b) $y' \tan^{-1} x - y(1+x^2)^{-1} = 0$;
> (c) $x^2 y' + xy^2 = 4y^2$.

In each case we re-arrange the equation so that all terms involving y appear on one side of an equality sign and all those involving x appear on the other. To save space we write two equations on each line.

(a)

$$y' = xy^3, \quad \Rightarrow \quad \frac{dy}{y^3} = x\,dx,$$

$$\Rightarrow \quad -\frac{1}{2y^2} = \frac{1}{2}x^2 + A, \quad \Rightarrow \quad y = \frac{\pm 1}{\sqrt{c - x^2}}.$$

(b)

$$y' \tan^{-1} x = \frac{y}{1+x^2}, \quad \Rightarrow \quad \frac{dy}{y} = \frac{dx}{(1+x^2)\tan^{-1} x},$$

$$\Rightarrow \quad \ln y = \ln(\tan^{-1} x) + A, \quad \Rightarrow \quad y = c \tan^{-1} x.$$

(c)

$$x^2 y' = y^2(4 - x), \quad \Rightarrow \quad \frac{dy}{y^2} = \left(\frac{4}{x^2} - \frac{1}{x}\right) dx,$$

$$\Rightarrow \quad -\frac{1}{y} = -\frac{4}{x} - \ln x + c, \quad \Rightarrow \quad y = \frac{x}{4 + x\ln x - cx}.$$

14.4 *Find the values of α and β that make*

$$dF(x, y) = \left(\frac{1}{x^2 + 2} + \frac{\alpha}{y} \right) dx + (xy^{\beta} + 1)\, dy$$

an exact differential. For these values solve $dF(x, y) = 0$.

For the differential to be exact we need

$$\frac{\partial}{\partial y} \left(\frac{1}{x^2 + 2} + \frac{\alpha}{y} \right) = \frac{\partial}{\partial x} (xy^{\beta} + 1),$$
$$-\frac{\alpha}{y^2} = y^{\beta}.$$

Thus if $\alpha = -1$ and $\beta = -2$ then dF will be an exact differential. Integrating the equation then leads to

$$c' = F(x, y) = \int \left(\frac{x}{y^2} + 1 \right) dy + g(x)$$
$$= -\frac{x}{y} + y + g(x),$$

where

$$\frac{1}{x^2 + 2} - \frac{1}{y} = \frac{\partial F}{\partial x} = -\frac{1}{y} + g'(x),$$

which implies that

$$g(x) = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + c''.$$

Collecting these results together, we can give the solution as

$$c = F(x, y) = -\frac{x}{y} + y + \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right).$$

14.6 *By finding an appropriate integrating factor, solve*

$$\frac{dy}{dx} = -\frac{2x^2 + y^2 + x}{xy}.$$

Arrange the equation in the form

$$xy\, dy + (2x^2 + y^2 + x)\, dx = 0.$$

Now apply the standard prescription for determining whether a suitable IF exists:

$$\frac{1}{xy}\left[\frac{\partial}{\partial y}(2x^2+y^2+x)-\frac{\partial}{\partial x}(xy)\right]=\frac{1}{xy}(2y-y)=\frac{1}{x}.$$

This is a function of x only, thus showing that one does and that the IF needed is

$$\mu(x)=\exp\left\{\int\frac{1}{x}\,dx\right\}=\exp(\ln x)=x.$$

The exact equation is thus

$$x^2y\,dy+(2x^3+xy^2+x^2)\,dx=0.$$

If this is to integrate to $f(x,y)=c$ then

$$\frac{\partial f}{\partial x}=2x^3+xy^2+x^2 \quad\Rightarrow\quad f(x,y)=\frac{1}{2}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3+g(y).$$

The further requirement that $\partial f/\partial y = x^2y$ shows that $g(y)=0$ and so, on multiplying through by 6, we obtain the solution

$$3x^4+2x^3+3x^2y^2=c.$$

14.8 *An electric circuit contains a resistance R and a capacitor C in series, and a battery supplying a time-varying electromotive force $V(t)$. The charge q on the capacitor therefore obeys the equation*

$$R\frac{dq}{dt}+\frac{q}{C}=V(t).$$

Assuming that initially there is no charge on the capacitor, and given that $V(t)=V_0\sin\omega t$, find the charge on the capacitor as a function of time.

In standard form the equation is

$$\frac{dq}{dt}+\frac{q}{RC}=\frac{V_0}{R}\sin\omega t.$$

The required IF is

$$\mu(t)=\exp\left\{\int^t\frac{1}{RC}\,du\right\}=e^{t/RC}\equiv e^{\omega_0 t},$$

thus defining ω_0.

Multiplying through by this IF and expressing the LHS as a total derivative gives

$$\frac{d}{dt}[\,e^{\omega_0 t}q(t)\,]=\frac{V_0\,e^{\omega_0 t}}{R}\sin\omega t.$$

Since $q(0) = 0$, this leads to

$$\begin{aligned} q(t) &= \frac{V_0 e^{-\omega_0 t}}{R} \int_0^t e^{\omega_0 u} \sin \omega u \, du \\ &= \frac{V_0 e^{-\omega_0 t}}{R} \text{Im} \left\{ \int_0^t e^{(\omega_0 + i\omega)u} \, du \right\} \\ &= \frac{V_0 e^{-\omega_0 t}}{R} \text{Im} \left[\frac{e^{(\omega_0 + i\omega)t} - 1}{\omega_0 + i\omega} \right] \\ &= \frac{V_0 e^{-\omega_0 t}}{R(\omega_0^2 + \omega^2)} \text{Im} \left[\omega_0 e^{(\omega_0 + i\omega)t} - i\omega e^{(\omega_0 + i\omega)t} - \omega_0 + i\omega \right] \\ &= \frac{V_0 e^{-\omega_0 t}}{R(\omega_0^2 + \omega^2)} \left[\omega_0 e^{\omega_0 t} \sin \omega t - \omega e^{\omega_0 t} \cos \omega t + \omega \right] \\ &= \frac{R^2 C^2 V_0}{R(1 + R^2 C^2 \omega^2)} \left[\frac{1}{RC} \sin \omega t - \omega \cos \omega t + \omega e^{-t/RC} \right] \\ &= \frac{C V_0}{1 + (RC\omega)^2} \left[\sin \omega t - \omega RC \cos \omega t + \omega RC e^{-t/RC} \right]. \end{aligned}$$

This gives the full time dependence of the charge on the capacitor. The first two terms give the long-term behaviour, whilst the final one is a transient arising from the intial conditions.

14.10 *Use the result of exercise 14.9 to find the law of force, acting towards the origin, under which a particle must move so as to describe the following trajectories:*

(a) *A circle of radius a that passes through the origin;*
(b) *An equiangular spiral, which is defined by the property that the angle α between the tangent and the radius vector is constant along the curve.*

(a) As shown in part (a) of figure 14.1, $p = r \sin \phi$ and, from simple geometry, $\sin \phi = \frac{1}{2} r/a$. It follows immediately that $r^2 = 2ap$ and

$$f = \frac{h^2}{mp^3} \frac{dp}{dr} = \frac{h^2}{m} \frac{8a^3}{r^6} \frac{2r}{2a} \propto \frac{1}{r^5}.$$

(b) By definition, and as shown in figure (b), $p = r \sin \alpha$ and therefore

$$f = \frac{h^2}{mp^3} \frac{dp}{dr} = \frac{h^2 \sin \alpha}{mr^3 \sin^3 \alpha} \propto \frac{1}{r^3}.$$

Note that for each case the constant h, which depends upon the initial conditions, will contain the parameter a or α; consequently only the r-dependence of f can be stated.

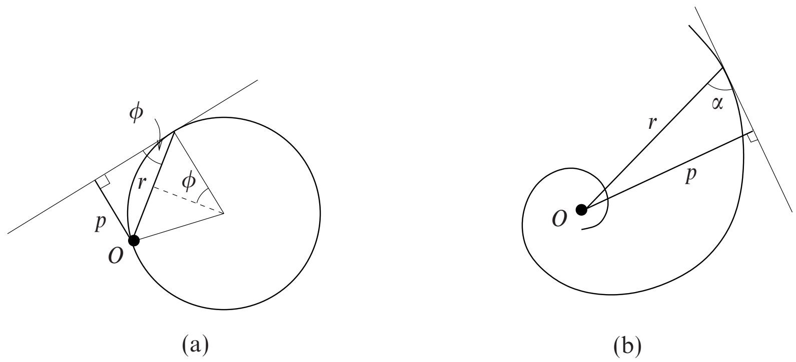

Figure 14.1 The trajectories discussed in exercise 14.10.

14.12 *A mass m is accelerated by a time-varying force $\alpha \exp(-\beta t)v^3$, where v is its velocity. It also experiences a resistive force ηv, where η is a constant, owing to its motion through the air. The equation of motion of the mass is therefore*

$$m\frac{dv}{dt} = \alpha \exp(-\beta t)v^3 - \eta v.$$

Find an expression for the velocity v of the mass as a function of time, given that it has an initial velocity v_0.

The equation can be written as

$$\frac{dv}{dt} + \frac{\eta}{m}v = \frac{\alpha e^{-\beta t}}{m}v^3,$$

which is Bernouilli's equation with $n = 3$. Therefore put $u = v^{1-3}$, i.e. $v = u^{-1/2}$; this leads to

$$-\frac{1}{2}\frac{1}{u^{3/2}}\frac{du}{dt} + \frac{\eta}{m}u^{-1/2} = \frac{\alpha e^{-\beta t}}{m}u^{-3/2},$$

$$\frac{du}{dt} - \frac{2\eta}{m}u = -\frac{2\alpha e^{-\beta t}}{m}.$$

The IF for this equation is clearly $e^{-2\eta t/m}$ and when applied gives

$$\frac{d}{dt}(ue^{-2\eta t/m}) = -\frac{2\alpha}{m}\exp\left[-\left(\beta + \frac{2\eta}{m}\right)t\right]$$

$$ue^{-2\eta t/m} = \frac{2\alpha}{\beta m + 2\eta}\exp\left[-\left(\beta + \frac{2\eta}{m}\right)t\right] + A,$$

or, in terms of v, $$\frac{1}{v^2} = \frac{2\alpha e^{-\beta t}}{\beta m + 2\eta} + Ae^{2\eta t/m}.$$

Using the initial velocity to determine the value of A then gives the solution at a general time t as $v(t)$ where

$$\frac{1}{v^2} = \frac{2\alpha}{\beta m + 2\eta}\left(e^{-\beta t} - e^{2\eta t/m}\right) + \frac{1}{v_0^2} e^{2\eta t/m}.$$

> ***14.14*** *Solve*
> $$\frac{dy}{dx} = \frac{1}{x + 2y + 1}.$$

Since the only linear combination of x and y to appear is $x + 2y + 1$, we set it equal to v with $dv/dx = 1 + 2dy/dx$. The equation then becomes

$$\frac{1}{2}\frac{dv}{dx} - \frac{1}{2} = \frac{1}{v} \quad \Rightarrow \quad \frac{dv}{dx} = \frac{v+2}{v}.$$

We can now separate the variables and integrate:

$$dx = \left(1 - \frac{2}{v+2}\right) dv \quad \Rightarrow \quad x + c = v - 2\ln(v+2).$$

Re-substitution for v gives the final answer as

$$\begin{aligned} & x + c = x + 2y + 1 - 2\ln(x + 2y + 3), \\ \Rightarrow \quad & k + y = \ln(x + 2y + 3). \end{aligned}$$

> ***14.16*** *If $u = 1 + \tan y$, calculate $d(\ln u)/dy$; hence find the general solution of*
> $$\frac{dy}{dx} = \tan x \cos y\ (\cos y + \sin y).$$

With $u = 1 + \tan y$, the derivative of $\ln u$ with respect to y is

$$\frac{d(\ln u)}{dy} = \frac{\sec^2 y}{1 + \tan y} = \frac{1}{\cos y(\cos y + \sin y)}.$$

Now, rearranging the equation given in the question:

$$\begin{aligned} \frac{dy}{\cos y(\cos y + \sin y)} &= \tan x\, dx, && \text{[separating variables]} \\ \ln(1 + \tan y) &= -\ln\cos x + A, && \text{[integrating, using the above result]} \\ \cos x(1 + \tan y) &= k, \end{aligned}$$

to give as the final solution

$$y = \tan^{-1}(k \sec x - 1).$$

14.18 *A reflecting mirror is made in the shape of the surface of revolution generated by revolving the curve $y(x)$ about the x-axis. In order that light rays emitted from a point source at the origin are reflected back parallel to the x-axis, the curve $y(x)$ must obey*

$$\frac{y}{x} = \frac{2p}{1-p^2},$$

where $p = dy/dx$. By solving this equation for x find the curve $y(x)$.

We first eliminate y, by differentiating it to obtain a first-order equation for p, as follows.

$$\begin{aligned}
y &= \frac{2p\,x}{1-p^2},\\
p = \frac{dy}{dx} &= \frac{(1-p^2)(2p+2xp') - (2p\,x)(-2p\,p')}{(1-p^2)^2},\\
(p-p^3)(1-p^2) &= 2p - 2p^3 + 2xp' - 2xp^2p' + 4xp^2p',\\
(1-p^2)(p-p^3-2p) &= (1+p^2)2xp',\\
p\,(p^2-1) &= 2xp'.
\end{aligned}$$

We now separate the variables, use partial fractions and integrate:

$$\begin{aligned}
\frac{dx}{x} &= \frac{2\,dp}{p\,(p-1)(p+1)}\\
&= \frac{-2\,dp}{p} + \frac{dp}{p-1} + \frac{dp}{p+1},\\
\Rightarrow \quad A + \ln x &= -2\ln p + \ln(p-1) + \ln(p+1).
\end{aligned}$$

This can be arranged as

$$Bx = \frac{p^2-1}{p^2} \qquad \text{or} \qquad p = \frac{\pm 1}{\sqrt{1-Bx}}.$$

We now substitute for p in the original equation and obtain

$$\frac{y}{x} = \frac{2p^{-1}}{p^{-2}-1} = \frac{\pm 2\sqrt{1-Bx}}{1-Bx-1},$$

which, in turn, can be rearranged as

$$y = \mp\frac{2\sqrt{1-Bx}}{B} \qquad \text{or} \qquad y^2 = \frac{4}{B^2} - \frac{4x}{B}.$$

This is a parabola, symmetric about the x-axis and with its apex at $x = 1/B$. The way it faces depends upon the sign of B.

14.20 *Find a parametric solution of*

$$x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - y = 0$$

as follows.

(a) *Write an equation for* y *in terms of* $p = dy/dx$ *and show that*

$$p = p^2 + (2px + 1)\frac{dp}{dx}.$$

(b) *Using* p *as the independent variable, arrange this as a linear first-order equation for* x.

(c) *Find an appropriate integrating factor to obtain*

$$x = \frac{\ln p - p + c}{(1-p)^2},$$

which, together with the expression for y *obtained in* (a), *gives a parameterisation of the solution.*

(d) *Reverse the roles of* x *and* y *in steps* (a) *to* (c), *putting* $dx/dy = p^{-1}$, *and show that essentially the same parameterisation is obtained.*

(a) Writing $p = dy/dx$, the equation becomes

$$\begin{aligned} y &= xp^2 + p, \\ p = \frac{dy}{dx} &= p^2 + 2xp\,p' + p' \\ &= p^2 + (2xp + 1)\frac{dp}{dx}. \end{aligned}$$

(b) In differential form, this equation reads

$$df = p\,(1-p)\,dx - (2xp + 1)\,dp = 0.$$

(c) We now apply the standard test for the existence of an IF for $f(x,p)\,dx + g(x,p)\,dp$:

$$\frac{1}{f}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial p}\right) = \frac{1}{p\,(1-p)}[-2p - (1-2p)] = -\frac{1}{p\,(1-p)}.$$

As this is a function of p alone, an IF exists and is given by

$$\exp\left[\int \frac{-1}{p\,(1-p)}\,dp\right] = \exp\left[\int \left(-\frac{1}{p} - \frac{1}{1-p}\right)\,dp\right]$$
$$= \exp[\,-\ln p + \ln(1-p)\,] = \frac{1-p}{p}.$$

With this IF, the equation becomes

$$(1-p)^2\,dx - \frac{(2xp+1)(1-p)}{p}\,dp = 0,$$
$$d[\,(1-p)^2 x\,] - \frac{dp}{p} + dp = 0,$$
$$(1-p)^2 x - \ln p + p = c.$$

This gives $x = (c + \ln p - p)(1-p)^{-2}$ and, together with $y = p + p^2 x$, gives a full parameterisation, $x = x(p)$, $y = y(p)$, of the solution.

(d) Now set $dx/dy = p^{-1} = q$. A parallel calculation to that in part (b) gives

$$0 = xp^2 + p - y,$$
$$x = -\frac{1}{p} + \frac{y}{p^2} = -q + q^2 y,$$
$$q = \frac{dx}{dy} = -\frac{dq}{dy} + q^2 + 2qy\frac{dq}{dy},$$
$$0 = (2qy - 1)\,dq + (q^2 - q)\,dy.$$

As in part (c), consider

$$\frac{1}{q(q-1)}[\,2q - (2q-1)\,] = -\frac{1}{q(1-q)}.$$

It follows that relevant IF is $(1-q)/q$ and that when it is applied the equation becomes

$$-(1-q)^2\,dy + \frac{(1-q)(2qy-1)}{q}\,dq = 0.$$

This leads to

$$d[\,(1-q)^2 y\,] + \frac{dq}{q} - dq = 0,$$

and then to

$$y = \frac{c' - \ln q + q}{(1-q)^2}.$$

Together with $x = -q + q^2 y$, this expression for y gives essentially the same parameterisation as obtained previously. This can be verified, if necessary, by writing $q = p^{-1}$ and substituting in the second parameterisation; it will be found that $c = 2 + c'$ makes the two forms identical.

14.22 *The action of the control mechanism on a particular system for an input $f(t)$ is described, for $t \geq 0$, by the coupled first-order equations:*

$$\dot{y} + 4z = f(t),$$
$$\dot{z} - 2z = \dot{y} + \tfrac{1}{2}y.$$

Use Laplace transforms to find the response $y(t)$ of the system to a unit step input $f(t) = H(t)$, given that $y(0) = 1$ and $z(0) = 0$.

We start by taking the Laplace transforms of the two equations, at the same time incorporating the initial conditions.

$$s\bar{y} - 1 + 4\bar{z} = \bar{f} \quad \Rightarrow \quad 4\bar{z} = \bar{f} + 1 - s\bar{y}$$

and

$$s\bar{z} - 0 - 2\bar{z} = s\bar{y} - 1 + \tfrac{1}{2}\bar{y} \quad \Rightarrow \quad (s-2)\bar{z} = (s + \tfrac{1}{2})\bar{y} - 1.$$

Eliminating $\bar{z}$ from these algebraic equations gives

$$4\frac{(s+\frac{1}{2})\bar{y} - 1}{s-2} = \bar{f} + 1 - s\bar{y},$$
$$\bar{y}[\,4s + 2 + s(s-2)\,] = (s-2)(\bar{f}+1) + 4,$$
$$\bar{y} = \frac{(\bar{f}+1)s + 2 - 2\bar{f}}{s^2 + 2s + 2}.$$

This is the transform of the response to a general input $f(t)$.

For the particular input $f(t) = H(t)$, $\bar{f} = 1/s$ and

$$\bar{y} = \frac{1 + s + 2 - 2s^{-1}}{s^2 + 2s + 2} = \frac{s^2 + 3s - 2}{s[\,(s+1)^2 + 1\,]} = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 1}.$$

Cross-multiplying and equating coefficients requires that $1 = A + B$, $3 = 2A + C$ and $-2 = 2A$. These have solution $A = -1$, $B = 2$ and $C = 5$ to give

$$\bar{y} = -\frac{1}{s} + \frac{2(s+1)}{(s+1)^2 + 1} + \frac{3}{(s+1)^2 + 1},$$
$$\Rightarrow \quad y(t) = -1 + 2e^{-t}\cos t + 3e^{-t}\sin t.$$

> ***14.24*** *Solve the following first-order equations for the boundary conditions given:*
> (a) $y' - (y/x) = 1, \quad y(1) = -1;$
> (b) $y' - y \tan x = 1, \quad y(\pi/4) = 3;$
> (c) $y' - y^2/x^2 = 1/4, \quad y(1) = 1;$
> (d) $y' - y^2/x^2 = 1/4, \quad y(1) = 1/2.$

(a) As in part (a) of the previous exercise (14.23), this equation needs an integrating factor, given in this case by

$$\mu(x) = \exp\left\{\int -\frac{1}{x}\,dx\right\} = \exp(-\ln x) = \frac{1}{x}.$$

We then have

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x} \quad \Rightarrow \quad \frac{y}{x} = \ln x + A.$$

Since $y(1) = -1$, A must have the value -1 and so $y = x \ln x - x$.

(b) Again, an IF is needed; this time given by

$$\mu(x) = \exp\left\{\int -\tan x\,dx\right\} = \exp(\ln \cos x) = \cos x.$$

The equation now reads

$$\frac{d}{dx}(y \cos x) = \cos x \quad \Rightarrow \quad y \cos x = \sin x + A.$$

The given boundary condition is that $3/\sqrt{2} = 1/\sqrt{2} + A$, establishing A as $\sqrt{2}$. The final answer is therefore $y = \tan x + \sqrt{2}\sec x$.

(c) This is not a linear equation, though it is homogeneous, and we therefore set $y = ux$. The equation then becomes

$$u + x\frac{du}{dx} = u^2 + \frac{1}{4},$$
$$x\frac{du}{dx} = \left(\frac{1}{2} - u\right)^2.$$

The equation is now separable and gives

$$\frac{du}{(\frac{1}{2} - u)^2} = \frac{dx}{x} \quad \Rightarrow \quad \frac{1}{\frac{1}{2} - u} = \ln x + A.$$

The boundary condition is that $u = 1/1 = 1$ when $x = 1$, implying that $A = -2$.

Now, substituting $u = y/x$ gives

$$\frac{1}{2} - \frac{y}{x} = \frac{1}{\ln x - 2} \quad \Rightarrow \quad y = \frac{x}{2 - \ln x} + \frac{x}{2}.$$

(d) For the boundary condition $y(1) = 1/2$ the method of solution used in part (c) fails, as it requires A to satisfy the impossible equation $1/0 = 0 + A$. We must therefore try to find a singular solution.

The first equation of the solution in part (c) would be automatically satisfied with u independent of x if

$$\frac{du}{dx} = 0 \quad \Rightarrow \quad u = u^2 + \frac{1}{4} \quad \Rightarrow \quad u = \frac{1}{2}.$$

The conclusion is that u is a constant and, since this satisfies the inital assumption that its derivative is zero, the 'circle is complete'. Hence there is a singular solution $u = \frac{1}{2}$, i.e. $y = \frac{1}{2}x$, that satisfies both the differential equation and the boundary condition.

14.26 *Solve the differential equation*

$$\sin x \frac{dy}{dx} + 2y \cos x = 1$$

subject to the boundary condition $y(\pi/2) = 1$.

Either
By inspection, the IF for this equation is $\sin x$.
or
After dividing through by $\sin x$ this becomes a standard first-order linear equation in need of the integrating factor

$$\exp\left\{\int \frac{2\cos x}{\sin x}\,dx\right\} = \exp(2\ln\sin x) = \sin^2 x.$$

By either method, multiplying the original equation through by $\sin x$ or the standardised one by $\sin^2 x$, the exact equation is

$$\sin^2 x \frac{dy}{dx} + 2y\cos x \sin x = \sin x,$$

$$\frac{d}{dx}(y\sin^2 x) = \sin x,$$

$$y\sin^2 x = -\cos x + A.$$

The condition $y(\pi/2) = 1$ implies that $A = 1$ and hence

$$y = \frac{1 - \cos x}{\sin^2 x} = \frac{1 - \cos x}{1 - \cos^2 x} = \frac{1}{1 + \cos x}.$$

> ***14.28*** *Find the solution of*
>
> $$(5x + y - 7)\frac{dy}{dx} = 3(x + y + 1).$$

The equation is not homogeneous and the two variables x and y appear in different linear combinations on the two sides of the equation. We therefore seek shifts in their origins that will make the expression for the derivative homogeneous, i.e. remove the constant terms from both its numerator and denominator. To do this we set

$$x = X + \alpha \qquad \text{and} \qquad y = Y + \beta.$$

We then require

$$3\alpha + 3\beta + 3 = 0 \quad \text{and} \quad 5\alpha + \beta - 7 = 0$$

These have the straightforward solution $\alpha = 2$ and $\beta = -3$; with these values the original equation reduces to

$$\frac{dX}{dY} = \frac{3X + 3Y}{5X + Y}.$$

This is now homogeneous and to solve it we set $Y = vX$ and obtain

$$\begin{aligned} \frac{dY}{dX} &= v + X\frac{dv}{dX}, \\ X\frac{dv}{dX} &= \frac{dY}{dX} - v = \frac{3X + 3Y}{5X + Y} - v, \\ &= \frac{3 + 3v - 5v - v^2}{5 + v}. \end{aligned}$$

We now separate the variables and use method (iii) for a partial fraction expansion, obtaining

$$\begin{aligned} \frac{dX}{X} &= \frac{5 + v}{3 - 2v - v^2} = \frac{A}{3 + v} + \frac{B}{1 - v}, \\ &= \frac{1}{2(3 + v)} + \frac{3}{2(1 - v)}, \\ \Rightarrow \quad \ln X &= \tfrac{1}{2}\ln(3 + v) - \tfrac{3}{2}\ln(1 - v) + k. \end{aligned}$$

Re-substituting for v, X and Y, gives

$$x - 2 = A\left(3 + \frac{y + 3}{x - 2}\right)^{1/2}\left(1 - \frac{y + 3}{x - 2}\right)^{-3/2} = A\frac{(3x + y - 3)^{1/2}(x - 2)}{(x - y - 5)^{3/2}}.$$

Finally, this result can be re-written as

$$(x - y - 5)^3 = B(3x + y - 3).$$

14.30 *Find the solution of*

$$(2\sin y - x)\frac{dy}{dx} = \tan y,$$

if (a) $y(0) = 0$, *and* (b) $y(0) = \pi/2$.

Since x appears only in the combination $x\,dy/dx$ it will probably make the solution simpler to take y as the independent variable and x as the dependent one. With this in mind, we re-arrange the equation as

$$\tan y \frac{dx}{dy} + x = 2\sin y,$$

or, in standard form, as

$$\frac{dx}{dy} + x\cot y = 2\cos y.$$

The IF is clearly $\exp(\ln \sin y) = \sin y$, and the equation can be written

$$\frac{d}{dy}(x \sin y) = \sin 2y,$$

$$x \sin y = -\tfrac{1}{2}\cos 2y + k.$$

(a) For $y(0) = 0$ we must have $k = \frac{1}{2}$, and the solution becomes

$$x = \frac{1 - \cos 2y}{2\sin y} = \frac{2\sin^2 y}{2\sin y} = \sin y.$$

(b) If $y(0) = \pi/2$ then $k = -\frac{1}{2}$ and the solution is

$$x = \frac{-1 - \cos 2y}{2\sin y} = \frac{-2\cos^2 y}{2\sin y} = -\cos y \cot y.$$

15

Higher-order ordinary differential equations

> ***15.2*** *Find the roots of the auxiliary equation for the following. Hence solve them for the boundary conditions stated.*
>
> $$\text{(a)}\ \frac{d^2f}{dt^2} + 2\frac{df}{dt} + 5f = 0 \quad \text{with } f(0) = 1, f'(0) = 0.$$
>
> $$\text{(b)}\ \frac{d^2f}{dt^2} + 2\frac{df}{dt} + 5f = e^{-t}\cos 3t \quad \text{with } f(0) = 0, f'(0) = 0.$$

The two equations have the same LHS and the trial function $f(x) = e^{mx}$ leads to the common auxiliary equation

$$m^2 + 2m + 5 = 0 \quad \Rightarrow \quad m = -1 \pm \sqrt{1-5} = -1 \pm 2i.$$

Thus the CF for both equations is $f(t) = e^{-t}(A\cos 2t + B\sin 2t)$.

(a) Since the RHS of the equation is zero, no particular integral is needed (formally it is $f(x) = 0$). For the CF the boundary conditions require

$$\begin{aligned}
f(0) = 1 \quad &\Rightarrow \quad 1 = 1(A + 0) \quad \Rightarrow \quad A = 1,\\
f'(0) = 0 \quad &\Rightarrow \quad 0 = -2e^{-t}\sin 2t - e^{-t}\cos 2t\\
&\qquad\qquad - Be^{-t}\sin 2t + 2Be^{-t}\cos 2t, \text{ at } t = 0,\\
&\qquad = -1 + 2B \quad \Rightarrow \quad B = \tfrac{1}{2}.
\end{aligned}$$

Resubstitution gives $\quad f(t) = e^{-t}(\cos 2t + \frac{1}{2}\sin 2t).$

(b) Since the CF does not contain a term involving $e^{-t}\cos 3t$ we may try a linear combination of $e^{-t}\cos 3t$ and $e^{-t}\sin 3t$ as the PI, as follows.

$$\begin{aligned}
f &= e^{-t}(C\cos 3t + D\sin 3t),\\
f' &= -e^{-t}(C\cos 3t + D\sin 3t) + e^{-t}(-3C\sin 3t + 3D\cos 3t),\\
f'' &= e^{-t}(C\cos 3t + D\sin 3t) - 2e^{-t}(-3C\sin 3t + 3D\cos 3t)\\
&\qquad - 9e^{-t}(C\cos 3t + D\sin 3t).
\end{aligned}$$

When these are substituted into the equation, the coefficients of $e^{-t}\sin 3t$ require that

$$D + 6C - 9D - 2D - 6C + 5D = 0 \quad \Rightarrow \quad D = 0.$$

Those for $e^{-t}\cos 3t$ imply that

$$C - 0 - 9C - 2C + 0 + 5C = 1 \quad \Rightarrow \quad C = -\tfrac{1}{5}.$$

With this PI the general solution becomes

$$f(t) = e^{-t}(A\cos 2t + B\sin 2t) - \tfrac{1}{5}e^{-t}\cos 3t.$$

The boundary condition $f(0) = 0$ requires that $A = \frac{1}{5}$, and the condition on the derivative, $f'(0) = 0$, implies (after multiplying all through by 5 for convenience) that

$$\begin{aligned}
0 &= -e^{-t}(\cos 2t + 5B\sin 2t - \cos 3t)\\
&\qquad + e^{-t}(-2\sin 2t + 10B\cos 2t + 3\sin 3t) \text{ at } t = 0,\\
\Rightarrow \quad 0 &= -(1 + 0 - 1) + (-0 + 10B + 0) \quad \Rightarrow \quad B = 0.
\end{aligned}$$

Thus, the final solution is

$$f(t) = \tfrac{1}{5}e^{-t}(\cos 2t - \cos 3t),$$

which can, if necessary, be checked by re-substitution.

15.4 *Solve the differential equation*

$$\frac{d^2f}{dt^2} + 6\frac{df}{dt} + 9f = e^{-t},$$

subject to the conditions $f = 0$ *and* $df/dt = \lambda$ *at* $t = 0$.

Find the equation satisfied by the positions of the turning points of $f(t)$ *and hence, by drawing suitable sketch graphs, determine the number of turning points the solution has in the range* $t > 0$ *if* (a) $\lambda = 1/4$, *and* (b) $\lambda = -1/4$.

The auxiliary equation and resulting CF are

$$m^2 + 6m + 9 = 0 \quad \Rightarrow \quad m = -3 \text{ (repeated root)},$$
$$\Rightarrow \quad f(t) = (A + Bt)e^{-3t}.$$

For a particular integral, since e^{-t} does not appear in the CF, we try $f = Ce^{-t}$ and obtain

$$Ce^{-t} - 6Ce^{-t} + 9Ce^{-t} = e^{-t} \quad \Rightarrow \quad C = \tfrac{1}{4}.$$

The general solution is therefore

$$f(t) = (A + Bt)e^{-3t} + \tfrac{1}{4}e^{-t}.$$

We now incorporate the boundary conditions:

$$f(0) = 0 \quad \Rightarrow \quad 0 = (A + 0)1 + \tfrac{1}{4} \quad \Rightarrow \quad A = -\tfrac{1}{4},$$
$$f'(0) = \lambda \quad \Rightarrow \quad \lambda = -3A + B - 0 - \tfrac{1}{4} \quad \Rightarrow \quad B = \lambda - \tfrac{1}{2}.$$

Therefore the solution matching these boundary conditions is

$$f(t) = [-\tfrac{1}{4} + (\lambda - \tfrac{1}{2})t]e^{-3t} + \tfrac{1}{4}e^{-t}.$$

The turning points of the solution are given by $f'(t) = 0$, i.e.

$$\tfrac{3}{4}e^{-3t} + (\lambda - \tfrac{1}{2})e^{-3t} - 3(\lambda - \tfrac{1}{2})te^{-3t} - \tfrac{1}{4}e^{-t} = 0,$$
$$e^{-2t}\left[(\lambda + \tfrac{1}{4}) + (\tfrac{3}{2} - 3\lambda)t\right] = \tfrac{1}{4},$$
$$(4\lambda + 1) + (6 - 12\lambda)t = e^{2t}.$$

(a) For $\lambda = \frac{1}{4}$ the equation becomes $e^{2t} = 2 + 3t$. Consider the behaviours of the functions on either side of this equation:

At $t = 0$, $\quad 2 + 3t = 2 > 1 = e^{2t}$.

For large t, $\quad e^{2t} > 2 + 3t$.

Both functions are monotonic and thus there is one, and only one, solution to $e^{2t} = 2 + 3t$ in $t > 0$. It follows that the solution to the original differential equation has only one turning point in this range.

(b) For $\lambda = -\frac{1}{4}$ the equation becomes $e^{2t} = 9t$. Again consider the behaviours of the two sides of the equation.

At $t = 0$, $\quad e^{2t} = 1 > 0 = 9t$.

At $t = 1$, $\quad e^{2t} = e^2 < 9 = 9t$.

For large t, $\quad e^{2t} > 9t$.

Both functions are monotonic and thus there are two solutions to $e^{2t} = 9t$ in $t > 0$. It follows that the solution to the original differential equation has two turning points in this range.

***15.6** Determine the values of α and β for which the following functions are linearly dependent:*

$$y_1(x) = x\cosh x + \sinh x,$$
$$y_2(x) = x\sinh x + \cosh x,$$
$$y_3(x) = (x+\alpha)e^x,$$
$$y_4(x) = (x+\beta)e^{-x}.$$

You will find it convenient to work with those linear combinations of the $y_i(x)$ that can be written the most compactly.

To make the working more compact, write

$$y_5(x) = y_1 + y_2 = (x+1)(\cosh x + \sinh x) = (x+1)e^x,$$
$$y_6(x) = y_1 - y_2 = (x-1)(\cosh x - \sinh x) = (x-1)e^{-x}.$$

We notice that $y_3 = y_5$ if $\alpha = 1$ and that $y_4 = y_6$ if $\beta = -1$. With these values the functions are linearly dependent and so give the answer to the original question.

However, we will continue as if this had not been noticed and compute the Wronskian $W(y_3, y_4, y_5, y_6)$. For this we need the derivatives (using Leibnitz' theorem)

$$\frac{d^n}{dx^n}[\,(x+\gamma)e^x\,] = (x+\gamma)e^x + ne^x = (x+\gamma+n)e^x,$$
$$\frac{d^n}{dx^n}[\,(x+\gamma)e^{-x}\,] = (-1)^n(x+\gamma)e^{-x} + n(-1)^{n-1}e^{-x}$$
$$= (-1)^n(x+\gamma-n)e^{-x}.$$

Each column of the Wronskian will have a common factor of $e^{\pm x}$ and we take these outside the determinant, writing $W(y_3, y_4, y_5, y_6)$ as

$$W = e^x\,e^{-x}\,e^x\,e^{-x}\begin{vmatrix} x+\alpha & x+\beta & x+1 & x-1 \\ x+\alpha+1 & -x-\beta+1 & x+2 & -x+2 \\ x+\alpha+2 & x+\beta-2 & x+3 & x-3 \\ x+\alpha+3 & -x-\beta+3 & x+4 & -x+4 \end{vmatrix}$$
$$= \begin{vmatrix} \alpha-1 & \beta+1 & x+1 & x-1 \\ \alpha-1 & -\beta-1 & x+2 & -x+2 \\ \alpha-1 & \beta+1 & x+3 & x-3 \\ \alpha-1 & -\beta-1 & x+4 & -x+4 \end{vmatrix}.$$

To obtain this last form, we have subtracted the third column from the first and the fourth from the second. The common factors $\alpha-1$ and $\beta+1$ can be taken out of the determinant which then becomes a function of x only. For the Wronskian to vanish for all x (and hence make the functions dependent) requires either $\alpha = 1$ or $\beta = -1$ or both. In fact, the remaining determinant has the value -16, independent of the value of x, but all that matters for drawing our conclusion is that it is non-zero.

15.8 *The two functions $x(t)$ and $y(t)$ satisfy the simultaneous equations*

$$\frac{dx}{dt} - 2y = -\sin t,$$
$$\frac{dy}{dt} + 2x = 5\cos t.$$

Find explicit expressions for $x(t)$ and $y(t)$, given that $x(0) = 3$ and $y(0) = 2$. Sketch the solution trajectory in the xy-plane for $0 \le t < 2\pi$, showing that the trajectory crosses itself at $(0, 1/2)$ and passes through the points $(0, -3)$ and $(0, -1)$ in the negative x-direction.

By differentiating the first equation and then substituting for dy/dt from the second we obtain

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = -\cos t,$$
$$\frac{d^2x}{dt^2} - 2(5\cos t - 2x) = -\cos t,$$
$$\frac{d^2x}{dt^2} + 4x = 9\cos t.$$

The RHS is not contained in the CF and so the general solution is of the form

$$x(t) = A\cos 2t + B\sin 2t + C\cos t.$$

Substituting the PI part of this into the equation to find the value of C, gives $C(-1+4) = 9$ and hence $C = 3$. Further, since $x(0) = 3$, we must have $A = 0$ and it follows that

$$x(t) = B\sin 2t + 3\cos t.$$

Now,

$$\begin{aligned} y(t) &= \frac{1}{2}\left(\frac{dx}{dt} + \sin t\right) \\ &= B\cos 2t - \tfrac{3}{2}\sin t + \tfrac{1}{2}\sin t \\ &= B\cos 2t - \sin t. \end{aligned}$$

Since $y(0) = 2$, $B = 2$ and so, in summary,

$$x(t) = 2\sin 2t + 3\cos t,$$
$$y(t) = 2\cos 2t - \sin t.$$

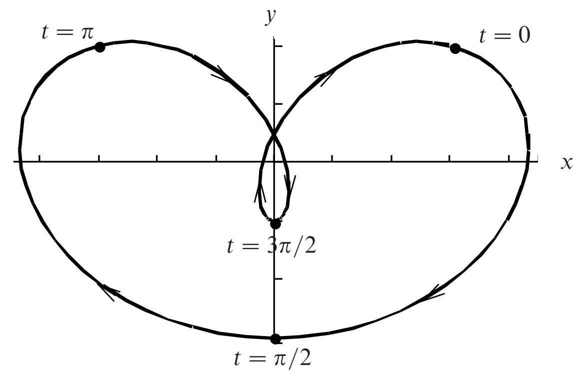

Figure 15.1 The closed curve generated by the equations in exercise 15.8.

The (closed) curve is shown in figure 15.1.

It crosses the y-axis when

$$2\sin 2t + 3\cos t = 0, \text{ i.e when } 4\sin t\cos t + 3\cos t = 0.$$

This has solutions when $\cos t = 0$, i.e. $t = \frac{1}{2}\pi$ and $t = \frac{3}{2}\pi$, as well as when $\sin t = -\frac{3}{4}$. The latter corresponds to two values of t, but with only one corresponding y-value given by

$$y(t) = 2\cos 2t - \sin t = 2 - 4\sin^2 t - \sin t = 2 - 4\left(-\frac{3}{4}\right)^2 + \frac{3}{4} = \frac{1}{2}.$$

Thus the curve crosses itself at $(0, \frac{1}{2})$.

Finally, consider the two other points on the trajectory at which $x = 0$.

When $t = \frac{1}{2}\pi$, $y(t) = 2(-1) - 1 = -3$ and $dx/dt = 4\cos 2t - 3\sin t = -4 - 3 = -7$.

When $t = \frac{3}{2}\pi$, $y(t) = 2(-1) - (-1) = -1$ and $dx/dt = -4 + 3 = -1$.

In both cases dx/dt is negative, showing that the trajectory passes through the points $(0, -3)$ and $(0, -1)$ in the negative x-direction.

15.10 *Use the method of Laplace transforms to solve*

(a) $\dfrac{d^2 f}{dt^2} + 5\dfrac{df}{dt} + 6f = 0, \qquad f(0) = 1,\ f'(0) = -4,$

(b) $\dfrac{d^2 f}{dt^2} + 2\dfrac{df}{dt} + 5f = 0, \qquad f(0) = 1,\ f'(0) = 0.$

(a) Recalling that $\mathscr{L}\left[f'\right] = s\bar{f} - f(0)$ and $\mathscr{L}\left[f''\right] = s^2\bar{f} - sf(0) - f'(0)$, we have

$$\begin{aligned}
0 &= s^2\bar{f} - sf(0) - f'(0) + 5[s\bar{f} - f(0)] + 6\bar{f} \\
&= s^2\bar{f} - s + 4 + 5(s\bar{f} - 1) + 6\bar{f}, \\
\Rightarrow \quad \bar{f} &= \frac{s+1}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3}, \\
&= \frac{-1}{s+2} + \frac{2}{s+3}, \text{ using one of the standard methods,} \\
\Rightarrow \quad f(t) &= 2e^{-3t} - e^{-2t}, \text{ from the look-up table.}
\end{aligned}$$

(b) Using the same method as in (a), we have

$$\begin{aligned}
0 &= s^2\bar{f} - sf(0) - f'(0) + 2[s\bar{f} - f(0)] + 5\bar{f} \\
&= s^2\bar{f} - s + 2(s\bar{f} - 1) + 5\bar{f}, \\
\Rightarrow \quad \bar{f} &= \frac{s+2}{s^2+2s+5} = \frac{s+2}{(s+1)^2+4} \\
&= \frac{(s+1)+1}{(s+1)^2+4}, \\
\Rightarrow \quad f(t) &= e^{-t}\cos 2t + \tfrac{1}{2}e^{-t}\sin 2t, \text{ from the look-up table.}
\end{aligned}$$

We note that this is the same result as that obtained in Exercise 15.2(a).

15.12 *Use Laplace transforms to solve, for $t \geq 0$, the differential equations*

$$\begin{aligned}
\ddot{x} + 2x + y &= \cos t, \\
\ddot{y} + 2x + 3y &= 2\cos t,
\end{aligned}$$

which describe a coupled system that starts from rest at the equilibrium position. Show that the subsequent motion takes place along a straight line in the xy-plane. Verify that the frequency at which the system is driven is equal to one of the resonance frequencies of the system; explain why there is no *resonant behaviour in the solution you have obtained.*

We start by taking the Laplace transforms of the equations with all initial values and first derivatives equal to zero.

$$\begin{aligned}
s^2\bar{x} + 2\bar{x} + \bar{y} &= \frac{s}{1+s^2} \qquad (*), \\
s^2\bar{y} + 2\bar{x} + 3\bar{y} &= \frac{2s}{1+s^2} \qquad (**).
\end{aligned}$$

Now consider the equation obtained by taking $2 \times (*) - (**)$.

$$\begin{aligned} \bar{x}(2s^2 + 4 - 2) + \bar{y}(2 - s^2 - 3) &= 0, \\ 2(s^2 + 1)\bar{x} - (s^2 + 1)\bar{y} &= 0, \\ 2\bar{x} - \bar{y} &= 0. \end{aligned}$$

This final equation, which is independent of s and hence of the t-dependence of x and y, means that $y(t)$ is a direct multiple of $x(t)$ and the motion takes place along a straight line in the x-y plane.

Setting $\bar{y} = 2\bar{x}$ in $(*)$ gives

$$(s^2 + 4)\bar{x} = \frac{s}{1 + s^2},$$

which, after rearrangement, gives the partial fraction expression for $\bar{x}$ as

$$\bar{x} = \frac{s}{3(s^2 + 1)} - \frac{s}{3(s^2 + 4)}.$$

This, in turn, implies (from the table of Laplace transforms) that

$$x(t) = \tfrac{1}{3}(\cos t - \cos 2t).$$

As in chapter 9 on Normal Modes, the natural frequencies of the system are given by

$$\begin{aligned} \begin{vmatrix} -\omega^2 + 2 & 1 \\ 2 & -\omega^2 + 3 \end{vmatrix} &= 0, \\ \omega^4 - 5\omega^2 + 4 &= 0, \\ (\omega^2 - 4)(\omega^2 - 1) &= 0. \end{aligned}$$

Thus the resonance frequencies are $\omega = 2$ and $\omega = 1$; the given driving frequency is the second of these.

However, for $\omega = 1$ the (x, y) eigenvector satisfies $(-1 + 2)x + (1)y = 0$, i.e. $y = -x$, whilst for $\omega = 2$ the (x, y) eigenvector satisfies $(-4 + 2)x + (1)y = 0$, i.e. $y = 2x$.

The driving terms in the given situation have frequency $\omega = 1$. But the solution obtained is purely that corresponding to $\omega = 2$ and contains no component of the $\omega = 1$ response. Consequently there is no resonant behaviour.

15.14 *For a lightly damped ($\gamma < \omega_0$) harmonic oscillator driven at its undamped resonance frequency ω_0, the displacement $x(t)$ at time t satisfies the equation*

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = F\sin\omega_0 t.$$

Use Laplace transforms to find the displacement at a general time if the oscillator starts from rest at its equilibrium position.

(a) *Show that ultimately the oscillation has amplitude $F/(2\omega_0\gamma)$ with a phase lag of $\pi/2$ relative to the driving force per unit mass F.*
(b) *By differentiating the original equation, conclude that if $x(t)$ is expanded as a power series in t for small t then the first non-vanishing term is $F\omega_0 t^3/6$. Confirm this conclusion by expanding your explicit solution.*

With no initial dispacement or motion, the Laplace transformed equation reads

$$s^2\bar{x} + 2\gamma s\bar{x} + \omega_0^2\bar{x} = \frac{F\omega_0}{s^2+\omega_0^2},$$

$$\frac{\bar{x}}{F\omega_0} = \frac{1}{(s^2+\omega_0^2)(s^2+2\gamma s+\omega_0^2)} = \frac{A+Bs}{s^2+\omega_0^2} + \frac{C+Ds}{(s+\gamma)^2+k^2},$$

where $k^2 = \omega_0^2 - \gamma^2$.

Cross-multiplying and equating the coefficients of the various powers of s yields

$$\begin{aligned} s^3 &: D + B = 0,\\ s^2 &: A + 2B\gamma + C = 0,\\ s^1 &: 2A\gamma + B\gamma^2 + Bk^2 + D\omega_0^2 = 0,\\ s^0 &: A\gamma^2 + Ak^2 + C\omega_0^2 = 1, \end{aligned}$$

with solutions

$$A = 0, \qquad D = -B = \frac{1}{2\gamma\omega_0^2}, \qquad C = \frac{1}{\omega_0^2}.$$

We can now rewrite the partial fraction expansion as

$$\frac{2\gamma\omega_0^2\bar{x}}{F\omega_0} = -\frac{s}{s^2+\omega_0^2} + \frac{(s+\gamma)+2\gamma-\gamma}{(s+\gamma)^2+k^2},$$

which integrates to

$$\frac{2\gamma\omega_0\,x(t)}{F} = -\cos\omega_0 t + e^{-\gamma t}\cos kt + \frac{\gamma}{k}e^{-\gamma t}\sin kt,$$

$$\text{i.e. } x(t) = \frac{F}{2\omega_0}\left[\frac{1}{\gamma}(e^{-\gamma t}\cos kt - \cos\omega_0 t) + \frac{1}{k}e^{-\gamma t}\sin kt\right].$$

This is the complete solution, valid for all times $t > 0$.

(a) As $t \to \infty$,

$$x(t) \approx -\frac{F}{2\omega_0\gamma}\cos\omega_0 t = \frac{F}{2\omega_0\gamma}\sin(\omega_0 t - \tfrac{1}{2}\pi).$$

Thus ultimately the oscillation has amplitude $F/(2\omega_0\gamma)$ with a phase lag of $\pi/2$ relative to the driving force F.

(b) At $t = 0$, both x and x' are zero, and so is $F\sin\omega_0 t$. It therefore follows from the original equation that $x''(0)$ is also zero. Thus, if $x(t)$ were expanded in a Taylor series about $t = 0$ the constant, linear and quadratic terms of the series would be missing.

Now consider the equation

$$x''' + 2\gamma x'' + \omega_0^2 x' = \omega_0 F\cos\omega_0 t,$$

obtained by differentiating the original one. At $t = 0$ this reduces to $x''' = \omega_0 F$, which is non-zero. Thus the leading term in the Taylor expansion of $x(t)$ is $F\omega_0 t^3/3!$.

From the explicit solution, the contributions to $2\omega_0 x(t)/F$ of the three terms, up to order t^3, are:

$$\begin{aligned} f_1 &= \frac{1}{\gamma}\left(1 - \frac{k^2t^2}{2!} + \cdots\right)\left(1 - \gamma t + \frac{\gamma^2 t^2}{2!} - \frac{\gamma^3 t^3}{3!} + \cdots\right) \\ &= \frac{1}{\gamma}\left(1 - \gamma t + \frac{(\gamma^2 - k^2)t^2}{2!} - \frac{(\gamma^3 - 3\gamma k^2)t^3}{3!} + \cdots\right), \end{aligned}$$

$$f_2 = -\frac{1}{\gamma}\left(1 - \frac{\omega_0^2 t^2}{2!} + \cdots\right),$$

$$\begin{aligned} f_3 &= \frac{1}{k}\left(kt - \frac{k^3t^3}{3!} + \cdots\right)\left(1 - \gamma t + \frac{\gamma^2 t^2}{2!} - \frac{\gamma^3 t^3}{3!} + \cdots\right) \\ &= \frac{1}{k}\left(kt - k\gamma t^2 + \frac{(3k\gamma^2 - k^3)t^3}{3!} + \cdots\right). \end{aligned}$$

Recalling that $k^2 = \omega_0^2 - \gamma^2$, we see that, when these contributions are added together, the constant term and the linear and quadratic terms in t all vanish. The cubic term in $2\omega_0 x(t)/F$ is

$$-\frac{(\gamma^2 - 3k^2)t^3}{3!} + \frac{(3\gamma^2 - k^2)t^3}{3!} = \frac{2(\gamma^2 + k^2)t^3}{3!} = \frac{\omega_0^2 t^3}{3},$$

and so the leading term in $x(t)$ is $F\omega_0 t^3/6$. This confirms our earlier conclusion based on the differential equation rather than its solution.

15.16 *In a particular scheme for modelling numerically one-dimensional fluid flow, the successive values, u_n, of the solution are connected for $n \geq 1$ by the difference equation*

$$c(u_{n+1} - u_{n-1}) = d(u_{n+1} - 2u_n + u_{n-1}),$$

where c and d are positive constants. The boundary conditions are $u_0 = 0$ and $u_M = 1$. Find the solution to the equation and show that successive values of u_n will have alternating signs if $c > d$.

We substitute the trial solution $u_n = A\lambda^n$ into the recurrence relation and obtain

$$-c(\lambda^{n+1} - \lambda^{n-1}) + d(\lambda^{n+1} - 2\lambda^n + \lambda^{n-1}) = 0,$$
$$(d - c)\lambda^2 - 2d\lambda + (d + c) = 0.$$

This is a quadratic equation for λ, with solution

$$\lambda = \frac{d \pm \sqrt{d^2 - (d^2 - c^2)}}{d - c} = \frac{d \pm c}{d - c} = 1 \text{ or } \frac{d + c}{d - c}.$$

The general solution, formed by taking a linear superposition of the trial solutions corresponding to the allowed values of λ, is thus

$$u_n = A1^n + B\left(\frac{d + c}{d - c}\right)^n \equiv A + B\mu^n, \text{ defining } \mu.$$

Now, imposing the boundary conditions:

$$u_0 = 0 \quad \Rightarrow \quad B = -A,$$
$$u_M = 1 \quad \Rightarrow \quad A(1 - \mu^M) = 1,$$
$$\Rightarrow \quad u_n \quad = \quad \frac{1 - \mu^n}{1 - \mu^M}.$$

This is the specific solution as a function of n.

If $c > d$ then μ is negative and has a magnitude > 1. The ratio of successive terms is

$$\frac{1 - \mu^{n+1}}{1 - \mu^n} = \frac{1}{(d - c)} \frac{[\,(d - c)^{n+1} - (d + c)^{n+1}\,]}{[\,(d - c)^n - (d + c)^n\,]}.$$

Since d and c are both positive, the terms in square brackets are necessarily both negative and the ratio has the same sign as $d - c$, i.e negative. Thus successive terms alternate in sign.

> ***15.18*** *Find an explicit expression for the* u_n *satisfying*
>
> $$u_{n+1} + 5u_n + 6u_{n-1} = 2^n,$$
>
> *given that* $u_0 = u_1 = 1$. *Deduce that* $2^n - 26(-3)^n$ *is divisible by 5 for all integer* n.

The characteristic equation of the recurrence relation, obtained by substituting $u_n = C\lambda^n$ into it with the RHS set equal to zero, is

$$\lambda^2 + 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda = -2 \text{ or } -3.$$

As neither value of λ is equal to 2, we may try $u_n = D\,2^n$ as a particular solution, leading to

$$D\,2^{n+1} + 5D\,2^n + 6D\,2^{n-1} = 2^n,$$
$$D(4 + 10 + 6) = 2, \quad \Rightarrow \quad D = \tfrac{1}{10}.$$

The general solution is thus

$$u_n = A(-2)^n + B(-3)^n + \frac{2^n}{10}.$$

Incorporating the two initial values:

$$u_0 = 1 \quad \Rightarrow \quad 1 = A + B + \tfrac{1}{10},$$
$$u_1 = 1 \quad \Rightarrow \quad 1 = -2A - 3B + \tfrac{2}{10},$$
$$\Rightarrow \quad A = \tfrac{35}{10} \text{ and } B = -\tfrac{26}{10}.$$

Thus, for general n,

$$u_n = \frac{1}{10}\left[35(-2)^n - 26(-3)^n + 2^n\right].$$

With these initial values and a recurrence relation that has integer coefficients (with that for the highest-index term equal to unity) all terms in the series *must* be integers. Thus, the expression in square brackets must divide by 10 for all $n \geq 2$, as well as for $n = 0$ and $n = 1$.

For $n > 0$, the first term in the bracket contains explicit factors of 2 and 5 and so divides by 10. We thus conclude that the sum of the remaining terms must also divide by 10, i.e. $2^n - 26(-3)^n$ divides by 10 and, therefore, also by 5. For $n = 0$, explicit evaluation of the expression gives -25, which is divisible by 5; this completes the proof.

15.20 *Consider the seventh-order recurrence relation*

$$u_{n+7} - u_{n+6} - u_{n+5} + u_{n+4} - u_{n+3} + u_{n+2} + u_{n+1} - u_n = 0.$$

Find the most general form of its solution, and show that:

(a) *if only the four initial values $u_0 = 0$, $u_1 = 2$, $u_2 = 6$ and $u_3 = 12$, are specified then the relation has one solution that cycles repeatedly through this set of four numbers;*
(b) *but if, in addition, it is required that $u_4 = 20$, $u_5 = 30$ and $u_6 = 42$ then the solution is unique, with $u_n = n(n+1)$.*

The characteristic equation is a seventh-order polynomial equation (but fortunately one with some obvious roots).

$$\begin{aligned}\lambda^7 - \lambda^6 - \lambda^5 + \lambda^4 - \lambda^3 + \lambda^2 + \lambda - 1 &= 0,\\ (\lambda - 1)(\lambda^6 - \lambda^4 - \lambda^2 + 1) &= 0,\\ (\lambda - 1)(\lambda^2 - 1)(\lambda^4 - 1) &= 0.\end{aligned}$$

The roots are therefore $\lambda = 1$ (triple), $\lambda = -1$ (double) and $\lambda = \pm i$. Consequently, the general solution is

$$(A + Bn + Cn^2)1^n + (D + En)(-1)^n + F(i)^n + G(-i)^n,$$

where the constants $A, B, \ldots, G$ must be consistent with any given values of particular u_n.

(a) If only the four initial values $u_0 = 0$, $u_1 = 2$, $u_2 = 6$ and $u_3 = 12$ are specified then we can choose all constants associated with linear or quadratic terms in n to be zero, i.e. $B = C = E = 0$ and solve for the remaining constants.

$$\begin{aligned}n &= 0, & 0 &= A + D + F + G,\\ n &= 1, & 2 &= A - D + iF - iG,\\ n &= 2, & 6 &= A + D - F - G,\\ n &= 3, & 12 &= A - D - iF + iG.\end{aligned}$$

Adding all the equations shows that $A = 5$, and adding the first and third shows that $A + D = 3$, i.e. $D = -2$. Putting these values into the first two equations then gives

$$F = -\tfrac{3}{2} + \tfrac{5}{2}i \quad \text{and} \quad G = -\tfrac{3}{2} - \tfrac{5}{2}i.$$

Thus the solution

$$u_n = 5 - 2(-1)^n - \tfrac{3}{2}(i)^n(1 + (-1)^n) + \tfrac{5}{2}(i)^{n+1}(1 - (-1)^n)$$

fits the first four given values and then cycles endlessly around them since $(-1)^n$, $(i)^n$ and $(-i)^n$ are all unchanged if n is increased by 4.

(b) With the first 7 values given and 7 unknown constants $A, B, \ldots, G$ to be determined, the solution will be unique (unless the determining equations turn out to be dependent). The simultaneous equations to be solved are:

$$\begin{aligned}
0 &= A + D + F + G,\\
2 &= A + B + C - D - E + iF - iG,\\
6 &= A + 2B + 4C + D + 2E - F - G,\\
12 &= A + 3B + 9C - D - 3E - iF + iG,\\
20 &= A + 4B + 16C + D + 4E + F + G,\\
30 &= A + 5B + 25C - D - 5E + iF - iG,\\
42 &= A + 6B + 36C + D + 6E - F - G.
\end{aligned}$$

It is clear from inspection and easily verified by substitution that they are satisfied by $B = C = 1$, with all other constants equal to zero. The direct solution of these equations, though tedious, gives the same result; it also provides assurance that the solution is unique. The general expression is therefore $u_n = n(n + 1)$.

15.22 *Find the general solution of*

$$(x + 1)^2 \frac{d^2 y}{dx^2} + 3(x + 1)\frac{dy}{dx} + y = x^2.$$

This is Legendre's linear equation and, as a first step, we set $x + 1 = e^t$ with

$$\frac{dx}{dt} = e^t, \qquad \frac{d}{dx} = e^{-t}\frac{d}{dt}, \qquad \frac{d^2}{dx^2} = e^{-t}\frac{d}{dt}\left(e^{-t}\frac{d}{dt}\right).$$

These substitutions give

$$\begin{aligned}
e^{2t}e^{-t}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right) + 3e^t e^{-t}\frac{dy}{dt} + y &= (e^t - 1)^2,\\
e^t\left(e^{-t}\frac{d^2 y}{dt^2} - e^{-t}\frac{dy}{dt}\right) + 3\frac{dy}{dt} + y &= (e^t - 1)^2,\\
\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + y &= (e^t - 1)^2.
\end{aligned}$$

This reduced equation with constant coefficients has the characteristic equation $m^2 + 2m + 1 = 0$, which has a repeated root and gives the CF as $y(t) = (A + Bt)e^{-t}$. This is not the same function as that in the equation's RHS (which contains a constant and t-dependent terms e^{2t} and e^t); we may therefore try substituting the simplest PI of $Ce^{2t} + De^t + E$ to obtain

$$4Ce^{2t} + De^t + 4Ce^{2t} + 2De^t + Ce^{2t} + De^t + E = e^{2t} - 2e^t + 1.$$

Clearly $C = \frac{1}{9}$, $D = -\frac{1}{2}$ and $E = 1$ and, after re-substituting for t, we have the general solution of the original equation as

$$\begin{aligned} y(x) &= \frac{A + B\ln(x+1)}{x+1} + \frac{(x+1)^2}{9} - \frac{x+1}{2} + 1 \\ &= \frac{A + B\ln(x+1)}{x+1} + \frac{x^2}{9} - \frac{5x}{18} + \frac{11}{18}. \end{aligned}$$

As expected, since the differential equation is second-order, its solution contains two arbitrary constants.

15.24 *Use the method of variation of parameters to find the general solutions of*

$$\text{(a)}\ \frac{d^2y}{dx^2} - y = x^n, \quad \text{(b)}\ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2xe^x.$$

(a) The CF is clearly

$$y(x) = Ae^x + Be^{-x},$$

and so we take as the PI

$$y(x) = k_1(x)e^x + k_2(x)e^{-x}.$$

The two simultaneous equations generated using the method of variation of parameters are

$$\begin{aligned} k_1'e^x + k_2'e^{-x} &= 0, \\ k_1'e^x - k_2'e^{-x} &= x^n. \end{aligned}$$

Solving for k_1' and integrating gives

$$\begin{aligned} k_1' &= \frac{x^n e^{-x}}{2}, \\ k_1 &= \left[-\frac{x^n e^{-x}}{2}\right]^x + \int^x \frac{nx^{n-1}}{2}\, e^{-x}\, dx \\ &= -\frac{e^{-x}}{2}(x^n + nx^{n-1} + n(n-1)x^{n-2} + \cdots + n!) \\ &= -\frac{e^{-x}}{2}\, n! \sum_{m=0}^{n} \frac{x^m}{m!}. \end{aligned}$$

Similarly, k_2 is given by

$$\begin{aligned}
k_2' &= -\frac{x^n e^x}{2}, \\
k_2 &= -\left[\frac{x^n e^x}{2}\right]^x - \int^x \frac{nx^{n-1}}{2}\, e^x\, dx \\
&= -\frac{e^x}{2}(x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^n n!) \\
&= -\frac{e^x}{2}\, n!(-1)^n \sum_{m=0}^{n} \frac{(-x)^m}{m!}.
\end{aligned}$$

The full PI, $k_1(x)e^x + k_2(x)e^{-x}$, has no explicit exponential factors, since each term in it contains the product $e^x e^{-x}$. It takes the form

$$\begin{aligned}
y(x) &= -\frac{n!}{2}\sum_{m=0}^{n}\frac{x^m}{m!} - \frac{n!}{2}(-1)^n \sum_{m=0}^{n}\frac{(-x)^m}{m!} \\
&= -\frac{n!}{2}\sum_{m=0}^{n}\frac{x^m}{m!}[\,1 + (-1)^{n+m}\,].
\end{aligned}$$

This n-th order polynomial is added to the CF, $y(x) = Ae^x + Be^{-x}$, to give the general solution.

(b) The auxiliary equation for the CF is $m^2 - 2m + 1 = 0$, which has repeated roots $m = 1$. Thus the Cf is $y(x) = (A + Bx)e^x$ and, since the RHS of the original equation is contained in this, the PI is to be taken as $y(x) = k_1(x)e^x + k_2(x)xe^x$. The simultaneous equations generated by the variation of parameters method are

$$\begin{aligned}
k_1'e^x + k_2'xe^x &= 0, \\
k_1'e^x + k_2'(e^x + xe^x) &= 2xe^x, \\
k_2'e^x &= 2xe^x \quad \Rightarrow \quad k_2(x) = x^2, \\
k_1' &= -k_2'x = -2x^2 \quad \Rightarrow \quad k_1(x) = -\tfrac{2}{3}x^3.
\end{aligned}$$

A PI is therefore

$$y(x) = -\tfrac{2}{3}x^3 e^x + x^2\, xe^x = \tfrac{1}{3}x^3 e^x,$$

giving the general solution as

$$y(x) = (A + Bx + \tfrac{1}{3}x^3)e^x.$$

15.26 *Consider the equation*

$$F(x, y) = x(x+1)\frac{d^2y}{dx^2} + (2 - x^2)\frac{dy}{dx} - (2+x)y = 0.$$

(a) *Given that $y_1(x) = 1/x$ is one of its solutions, find a second linearly independent one,*
(i) *by setting $y_2(x) = y_1(x)u(x)$, and*
(ii) *by noting the sum of the coefficients in the equation.*

(b) *Hence, using the variation of parameters method, find the general solution of*

$$F(x, y) = (x+1)^2.$$

(a)(i) Set $y_2(x) = u(x)/x$ and substitute:

$$x(x+1)\left(\frac{2u}{x^3} - \frac{2u'}{x^2} + \frac{u''}{x}\right) + (2 - x^2)\left(-\frac{u}{x^2} + \frac{u'}{x}\right) - \frac{2+x}{x}u = 0,$$

$$(1+x)u'' + \left[-\frac{2(1+x)}{x} + \frac{2-x^2}{x}\right]u' + 0u = 0,$$

$$(1+x)u'' - (2+x)u' = 0.$$

Hence, on separating variables and integrating once, we have

$$\frac{u''}{u'} = \frac{2+x}{1+x} = 1 + \frac{1}{1+x},$$

$$\ln u' = x + \ln(1+x),$$

$$u' = (1+x)e^x.$$

A second integration then gives

$$u = e^x + [\,xe^x\,]^x - \int^x e^x\,dx,$$

$$= e^x + xe^x - e^x = xe^x,$$

$$\text{i.e.} \qquad y_2(x) = \frac{1}{x}xe^x = e^x \text{ is the second solution.}$$

(a)(ii) The sum of the coefficients of the various terms in the linear equation is

$$x(x+1) + (2 - x^2) - (2+x) = 0.$$

It follows immediately (see subsection 15.3.6) that $y(x) = e^x$ is a solution of the equation, as we have already found.

(b) We already have the two independent solutions needed to form the CF, x^{-1} and e^x. So we take for the PI

$$y(x) = k_1(x)x^{-1} + k_2(x)e^x$$

and require that

$$\begin{aligned}
k_1' x^{-1} + k_2' e^x &= 0, \\
-k_1' x^{-2} + k_2' e^x &= \frac{(x+1)^2}{x(x+1)} = \frac{x+1}{x}, \\
-k_1' \frac{1+x}{x^2} &= \frac{x+1}{x}, \qquad \text{by subtraction,} \\
\Rightarrow \quad k_1' &= -x \quad \Rightarrow \quad k_1(x) = -\tfrac{1}{2}x^2, \\
\Rightarrow \quad k_2' &= -\frac{e^{-x}(-x)}{x} = e^{-x} \quad \Rightarrow \quad k_2(x) = -e^{-x}.
\end{aligned}$$

The complete PI is thus

$$y(x) = -\frac{1}{2}\frac{x^2}{x} - e^{-x}\,e^x = -\frac{x}{2} - 1,$$

and the general solution of the inhomogeneous equation $F(x, y) = (x+1)^2$ is

$$y(x) = \frac{A}{x} + Be^x - \frac{x}{2} - 1,$$

for arbitrary constants A and B.

15.28 *Use the result of the previous exercise (15.27) to find the Green's function $G(x, \xi)$ that satisfies*

$$\frac{d^2G}{dx^2} + 3\frac{dG}{dx} + 2G = \delta(x - x),$$

in the interval $0 \le x, \xi \le 1$ with $G(0, \xi) = G(1, \xi) = 0$. Hence obtain integral expressions for the solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \begin{cases} 0 & 0 < x < x_0, \\ 1 & x_0 < x < 1, \end{cases}$$

distinguishing between the cases (a) $x < x_0$, *and* (b) $x > x_0$.

The auxiliary equation is $m^2 + 3m + 2 = 0$ and two independent solutions are $y(x) = e^{-x}$ and $y(x) = e^{-2x}$. We need linear combinations of these that satisfy $y_1(0) = 0$ and $y_2(1) = 0$. The former is clearly satisfied by taking $y_1(x) = e^{-x} - e^{-2x}$. For the latter, take

$$y_2(x) = e^{-x} + \alpha e^{-2x} \text{ and require } y_2(1) = 0 \quad \Rightarrow \quad \alpha = -e.$$

Thus $y_2(x) = e^{-x} - e^{-2x+1}$ is the appropriate linear combination for the region containing $x = 1$.

The Wronskian of these two functions is

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ &= (e^{-x} - e^{-2x})(-e^{-x} + 2e^{-2x+1}) \\ &\quad\quad - (e^{-x} - e^{-2x+1})(-e^{-x} + 2e^{-2x}) \\ &= e^{-3x+1} - e^{-3x} \\ &= (e-1)e^{-3x}. \end{aligned}$$

Hence, using the result from the previous question in the main text, the Green's function is

$$G(x, \xi) = \begin{cases} G_1(x, \xi) = \dfrac{(e^{-x} - e^{-2x})(e^{-\xi} - e^{-2\xi+1})}{(e-1)e^{-3\xi}} & 0 < x < \xi \\ G_2(x, \xi) = \dfrac{(e^{-x} - e^{-2x+1})(e^{-\xi} - e^{-2\xi})}{(e-1)e^{-3\xi}} & \xi < x < 1. \end{cases}$$

Now, in general, the solution of the given equation with its RHS equal to $f(x)$ is

$$y(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi = \int_{x_0}^1 G(x, \xi)\, d\xi.$$

The second equality follows because $f(\xi) = 0$ for $\xi < x_0$.

(a) For $x < x_0$, the variable of integration ξ is greater than x throughout the integration range $x_0 \le \xi \le 1$ and so $G = G_1$ throughout, i.e.

$$y(x) = \int_{x_0}^1 G_1(x, \xi)\, d\xi.$$

(b) However, for $x > x_0$ the integral is divided into two parts. For $x_0 < \xi < x$, G_2 is the appropriate Green's function, whilst for the remainder of the integral range, $x < \xi < 1$, G_1 must be used. Thus

$$y(x) = \int_{x_0}^x G_2(x, \xi)\, d\xi + \int_x^1 G_1(x, \xi)\, d\xi.$$

15.30 *Show that the Green's function for the equation*

$$\frac{d^2 y}{dx^2} + \frac{y}{4} = f(x),$$

subject to the boundary conditions $y(0) = y(\pi) = 0$, *is given by*

$$G(x, z) = \begin{cases} -2\cos\frac{1}{2}x \sin\frac{1}{2}z & 0 \le z \le x, \\ -2\sin\frac{1}{2}x \cos\frac{1}{2}z & x \le z \le \pi. \end{cases}$$

This result could be written down almost immediately using the general result of exercise 15.27. It could also be derived more specifically using the continuity/discontinuity conditions at $x = z$. By way of illustration, we will use yet another approach, based on the variation of parameters.

It is clear that the CF of the given equation is $y(x) = A\sin(x/2) + B\cos(x/2)$. So we take $y(x) = A(x)\sin(x/2) + B(x)\cos(x/2)$, with $A(\pi) = 0$ and $B(0) = 0$, and impose the usual constraints of the method, namely

$$A'\sin\frac{x}{2} + B'\cos\frac{x}{2} = 0,$$
$$\frac{1}{2}A'\cos\frac{x}{2} - \frac{1}{2}B'\sin\frac{x}{2} = f(x).$$

These simultaneous equations have the solution

$$A' = 2f(x)\cos\frac{x}{2} \quad \text{and} \quad B' = -2f(x)\sin\frac{x}{2}.$$

Integrating these two differential equations, incorporating the boundary values of A and B given above, and resubstituting in the assumed form of solution gives

$$y(x) = -\cos\frac{x}{2}\int_0^x 2f(z)\sin\frac{z}{2}\,dz - \sin\frac{x}{2}\int_x^\pi 2f(z)\cos\frac{z}{2}\,dz.$$

Note that the expression for $A(x)$ has to be obtained by integrating from π, where its value is known, to x.

From this integral form of solution the Green's function can be read off as

$$G(x,z) = \begin{cases} -2\cos\frac{1}{2}x\sin\frac{1}{2}z & 0 \le z \le x, \\ -2\sin\frac{1}{2}x\cos\frac{1}{2}z & x \le z \le \pi. \end{cases}$$

15.32 *Consider the equation*

$$\frac{d^2y}{dx^2} + f(y) = 0,$$

where $f(y)$ can be any function.

(a) *By multiplying through by dy/dx, obtain the general solution relating x and y.*

(b) *A mass m, initially at rest at the point $x = 0$, is accelerated by a force*

$$f(x) = A(x_0 - x)\left[1 + 2\ln\left(1 - \frac{x}{x_0}\right)\right].$$

Its equation of motion is $m\,d^2x/dt^2 = f(x)$. Find x as a function of time and show that ultimately the particle has travelled a distance x_0.

(a) We multiply the equation through by y' and integrate twice obtaining

$$\begin{aligned}
y'y'' &= -fy', \\
\frac{1}{2}(y')^2 &= -\int f(y)\,\frac{dy}{dx}\,dx + A, \\
\frac{dy}{dx} &= \left(B - 2\int^y f(u)\,du\right)^{1/2}, \\
\int^x dx &= \int^y \frac{dz}{\left(B - 2\int^z f(u)\,du\right)^{1/2}}, \\
C + x &= \int^y \frac{dz}{\left(B - 2\int^z f(u)\,du\right)^{1/2}}.
\end{aligned}$$

This is the general equation relating y to x, whatever the form of the function f.

(b) Here x is the dependent variable, the independent one being t.

To use the result of part (a) we first need to evaluate

$$\int A(x_0 - u)\left[1 + 2\ln\left(1 - \frac{u}{x_0}\right)\right] du.$$

Either by observation or by integration by parts the integrand is the derivative of $-A(x_0 - x)^2 \ln[\,(x_0 - x)/x_0\,]$. The force is related to the 'f' of part (a) by a factor of $-1/m$. Consequently the solution given in part (a) becomes

$$C + t = \int^x \frac{dz}{\left(B - \frac{2A}{m}(x_0 - z)^2 \ln[\,(x_0 - z)/x_0\,]\right)^{1/2}}$$

Referring back to the derivation in part (a), we see that if $y' = 0$ when $y = 0$ then $A = B = 0$; the corresponding situation, $\dot{x} = 0$ when $x = 0$, holds good here and B in the denominator of the integrand is zero. More obviously, since $x = 0$ when $t = 0$ we must have $C = 0$. Thus

$$t = \int^x \frac{dz}{\left(\frac{2A}{m}(x_0 - z)^2 \ln[\,x_0/(x_0 - z)\,]\right)^{1/2}}.$$

Note that we have inverted the argument of the logarithm.

All that remains is to evaluate the integral.

$$\begin{aligned}
\sqrt{\frac{2A}{m}}\,t &= \int^{x} \frac{dz}{(x_0 - z)\sqrt{\ln x_0 - \ln(x_0 - z)}} \\
&= \left[2\sqrt{\ln x_0 - \ln(x_0 - z)} \right]_0^x, \\
&= 2\sqrt{\ln x_0 - \ln(x_0 - x)}, \\
\frac{2At^2}{m} &= 4[\ln x_0 - \ln(x_0 - x)], \\
\frac{x_0 - x}{x_0} &= e^{-At^2/2m}, \\
x &= x_0 \left(1 - e^{-At^2/2m}\right).
\end{aligned}$$

Clearly as $t \to \infty$ the value of x approaches x_0.

15.34 *Find the general solution of the equation*

$$x\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = Ax.$$

This third-order equation is one in which y does not appear and so we set $dy/dx = p$ and rewrite the equation as one of second order.

$$\begin{aligned}
x\frac{d^2p}{dx^2} + 2\frac{dp}{dx} &= Ax, \\
x^2\frac{d^2p}{dx^2} + 2x\frac{dp}{dx} &= Ax^2, \qquad \text{using the obvious IF,} \\
\frac{d}{dx}\left(x^2\frac{dp}{dx}\right) &= Ax^2.
\end{aligned}$$

Successive integrations then give

$$\begin{aligned}
x^2\frac{dp}{dx} &= \frac{Ax^3}{3} + B, \\
\frac{dp}{dx} &= \frac{Ax}{3} + \frac{B}{x^2}, \\
p &= \frac{Ax^2}{6} - \frac{B}{x} + C, \\
y &= \frac{Ax^3}{18} - B\ln x + Cx + D.
\end{aligned}$$

Recall that A is given in the question; there are only three arbitray constants, B, C and D, as is to be expected for the solution of a third-order equation.

15.36 *Find the form of the solutions of the equation*

$$\frac{dy}{dx}\frac{d^3y}{dx^3} - 2\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = 0$$

that have $y(0) = \infty$.

[*You will need the result* $\int^z$ cosech $u\,du = -\ln(\text{cosech } z + \coth z)$.]

Since y does not appear in the equation (the same is true of x) we set $dy/dx = p$ and reformulate it. The required derivatives are

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$
$$\frac{d^3y}{dx^3} = \frac{dy}{dx}\frac{d}{dy}\left(p\frac{dp}{dy}\right) = p\left(\frac{dp}{dy}\right)^2 + p^2\frac{d^2p}{dy^2},$$

and the re-formulated equation is

$$0 = p^2\left(\frac{dp}{dy}\right)^2 + p^3\frac{d^2p}{dy^2} - 2p^2\left(\frac{dp}{dy}\right)^2 + p^2,$$
$$0 = p\frac{d^2p}{dy^2} - \left(\frac{dp}{dy}\right)^2 + 1.$$

This non-linear equation can be simplified by setting $dp/dy = q$ with, in the same way as above, $d^2p/dy^2 = q\,dq/dp$.

$$p\,q\frac{dq}{dp} - q^2 + 1 = 0,$$
$$\frac{q\,dq}{q^2-1} = \frac{dp}{p}, \qquad \text{separating variables,}$$
$$\Rightarrow \quad \tfrac{1}{2}\ln(q^2-1) = \ln p + A,$$
$$q^2 - 1 = B^2p^2.$$

Now set $Bp = \sinh\theta$, with

$$q = \frac{dp}{dy} = \frac{\cosh\theta}{B}\frac{d\theta}{dy},$$

to obtain

$$\frac{\cosh^2\theta}{B^2}\left(\frac{d\theta}{dy}\right)^2 - 1 = \sinh^2\theta,$$
$$\frac{\cosh^2\theta}{B^2}\left(\frac{d\theta}{dy}\right)^2 = \cosh^2\theta,$$
$$\Rightarrow \quad \theta = By + C.$$

Now,

$$
\begin{aligned}
B\frac{dy}{dx} &= Bp = \sinh(By + C), \\
dx &= \frac{B\,dy}{\sinh(By + C)}, \qquad \text{separating variables}, \\
\Rightarrow \quad x + c &= -\ln[\operatorname{cosech}(By + C) + \coth(By + C)].
\end{aligned}
$$

Since $y(0) = \infty$,

$$0 + c = -\ln[0 + 1] \quad \Rightarrow \quad c = 0$$

and the final answer is

$$\operatorname{cosech}(By + C) + \coth(By + C) = e^{-x}.$$

16

Series solutions of ordinary differential equations

16.2 *Find solutions, as power series in z, of the equation*

$$4zy'' + 2(1-z)y' - y = 0.$$

Identify one of the solutions and verify it by direct substitution.

Putting the equation in its standard form shows that $z = 0$ is a singular point of the equation but, as $2z(1-z)/4z$ and $-z^2/4z$ are finite as $z \to 0$, it is a regular singular point. We therefore substitute a Frobenius type solution,

$$y(z) = z^{\sigma} \sum_{n=0}^{\infty} a_n z^n \text{ with } a_0 \neq 0,$$

and obtain

$$4\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-1} + 2(1-z)\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1} - \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0.$$

Equating the coefficient of $z^{\sigma-1}$ to zero gives the indicial equation as

$$4\sigma(\sigma-1)a_0 + 2\sigma a_0 = 0 \quad \Rightarrow \quad \sigma = 0,\ \tfrac{1}{2}.$$

These two indicial values do not differ by an integer and so we expect both to yield (independent) series solutions.

(a) $\sigma = 0$.

Equating the general coefficient of z^m to zero (with $\sigma = 0$),

$$4(m+1)ma_{m+1} + 2(m+1)a_{m+1} - 2ma_m - a_m = 0,$$

gives the recurrence relation as

$$a_{m+1} = \frac{2m+1}{2(m+1)(2m+1)} = \frac{a_m}{2(m+1)},$$
$$\Rightarrow \quad a_m = \frac{a_0}{2^m\, m!},$$
$$\Rightarrow \quad y(x) = a_0 z^0 \sum_{n=0}^{\infty} \frac{z^n}{2^n\, n!} = a_0 e^{z/2}.$$

Substituting this into the original equation yields

$$4a_0 z \frac{e^{z/2}}{4} + 2a_0(1-z)\frac{e^{z/2}}{2} - a_0 e^{z/2} = 0,$$

which is a valid equation and thus verifies the solution.

(b) $\sigma = \frac{1}{2}$

Equating the general coefficient of $z^{m+\frac{1}{2}}$ to zero (with $\sigma = \frac{1}{2}$),

$$4(m+\tfrac{3}{2})(m+\tfrac{1}{2})a_{m+1} + 2(m+\tfrac{3}{2})a_{m+1} - 2(m+\tfrac{1}{2})a_m - a_m = 0,$$

gives the recurrence relation as

$$a_{m+1} = \frac{2m+2}{(2m+3)(2m+2)}\, a_m = \frac{a_m}{2m+3},$$
$$\Rightarrow \quad a_m = \frac{a_0}{1\,3\,5\,\cdots\,(2m+1)} = \frac{2^m\, m!\, a_0}{(2m+1)!},$$

and the second series solution as

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{2^n\, n!\, z^{n+\frac{1}{2}}}{(2n+1)!}.$$

16.4 *Change the independent variable in the equation*

$$\frac{d^2 f}{dz^2} + 2(z-\alpha)\frac{df}{dz} + 4f = 0 \qquad (*)$$

from z to $x = z - \alpha$, and find two independent series solutions, expanded about $x = 0$, of the resulting equation. Deduce that the general solution of $(*)$ is

$$f(z,\alpha) = A(z-\alpha)e^{-(z-\alpha)^2} + B\sum_{m=0}^{\infty} \frac{(-4)^m m!}{(2m)!}\,(z-\alpha)^{2m},$$

with A and B arbitrary constants.

We start with $f'' + 2(z-\alpha)f' + 4f = 0$ and set $f(z) = g(x)$, where $x = z - \alpha$.

The differential operators are unchanged, i.e. $d/dx = d/dz$ and $d^2/dx^2 = d^2/dz^2$. The resulting equation for $g(x)$ is $g'' + 2xg' + 4g = 0$, where now a prime denotes differentiation with respect to x.

The origin $x = 0$ is an ordinary point of this equation and we can write its two solutions as ordinary power series (formally, its indicial roots are 0 and 1),

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Making this substitution yields

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} na_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0.$$

Equating the coefficient of x^m to zero then gives

$$a_{m+2} = -\frac{2m+4}{(m+2)(m+1)}\,a_m = -\frac{2}{m+1}\,a_m,$$

or, on re-indexing,

$$a_m = -\frac{2}{m-1}a_{m-2}.$$

Since this recurrence relation relates indices differing by 2, we can (as expected) obtain two linearly independent solutions by taking (i) $a_0 = 1$, $a_1 = 0$ and (ii) $a_0 = 0$, $a_1 = 0$.

(i) $a_0 = 1$, $a_1 = 0$

Here, only even powers of x appear and we can write $m = 2p$. The expansion coefficients are then given by

$$a_{2p} = -\frac{2}{2p-1}\,a_{2p-2} = \frac{(-1)^p\,2^p\,2^p\,p\,!}{(2p)!}\,a_0,$$

and one solution of the original equation is

$$f_0(z) = g_0(x) = a_0\sum_{n=0}^{\infty}\frac{(-4)^n\,n!}{(2n)!}x^{2n} = a_0\sum_{n=0}^{\infty}\frac{(-4)^n\,n!}{(2n)!}(z-\alpha)^{2n}.$$

(ii) $a_0 = 0$, $a_1 = 1$

Here, only odd powers of x appear and we can write $m = 2p+1$. The expansion coefficients are then given by

$$a_{2p+1} = -\frac{2}{2p}\,a_{2p-1} = \frac{(-1)^p}{p\,!}\,a_1,$$

and a second solution of the original equation is

$$f_1(z) = g_1(x) = a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}$$
$$= a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (z-\alpha)^{2n+1}$$
$$= a_1 (z-\alpha) e^{-(z-\alpha)^2}.$$

The general solution of (∗) is any linear combination of $f_0(z)$ and $f_1(z)$, as stated.

> ***16.6*** *Verify that $z = 0$ is a regular singular point of the equation*
>
> $$z^2 y'' - \tfrac{3}{2} z y' + (1+z) y = 0,$$
>
> *and that the indicial equation has roots 2 and $1/2$. Show that the general solution is*
>
> $$y(z) = 6a_0 z^2 \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) 2^{2n} z^n}{(2n+3)!} + b_0 \left(z^{1/2} + 2z^{3/2} - \frac{z^{1/2}}{4} \sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n} z^n}{n(n-1)(2n-3)!} \right).$$

In standard form the equation is

$$y'' - \frac{3}{2z} y' + \frac{1+z}{z^2} = 0.$$

Now,

$$-\frac{3z}{2z} \to -\frac{3}{2} \text{ and } \frac{z^2(1+z)}{z^2} \to 1 \text{ as } z \to 0.$$

Both limits are finite and so $z = 0$ is a regular singular point of the equation.

Substituting a Frobenius solution gives

$$\sum_{n=0}^{\infty} (\sigma+n)(\sigma+n-1) a_n z^{\sigma+n} - \frac{3}{2} \sum_{n=0}^{\infty} (\sigma+n) a_n z^{\sigma+n} + (1+z) \sum_{n=0}^{\infty} a_n z^{\sigma+n} = 0.$$

Equating the coefficient of z^σ to zero shows that

$$[\,\sigma(\sigma-1) - \tfrac{3}{2}\sigma + 1\,] a_0 = 0,$$
$$\sigma^2 - \tfrac{5}{2}\sigma + 1 = 0 \quad \Rightarrow \quad \sigma = 2,\ \tfrac{1}{2}.$$

These roots do not differ by an integer and so we expect two linearly independent power series solutions.

(a) Setting $\sigma = 2$ and equating the coefficient of z^{m+2} to zero gives

$$\begin{aligned}(2+m)(1+m)a_m - \tfrac{3}{2}(2+m)a_m + a_m + a_{m-1} &= 0,\\ \Rightarrow \quad a_m(2+3m+m^2-3-\tfrac{3}{2}m+1) &= -a_{m-1},\\ \Rightarrow \quad a_m[\,m(\tfrac{3}{2}+m)\,] &= -a_{m-1}.\end{aligned}$$

Repeated application of this relation gives

$$\begin{aligned}a_m = -\frac{2}{m(2m+3)}\,a_{m-1} &= \frac{(-1)^m\,2^m}{m!\,[\,(2m+3)(2m+1)\cdots 5\,]}\,a_0\\ &= \frac{(-1)^m\,2^m\,3\,2^{m+1}\,(m+1)!}{m!\,(2m+3)!}\,a_0\\ &= \frac{6(-1)^m\,2^{2m}\,(m+1)}{(2m+3)!}\,a_0,\\ \text{leading to } y_1(z) &= Az^2\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{2n}\,(n+1)\,z^n}{(2n+3)!}.\end{aligned}$$

(b) Setting $\sigma = \frac{1}{2}$ and following the same procedure (for $z^{m+\frac{1}{2}}$) yields

$$\begin{aligned}(\tfrac{1}{2}+m)(-\tfrac{1}{2}+m)a_m - \tfrac{3}{2}(\tfrac{1}{2}+m)a_m + a_m + a_{m-1} &= 0,\\ a_m(4m^2-1-3-6m+4) &= -4a_{m-1},\end{aligned}$$

which reduces to

$$a_m = -\frac{2}{m(2m-3)}\,a_{m-1}.$$

In particular, for $m = 1$,

$$a_1 = -\frac{2}{1(-1)}\,a_0 = 2a_0.$$

For $m \geq 2$,

$$a_m = \frac{(-1)^{m-1}\,2^{m-1}\,2^{m-2}\,(m-2)!}{m!\,(2m-3)!}\,a_1 = \frac{(-1)^{m-1}\,2^{2m}\,2a_0}{8m(m-1)\,(2m-3)!}$$

and the full expression for the second independent solution is

$$y_2(z) = Bz^{1/2}\left(1 + 2z - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{2n}\,z^n}{n(n-1)\,(2n-3)!}\right).$$

As stated in the question, the general solution is $y_1(z) + y_2(z)$ for arbitrary values of A and B.

> ***16.8*** *Consider a series solution of the equation*
>
> $$zy'' - 2y' + yz = 0 \qquad (*)$$
>
> *about its regular singular point.*
>
> (a) *Show that its indicial equation has roots that differ by an integer but that the two roots nevertheless generate linearly independent solutions*
>
> $$y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ 2nz^{2n+1}}{(2n+1)!},$$
>
> $$y_2(z) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)z^{2n}}{(2n)!}.$$
>
> (b) *Show that $y_1(z)$ is equal to $3a_0(\sin z - z\cos z)$ by expanding the sinusoidal functions. Then, using the Wronskian method, find an expression for $y_2(z)$ in terms of sinusoids. You will need to write z^2 as $(z/\sin z)(z \sin z)$ and integrate by parts to evaluate the integral involved.*
>
> (c) *Confirm that the two solutions are linearly independent by showing that their Wronskian is equal to $-z^2$, as would be expected from the form of $(*)$.*

(a) Substituting a Frobenius solution gives

$$\sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n z^{\sigma+n-1} - 2\sum_{n=0}^{\infty}(\sigma+n)a_n z^{\sigma+n-1} + \sum_{n=0}^{\infty} a_n z^{\sigma+n+1} = 0.$$

Equating the coefficient of $z^{\sigma-1}$ to zero shows that

$$[\,\sigma(\sigma-1) - 2\sigma\,]a_0 = 0, \quad \Rightarrow \quad \sigma = 0 \text{ or } 3.$$

Setting the coefficient of $z^{m+\sigma-1}$ to zero now gives the required recurrence relation:

$$(\sigma+m)(\sigma+m-1)a_m - 2(\sigma+m)a_m + a_{m-2} = 0,$$
$$(\sigma+m)(\sigma+m-3)a_m + a_{m-2} = 0,$$

i.e.

$$a_m = \frac{-1}{(\sigma+m)(\sigma+m-3)}\, a_{m-2}.$$

For $\sigma = 3$, $a_m = -a_{m-2}/[\,m(m+3)\,]$ and

$$y_1(z) = a_0 z^3 \left(1 - \frac{z^2}{(2)(5)} + \frac{z^4}{(2)(5)(4)(7)} - \cdots\right)$$
$$= 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2n\, z^{2n+1}}{(2n+1)!}.$$

For $\sigma = 0$, $a_m = -a_{m-2}/[\,m(m-3)\,]$ and

$$y_2(z) = a_0 z^0 \left(1 + \frac{z^2}{2} - \frac{z^4}{(2)(4)(1)} + \cdots\right)$$
$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(2n-1)\,z^{2n}}{(2n)!}.$$

Thus, although the indicial roots, 0 and 3, differ by an integer, two linearly independent series solutions are produced.

(b) We write the given function in series form:

$$3a_0(\sin z - z\cos z) = 3a_0 \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} - 3a_0 z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$
$$= 3a_0 \sum_{n=0}^{\infty} \frac{(-1)^n\,(1-2n-1)\,z^{2n+1}}{(2n+1)!}$$
$$= 3a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,2n\,z^{2n+1}}{(2n+1)!}$$
$$= 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,2n\,z^{2n+1}}{(2n+1)!} = y_1(z).$$

This confirms that the the first series is a multiple of $\sin z - z\cos z$ and the Wronskian method now gives the second solution as

$$y_2(z) = y_1(z) \int^z \frac{1}{y_1^2(u)} \exp\left\{-\int^u \left(\frac{-2}{v}\right) dv\right\} du$$
$$= y_1(z) \int^z \frac{1}{y_1^2(u)} \exp(2\ln u)\,du$$
$$= y_1(z) \int^z \frac{u^2}{(\sin u - u\cos u)^2}\,du$$
$$= y_1(z) \int^z \frac{u}{\sin u}\,\frac{u\sin u}{(\sin u - u\cos u)^2}\,du$$
$$= y_1(z) \left\{ \left[\frac{u}{\sin u}\,\frac{-1}{(\sin u - u\cos u)}\right]^z + \int^z \left(\frac{1}{\sin u} - \frac{u\cos u}{\sin^2 u}\right)\left(\frac{1}{\sin u - u\cos u}\right) du \right\}$$
$$= -\frac{z}{\sin z} + y_1(z) \int^z \frac{1}{\sin^2 u}\,du$$
$$= -\frac{z}{\sin z} + (\sin z - z\cos z)(-\cot z)$$
$$= \frac{-z - \sin z\cos z + z\cos^2 z}{\sin z}$$
$$= -z\sin z - \cos z.$$

Comparing the constant terms (i.e. the z^0 term) in this solution and the series obtained earlier, shows that this sinusoidal form is the negative of the series solution. Thus the series $y_2(z) = \cos z + z \sin z$.

(c) From the way the sinusoidal form was calculated, the solutions must satisfy the Wronskian condition, but an explicit verification is as follows.

$$\begin{aligned} W(y_1, y_2) &= (\sin z - z\cos z)(-\sin z + \sin z + z\cos z) \\ &\quad - (\cos z + z\sin z)(\cos z - \cos z + z\sin z) \\ &= z\cos z\sin z - z^2\cos^2 z - z\sin z\cos z - z^2\sin^2 z \\ &= -z^2. \end{aligned}$$

This is in accord with what is expected from (∗), where the factor multiplying y' is $-2/z$ when the equation is put in standard form. Explicitly,

$$W(y_1, y_2) = \exp\left\{-\int^z \left(\frac{-2}{v}\right) dv\right\} = z^2,$$

an arbitrary constant multiplying factor being irrelevant.

16.10 *Solve the equation*

$$z(1-z)\frac{d^2y}{dz^2} + (1-z)\frac{dy}{dz} + \lambda y = 0,$$

as follows.

(a) *Identify and classify its singular points and determine their indices.*
(b) *Find one series solution in powers of z. Give a formal expression for a second linearly independent solution.*
(c) *Deduce the values of λ for which there is a polynomial solution $P_N(z)$ of degree N. Evaluate the first four polynomials, normalised in such a way that $P_N(0) = 1$.*

(a) In standard form the equation reads

$$y'' + \frac{1}{z}y' + \frac{\lambda y}{z(1-z)} = 0.$$

By inspection, there are singular points at $z = 0$ and $z = 1$. However, the denominators of the functions that make the points singular have only first-order zeros; consequently, the points are both regular singular points.

Substituting a Frobenius solution about $z = 0$ gives

$$z(1-z)\sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n z^{\sigma+n-2} + (1-z)\sum_{n=0}^{\infty}(\sigma+n)a_n z^{\sigma+n-1} + \lambda\sum_{n=0}^{\infty}a_n z^{\sigma+n} = 0.$$

Equating the coefficient of $z^{\sigma-1}$ to zero shows that

$$[\,\sigma(\sigma-1)+1\sigma\,]a_0 = 0, \quad \Rightarrow \quad \sigma = 0 \text{ (repeated root)}.$$

For the point $z = 1$, we set $z - 1 = u$ with $f(u) = y(z)$ and obtain

$$(u+1)(-u)f'' + (-u)f' + \lambda f = 0.$$

If a Frobenius solution about $u = 0$ is substituted, the lowest power of u present will be $u^{\sigma-1}$; it will arise only from the first term and have coefficient $-\sigma(\sigma-1)a_0$. For this to be zero requires that $\sigma = 0$ or $\sigma = 1$; these are the indices of the point $u = 0$, i.e. of the point $z = 1$.

(b) For the solution about $z = 0$ we have a repeated indicial root $\sigma = 0$. We therefore need to use the derivative method to obtain two solutions. The recurrence relation generated by setting the coefficient of $z^{m+\sigma}$ to zero in the equation in part (a) is

$$(m+1+\sigma)(m+\sigma)a_{m+1} - (m+\sigma)(m+\sigma-1)a_m + (m+1+\sigma)a_{m+1} - (m+\sigma)a_m + \lambda a_m = 0,$$

$$\Rightarrow \quad a_{m+1} = \frac{(m+\sigma)^2 - \lambda}{(m+1+\sigma)^2}\, a_m.$$

The first solution is obtained by setting σ equal to 0 in this relation to yield

$$a_m = \frac{\prod_{r=0}^{m-1}(r^2-\lambda)}{(m!)^2}\, a_0$$

and

$$y_1(z) = a_0 + a_0\sum_{n=1}^{\infty}\frac{\prod_{r=0}^{n-1}(r^2-\lambda)}{(n!)^2}\, z^n.$$

The second solution $y_2(z)$ is obtained by differentiating the general solution with respect to σ before setting $\sigma = 0$.

$$\frac{\partial y}{\partial \sigma} = \frac{d(z^\sigma)}{d\sigma} \sum_{n=0}^{\infty} a_n(\sigma) z^n + z^\sigma \sum_{n=1}^{\infty} \frac{da_n(\sigma)}{d\sigma} z^n,$$

$$y_2(z) = z^0 \ln z \sum_{n=0}^{\infty} a_n(0) z^n + z^0 \sum_{n=1}^{\infty} \frac{da_n(0)}{d\sigma} z^n$$

$$= y_1(z) \ln z + \sum_{n=1}^{\infty} \frac{\partial}{\partial \sigma} \prod_{r=0}^{n-1} \left. \frac{(r+\sigma)^2 - \lambda}{(r+1+\sigma)^2} \right|_{\sigma=0} z^n.$$

(c) From the recurrence relation it is clear that the condition for the series solution to terminate is that $\lambda = (m+\sigma)^2$ for some integer m. Since $\sigma = 0$, this means that $\lambda = N^2$ for a polynomial of degree N. The first four polynomials, constucted using the recurrence relation, are

$$\lambda = 0, \qquad P_0(z) = 1,$$

$$\lambda = 1, \qquad P_1(z) = 1 + \frac{-1}{1^2} z = 1 - z,$$

$$\lambda = 4, \qquad P_2(z) = 1 + \frac{-4}{1^2} z + \frac{(-3)(-4)}{2^2} z^2 = 1 - 4z + 3z^2,$$

$$\lambda = 9, \qquad P_3(z) = 1 + \frac{-9}{1^2} z + \frac{(-8)(-9)}{2^2} z^2 + \frac{(-5)(-8)(-9)}{6^2} z^3$$

$$= 1 - 9z + 18z^2 - 10z^3.$$

By choosing $a_0 = 1$ in each case, we have ensured that $P_N(0) = 1$ for all N.

16.12 *Find the radius of convergence of a series solution about the origin for the equation* $(z^2 + az + b)y'' + 2y = 0$ *in the following cases:*

$$\text{(a) } a = 5,\ b = 6; \qquad \text{(b) } a = 5,\ b = 7.$$

Show that if a and b are real and $4b > a^2$ *then the radius of convergence is always given by* $b^{1/2}$.

The two roots of $z^2 + az + b = 0$ give the singular points, z_1 and z_2, of the equation. The radius of convergence R of the series solution about the origin is equal to the smaller of their two moduli.

If $4b > a^2$ then the roots are necessarily complex conjugates and

$$R^2 = |z_1|^2 = |z_2|^2 = \left(\frac{-a}{2}\right)^2 + \left(\frac{\sqrt{4b - a^2}}{2}\right)^2 = b \quad \Rightarrow \quad R = \sqrt{b}.$$

This is case (b), for which therefore $R = b = 7$.

If $a^2 > 4b$ the roots are real and the smaller of their two magnitudes gives the value of R. In case (a) the roots are $\frac{1}{2}(-5 \pm \sqrt{25-24}) = -2$ or -3, implying that $R = 2$.

16.14 *Prove that the Laguerre equation*

$$z\frac{d^2y}{dz^2} + (1-z)\frac{dy}{dz} + \lambda y = 0$$

has polynomial solutions $L_N(z)$ if λ is a non-negative integer N, and determine the recurrence relationship for the polynomial coefficients. Hence show that an expression for $L_N(z)$, normalised in such a way that $L_N(0) = N!$, is

$$L_N(z) = \sum_{n=0}^{N} \frac{(-1)^n (N!)^2}{(N-n)!(n!)^2} z^n.$$

Evaluate $L_3(z)$ explicitly.

We assume that there is a polynomial solution $L_N(z) = \sum_{n=0}^{N} a_n z^n$ with $a_N \neq 0$ and substitute this form into the differential equation:

$$z\sum_{n=0}^{N} n(n-1)a_n z^{n-2} + (1-z)\sum_{n=0}^{N} n a_n z^{n-1} + \lambda \sum_{n=0}^{N} a_n z^n = 0.$$

Consideration of the coefficient of z^N shows that we require $\lambda = N$.

The recurrence relation comes from equating the coefficient of z^{m-1} to zero:

$$m(m-1)a_m + m a_m - (m-1)a_{m-1} + N a_{m-1} = 0,$$

$$a_m = \frac{m-1-N}{m^2}\, a_{m-1} = \frac{(-1)^n\, N!\, a_0}{(N-n)!\,(n!)^2} = \frac{(-1)^n\,(N!)^2}{(N-n)!\,(n!)^2},$$

where, in the last step, we have used the requirement that $a_0 = L_N(0) = N!$.

Hence

$$L_N(z) = \sum_{n=0}^{N} \frac{(-1)^n\,(N!)^2\, z^n}{(N-n)!\,(n!)^2}.$$

Explicitly, for $N = 3$,

$$L_3(z) = 3! - \frac{6^2}{2!\,1^2}\, z + \frac{6^2}{1!\,2^2}\, z^2 - \frac{6^2}{0!\,6^2}\, z^3 = 6 - 18z + 9z^2 - z^3.$$

Essentially the same proof, but with a different normalisation of the polynomials, is given in the main text in section 18.7.

16.16 *Obtain the recurrence relations for the solution of Legendre's equation*

$$(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0.$$

in inverse *powers of z, i.e. set $y(z) = \sum a_n z^{\sigma-n}$, with $a_0 \neq 0$. Deduce that if ℓ is an integer then the series with $\sigma = \ell$ will terminate and hence converge for all $z \neq 0$ whilst that with $\sigma = -(\ell+1)$ does not terminate and hence converges only for $|z| > 1$.*

We substitute $y = \sum_{n=0}^{\infty} a_n z^{\sigma-n}$ with $a_0 \neq 0$ into Legendre's equation

$$(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0.$$

and obtain

$$(1-z^2)\sum_{n=0}^{\infty}(\sigma-n)(\sigma-n-1)a_n z^{\sigma-n-2} - 2z\sum_{n=0}^{\infty}(\sigma-n)a_n z^{\sigma-n-1} + \ell(\ell+1)\sum_{n=0}^{\infty} a_n z^{\sigma-n} = 0.$$

For the terms containing z^σ,

$$-\sigma(\sigma-1)a_0 - 2\sigma a_0 + \ell(\ell+1)a_0 = 0$$
$$\Rightarrow \quad -\sigma(\sigma-1+2) + \ell(\ell+1) = 0$$
$$\Rightarrow \quad \sigma = \ell,\ -(\ell+1).$$

These are the two indicial roots.

The recurrence relation is obtained by equating the coefficient of $z^{\sigma-m}$, i.e.

$$(\sigma-m+2)(\sigma-m+1)a_{m-2} - (\sigma-m)(\sigma-m-1)a_m - 2(\sigma-m)a_m + \ell(\ell+1)a_m,$$

to zero. The relation is thus

$$a_m = \frac{(\sigma-m+2)(\sigma-m+1)}{(\sigma-m)(\sigma-m-1+2)-\ell(\ell+1)}\,a_{m-2} \text{ with } m \geq 2.$$

For $\sigma = \ell$

$$\begin{aligned} a_n &= \frac{(\ell-n+2)(\ell-n+1)}{(\ell-n)(\ell-n-1+2)-\ell(\ell+1)}\,a_{n-2} \\ &= \frac{(\ell-n+2)(\ell-n+1)}{-n(\ell+1+\ell)+n^2}\,a_{n-2} \\ &= \frac{(\ell-n+2)(\ell-n+1)}{n(n-2\ell-1)}\,a_{n-2}. \end{aligned}$$

If ℓ is a positive integer then, irrespective of whether ℓ is even or odd, n will pass through either $\ell+1$ or $\ell+2$ and at that point one of the factors in the numerator will become zero. The series of coefficients will then terminate producing a function with a finite number of terms, each of which is a positive power of z^{-1}; such a function must be finite for all (non-zero) z. Although the denominator of the recurrence relation would become zero when $n = 2\ell+1$, the series will have terminated before that value of n is reached.

For $\sigma = -(\ell+1)$

$$\begin{aligned} a_n &= \frac{(-\ell-1-n+2)(-\ell-1-n+1)}{(-\ell-1-n)(-\ell-1-n-1+2)-\ell(\ell+1)}\, a_{n-2} \\ &= \frac{(\ell+n)(\ell+n-1)}{(\ell+n+1)(\ell+n)-\ell(\ell+1)}\, a_{n-2} \\ &= \frac{(\ell+n)(\ell+n-1)}{n(n+2\ell+1)}\, a_{n-2}. \end{aligned}$$

This series will not terminate because $(\ell+n)(\ell+n-1)$ cannot be equal to zero for $\ell > 0$ and $n \geq 2$. The denominator of the recurrence relation can never become zero.

Since the series is an infinite one in *inverse* powers of z, it will only converge for

$$\left|\frac{1}{z^2}\right| < \lim_{n\to\infty}\left|\frac{a_{n-2}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(\ell-n+2)(\ell-n+1)}{n(n-2\ell-1)}\right| = 1,$$

i.e. for $|z| > 1$.

17

Eigenfunction methods for differential equations

17.2 *Write the homogeneous Sturm-Liouville eigenvalue equation for which* $y(a) = y(b) = 0$ *as*

$$\mathcal{L}(y;\lambda) \equiv (py')' + qy + \lambda\rho y = 0,$$

where $p(x), q(x)$ *and* $\rho(x)$ *are continuously differentiable functions. Show that if* $z(x)$ *and* $F(x)$ *satisfy* $\mathcal{L}(z;\lambda) = F(x)$ *with* $z(a) = z(b) = 0$ *then*

$$\int_a^b y(x)F(x)\,dx = 0.$$

Demonstrate the validity of this general result by direct calculation for the specific case in which $p(x) = \rho(x) = 1$, $q(x) = 0$, $a = -1$, $b = 1$ *and* $z(x) = 1 - x^2$.

Write the integral I (omitting all arguments of functions) as follows:

$$\begin{aligned}
I &= \int_a^b yF\,dx \\
&= \int_a^b y\mathcal{L}(z;\lambda)\,dx \\
&= \int_a^b \left[\, y(pz')' + yqz + y\lambda\rho z \,\right]\,dx \\
&= \left[\, ypz' \,\right]_a^b - \int_a^b \left[\, y'pz' - z(qy + \lambda\rho y) \right]\,dx, \text{ with } y(a) = y(b) = 0, \\
&= 0 - \left[\, y'pz \,\right]_a^b + \int_a^b \left[\, (y'p)'z + z(qy + \lambda\rho y) \right]\,dx, \text{ with } z(a) = z(b) = 0, \\
&= 0 + \int_a^b z\mathcal{L}(y;\lambda)\,dx = \int_a^b z\,0\,dx = 0.
\end{aligned}$$

For the special case in which $p(x) = \rho(x) = 1$, $a = -1$, $b = 1$ and $q(x) = 0$, the equation reduces to

$$\begin{aligned} y'' + \lambda y = 0 &\text{ with } y(\pm 1) = 0. \\ \Rightarrow \quad y(x) &= A\cos(\sqrt{\lambda}x), \end{aligned}$$

with $\lambda = \dfrac{(2n+1)^2\pi^2}{4}$ and n a non-negative integer.

With the given form of $z(x)$,

$$\mathcal{L}(z\,;\lambda) = (1-x^2)'' + \lambda(1-x^2) = -2 + \lambda(1-x^2).$$

To verify the result we need to prove that $I = 0$, where

$$\begin{aligned} I &= \int_{-1}^{1} \cos(\mu x)[-2 + \mu^2(1-x^2)]\,dx, \text{ with } \mu = \sqrt{\lambda} = \frac{(2n+1)\pi}{2}, \\ &= \int_{-1}^{1} (\mu^2 - 2)\cos\mu x\,dx - \mu^2 \int_{-1}^{1} x^2 \cos\mu x\,dx \\ &= (\mu^2 - 2)\left[\frac{\sin\mu x}{\mu}\right]_{-1}^{1} - \mu^2 J \\ &= \frac{2(-1)^n(\mu^2-2)}{\mu} - \mu^2 J. \end{aligned}$$

Here J is the integral

$$\begin{aligned} J &= \int_{-1}^{1} x^2 \cos\mu x\,dx \\ &= \left[\frac{x^2\sin\mu x}{\mu}\right]_{-1}^{1} - 2\int_{-1}^{1} x\frac{\sin\mu x}{\mu}\,dx \\ &= \frac{2(-1)^n}{\mu} + \left[\frac{2x\cos\mu x}{\mu^2}\right]_{-1}^{1} - \int_{-1}^{1} \frac{2\cos\mu x}{\mu^2}\,dx \\ &= \frac{2(-1)^n}{\mu} + 0 - 0 - \left[\frac{2\sin\mu x}{\mu^3}\right]_{-1}^{1} \\ &= (-1)^n\left(\frac{2}{\mu} - \frac{4}{\mu^3}\right). \end{aligned}$$

Thus

$$I = (-1)^n\frac{2(\mu^2-2)}{\mu} - \mu^2(-1)^n\left(\frac{2}{\mu} - \frac{4}{\mu^3}\right) = 0,$$

as expected.

***17.4** Show that the equation*

$$y'' + a\delta(x)y + \lambda y = 0,$$

with $y(\pm\pi) = 0$ and a real, has a set of eigenvalues λ satisfying

$$\tan(\pi\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{a}.$$

Investigate the conditions under which negative eigenvalues, $\lambda = -\mu^2$ with μ real, are possible.

The problem is that of finding the Green's function $G(x, x_0)$ for the point $x_0 = 0$ over the range $-\pi \le x \le \pi$ with boundary values $y(\pm\pi) = 0$. We assume first that $\lambda > 0$. Continuity of the solution is needed at $x = 0$ but its derivative will have a step increase of magnitude $-ay(0)$.

Let

$$y(x) = \begin{cases} A \sin \nu x + B \cos \nu x, & -\pi \le x < 0, \\ C \sin \nu x + D \cos \nu x, & 0 \le x \le \pi. \end{cases},$$

where $\nu = \sqrt{\lambda}$. Then, continuity at $x = 0$ implies that $D = B$, whilst the step condition can be written

$$(\nu C + 0) - (\nu A + 0) = -a(0 + B).$$

The boundary values require

$$\frac{B}{A} = \tan \nu\pi = -\frac{D}{C} \quad \Rightarrow \quad A = -C.$$

Thus, substituting in the step condition gives

$$-\nu A - \nu A = -aA \tan \nu\pi \quad \Rightarrow \quad \tan \nu\pi = \frac{2\nu}{a}, \text{ i.e. } \tan \sqrt{\lambda}\pi = \frac{2\sqrt{\lambda}}{a}.$$

We note that, since the operator $\dfrac{d^2}{dx^2} + a\delta(x)$ is Hermitian, its eigenvalues can only be real. But this does not rule out the possibility of negative eigenvalues $\lambda = -\mu^2$ with μ real.

Putting the calculated values of B, C and D back into the assumed forms in part (a) shows that the explicit solution for that part is

$$y(x) = \begin{cases} E \sin[\sqrt{\lambda}(\pi + x)] & -\pi \le x < 0, \\ E \sin[\sqrt{\lambda}(\pi - x)] & 0 \le x \le \pi. \end{cases}$$

The corresponding result for $\lambda = -\mu^2$ is

$$y(x) = \begin{cases} E \sinh[\mu(\pi + x)] & -\pi \le x < 0, \\ E \sinh[\mu(\pi - x)] & 0 \le x \le \pi. \end{cases},$$

leading to the condition

$$\tanh \mu\pi = \frac{2\mu}{a}.$$

A simple sketch shows that this equation can only have a real solution for μ if the slope of $f(\mu) = \tanh(\mu\pi)$ at $\mu = 0$ is greater than the slope of $g(\mu) = 2\mu/a$ at the same place. The former slope is π and the latter $2/a$. Thus the condition for negative eigenvalues of the original equation is $a > 2/\pi$.

17.6 *Starting from the linearly independent functions* $1, x, x^2, x^3, \ldots$ *, in the range* $0 \le x < \infty$*, find the first three orthonormal functions* ϕ_0*,* ϕ_1 *and* ϕ_2*, with respect to the weight function* $\rho(x) = e^{-x}$*. By comparing your answers with the Laguerre polynomials generated by the recurrence relation*

$$(n+1)L_{n+1} - (2n+1-x)L_n + nL_{n-1} = 0,$$

deduce the form of $\phi_3(x)$*.*

We aim to construct the orthonormal functions using the Gram–Schmidt procedure. To evaluate the integrals involved we will make repeated use of the general result

$$\int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1) = n!.$$

Starting with $\phi_0 = 1$, all we need to check is its normalisation. Since

$$\int_0^\infty 1^2 e^{-x}\,dx = 1,$$

ϕ_0 is already correctly normalised.

We next calculate ϕ_1 as

$$\begin{aligned}\phi_1 &= x - \phi_0\langle\phi_0|x\rangle \\ &= x - 1\int_0^\infty 1\,z\,e^{-z}\,dz \\ &= x - 1,\end{aligned}$$

and check its normalisation:

$$\langle\phi_1|\phi_1\rangle = \int_0^\infty (x-1)^2 e^{-x}\,dx = \int_0^\infty (x^2 - 2x + 1)e^{-x}\,dx = 2! - 2(1!) + 1 = 1.$$

It too is already correctly normalised.

To find ϕ_2 we continue with the Gram–Schmidt construction using the expressions already derived for ϕ_0 and ϕ_1, as follows.

$$\begin{aligned}\phi_2(x) &= x^2 - 1\int_0^\infty 1\, z^2 e^{-z}\,dz - (x-1)\int_0^\infty (z-1)z^2 e^{-z}\,dz \\ &= x^2 - 2! - (x-1)\int_0^\infty (z^3 - z^2)e^{-z}\,dz \\ &= x^2 - 2 - (x-1)(3! - 2!) = x^2 - 4x + 2.\end{aligned}$$

Determining its normalisation constant is a little more complicated than for the first two functions, but to do so we evaluate

$$\begin{aligned}\langle\phi_2|\phi_2\rangle &= \int_0^\infty (x^4 + 16x^2 + 4 - 8x^3 + 4x^2 - 16x)e^{-x}\,dx \\ &= 4! + 20(2!) - 8(3!) - 16(1!) + 4 \\ &= 24 + 40 - 48 - 16 + 4 = 4.\end{aligned}$$

It is then clear that the correctly normalised ϕ_2 is $\phi_2(x) = \frac{1}{2}(x^2 - 4x + 2)$.

Next we explicitly generate the Laguerre polynomials using the recurrence relation

$$(n+1)L_{n+1} - (2n + 1 - x)L_n + nL_{n-1} = 0,$$

starting with $L_0(x) = 1$ (and L_{-1} conventionally equal to zero; it is multiplied by zero in any case). The equations for $n = 0, 1, 2$ read

$$\begin{aligned}L_1 - (0 + 1 - x)L_0 + 0 = 0 \quad &\Rightarrow \quad L_1(x) = 1 - x, \\ 2L_2 - (2 + 1 - x)L_1 + L_0 = 0 \quad &\Rightarrow \quad L_2(x) = \tfrac{1}{2}[\,(3-x)(1-x) - 1\,] \\ &\Rightarrow \quad L_2(x) = \tfrac{1}{2}[\,x^2 - 4x + 2\,], \\ 3L_3 - (4 + 1 - x)L_2 + 2L_1 = 0 \quad &\Rightarrow \quad 3L_3(x) = (5 - x)\tfrac{1}{2}(x^2 - 4x + 2) \\ &\qquad\qquad -2(1-x) \\ &\Rightarrow \quad L_3(x) = \tfrac{1}{6}(-x^3 + 9x^2 - 18x + 6).\end{aligned}$$

Comparing these results with the corresponding $\phi_n(x)$ for $n = 0, 1, 2$ shows that the ϕ_n are the same as the L_n, but their relative signs alternate. Although it is not conclusive on the basis of only three comparisons, a connection $\phi_n(x) = (-1)^n L_n(x)$ seems plausible. This is, in fact, correct and indicates that $\phi_3(x) = (-1)^3\frac{1}{6}(-x^3 + 9x^2 - 18x + 6) = \frac{1}{6}(x^3 - 9x^2 + 18x - 6)$, a conclusion that can be checked by direct, but tedious, calculation.

17.8 *A particle moves in a parabolic potential in which its natural angular frequency of oscillation is 1/2. At time $t = 0$ it passes through the origin with velocity v and is suddenly subjected to an additional acceleration of $+1$ for $0 \leq t \leq \pi/2$, and then -1 for $\pi/2 < t \leq \pi$. At the end of this period it is at the origin again. By making an eigenfunction expansion of the solution to the equation of motion, show that*

$$v = -\frac{8}{\pi}\sum_{m=0}^{\infty}\frac{1}{(4m+2)^2 - \frac{1}{4}} \approx -0.81.$$

The equation of motion is

$$\ddot{y} + \frac{1}{4}y = f(t) = \begin{cases} 1, & 0 \leq t < \pi/2, \\ -1, & \pi/2 \leq t < \pi, \end{cases}$$

with $y(0) = y(\pi) = 0$.

The eigenfunctions of the operator $\mathcal{L} = \dfrac{d^2}{dt^2} + \dfrac{1}{4}$ are obviously

$$y_n(t) = A_n \sin nt + B_n \cos nt$$

with corresponding eigenvalues $\lambda_n = n^2 - \frac{1}{4}$.

The boundary conditions, $y(0) = y(\pi) = 0$, require that n is a positive integer and that $B_n = 0$, i.e.

$$y_n(t) = A_n \sin nt = \sqrt{\frac{2}{\pi}} \sin nt \text{ (when normalised) with } n \geq 1.$$

If the required solution is $y(t) = \sum_n a_n y_n(t)$, then direct substitution yields

$$\sum_{n=1}^{\infty}(\tfrac{1}{4} - n^2)a_n y_n(t) = f(t).$$

Remembering that the y_n are sine functions, we apply the normal procedure for Fourier analysis, and obtain

$$a_m = \frac{1}{\frac{1}{4} - m^2}\int_0^{\pi} f(z) y_m(z)\, dz$$

and, consequently, that

$$y(t) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin nt}{\frac{1}{4} - n^2}\sqrt{\frac{2}{\pi}}\int_0^{\pi} f(z)\sin(nz)\, dz.$$

Now, from the given data, $f(z) = +1$ for $t < \pi/2$ and $f(z) = -1$ for $t > \pi/2$. So,

$$\begin{aligned}\int_0^{\pi} f(z)\sin(nz)\,dz &= \int_0^{\pi/2} \sin(nz)\,dz - \int_{\pi/2}^{\pi} \sin(nz)\,dz \\ &= \left[\frac{-\cos nz}{n}\right]_0^{\pi/2} - \left[\frac{-\cos nz}{n}\right]_{\pi/2}^{\pi} \\ &= \begin{cases} \dfrac{1}{n}(1-1) = 0, & \text{for } n \text{ odd}, \\ \dfrac{1}{2m}[\,-(-1)^m + 1 + 1 - (-1)^m\,], & \text{for } n = 2m, \end{cases} \\ &= \begin{cases} \dfrac{4}{2m} & \text{for } m \text{ odd, i.e. } n = 2m = 4r+2, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Thus,

$$y(t) = \frac{2}{\pi}\sum_{r=0}^{\infty} \frac{4\sin[\,(4r+2)t\,]}{(4r+2)[\,\frac{1}{4} - (4r+2)^2\,]},$$

and, by differentiation with respect to t and then setting $t = 0$,

$$v = \dot{y}(0) = -\frac{8}{\pi}\sum_{r=0}^{\infty} \frac{\cos(0)}{(4r+2)^2 - \frac{1}{4}} \approx -0.81,$$

as stated in the question.

17.10 *Consider the following two approaches to constructing a Green's function.*

(a) *Find those eigenfunctions $y_n(x)$ of the self-adjoint linear differential operator d^2/dx^2 that satisfy the boundary conditions $y_n(0) = y_n(\pi) = 0$, and hence construct its Green's function $G(x, z)$.*

(b) *Construct the same Green's function using a method based on the complementary function of the appropriate differential equation and the boundary coditions to be satisfied at the position of the δ-function, showing that it is*

$$G(x,z) = \begin{cases} x(z-\pi)/\pi, & 0 \le x \le z, \\ z(x-\pi)/\pi, & z \le x \le \pi. \end{cases}$$

(c) *By expanding the function given in* (b) *in terms of the eigenfunctions $y_n(x)$, verify that it is the same function as that derived in* (a).

Recalling that we have chosen to define the eigenvalue of a linear operator by

$$\mathcal{L}y_n = \lambda_n \rho y_n,$$

the eigenfunctions satisfying the given boundary conditions are

$$y_n(x) = \sqrt{\frac{2}{\pi}} \sin nx,$$

with corresponding eigenvalues $\lambda_n = -n^2$ for integer n. The Green's function is thus

$$G(x,z) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} y_n(x) y_n^*(z) = -\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n^2} \sin nx \sin nz.$$

(b) The differential equation defining the Green's function is

$$y'' = \delta(x - z) \qquad \text{with} \qquad y(0) = y(\pi) = 0.$$

It's solution takes the form

$$H(x,z) = \begin{cases} A + Bx & 0 \le x \le z, \\ C + Dx & z < x \le \pi. \end{cases}$$

From the boundary conditions it is clear that $A = 0$ and that $C = -D\pi$. Continuity at $x = z$ implies $Bz = C + Dz$ whilst the required unit step in the derivative implies $D - B = 1$. Together, these give $C = -z$, $D = z/\pi$ and $B = (z/\pi) - 1$. Resubstitution then gives as the Green's function

$$H(x,z) = \begin{cases} \left(\dfrac{z}{\pi} - 1\right) x = \dfrac{x(z-\pi)}{\pi} & 0 \le x \le z, \\ -z + \dfrac{z}{\pi} x = \dfrac{z(x-\pi)}{\pi} & z < x \le \pi. \end{cases}$$

(c) This verification is tantamount to finding a Fourier sine-series for the answer found in part (b):

$$H(x,z) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{\pi}} \sin nx \qquad \text{with}$$

$$\begin{aligned}
\pi \frac{\pi}{2} \sqrt{\frac{2}{\pi}} a_n &= \int_0^z (z - \pi) x \sin nx \, dx + \int_z^\pi z(x - \pi) \sin nx \, dx \\
&= (z - \pi) \left\{ \left[\frac{-x \cos nx}{n} \right]_0^z + \int_0^z \frac{\cos nx}{n} \, dx \right\} \\
&\quad - z\pi \left[\frac{-\cos nx}{n} \right]_z^\pi + z \left\{ \left[\frac{-x \cos nx}{n} \right]_z^\pi + \int_z^\pi \frac{\cos nx}{n} \, dx \right\} \\
&= (z - \pi) \left(-\frac{z \cos nz}{n} - 0 + \frac{\sin nz}{n^2} - 0 \right) + \frac{(-1)^n z\pi}{n} \\
&\quad - \frac{z\pi \cos nz}{n} + z \left(-\frac{(-1)^n \pi}{n} + \frac{z \cos nz}{n} + 0 - \frac{\sin nz}{n^2} \right) \\
&= -\frac{\pi \sin nz}{n^2}.
\end{aligned}$$

Thus,

$$a_n = -\sqrt{\frac{2}{\pi}}\frac{\sin nz}{n^2},$$

and resubstituting this expression for a_n shows that

$$H(x,z) = -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin nz\ \sin nx}{n^2},$$

so confirming that this is the same function as that derived in part (a).

17.12 *Show that the linear operator*

$$\mathcal{L} \equiv \tfrac{1}{4}(1+x^2)^2\frac{d^2}{dx^2} + \tfrac{1}{2}x(1+x^2)\frac{d}{dx} + a,$$

acting upon functions defined in $-1 \le x \le 1$ *and vanishing at the endpoints of the interval, is Hermitian with respect to the weight function* $(1+x^2)^{-1}$.

By making the change of variable $x = \tan(\theta/2)$, *find two even eigenfunctions,* $f_1(x)$ *and* $f_2(x)$, *of the differential equation*

$$\mathcal{L}u = \lambda u.$$

We take as our general functions $u(x)$ and $v(x)$ with $u(\pm 1) = v(\pm 1) = 0$. The operator $\mathcal{L}$ will be Hermitian with respect to the given weight function $w(x)$ if its adjoint, $\mathcal{L}^\dagger$ defined by

$$\int_{-1}^{1} v^*(\mathcal{L}u)w\,dx = \int_{-1}^{1}(\mathcal{L}^\dagger v)^* uw\,dx,$$

is equal to $\mathcal{L}$ and certain boundary contributions vanish.

Now consider

$$\begin{aligned}
I &= \int_{-1}^{1}\frac{v^*\mathcal{L}u}{1+x^2}\,dx \\
&= \int_{-1}^{1} v^*\left[\frac{1}{4}(1+x^2)\frac{d^2u}{dx^2} + \frac{1}{2}x\frac{du}{dx} + \frac{au}{1+x^2}\right]dx \\
&= \int_{-1}^{1} v^*\left\{\left[\tfrac{1}{4}(1+x^2)u'\right]' + \frac{au}{1+x^2}\right\}dx.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
I &= \left[v^* \tfrac{1}{4}(1+x^2)u' \right]_{-1}^{1} - \int_{-1}^{1} \left\{ v^{*\prime} \tfrac{1}{4}(1+x^2)u' - \frac{auv^*}{1+x^2} \right\} dx \\
&= 0 - \left[v^{*\prime} \tfrac{1}{4}(1+x^2)u \right]_{-1}^{1} + \int_{-1}^{1} \left\{ \left[\tfrac{1}{4}(1+x^2)v^{*\prime} \right]' u + \frac{auv^*}{1+x^2} \right\} dx \\
&= 0 + \int_{-1}^{1} u \left\{ \left[\tfrac{1}{4}(1+x^2)v' \right]' + \frac{a^* v}{1+x^2} \right\}^* dx \\
&= \int_{-1}^{1} \frac{(\mathcal{L}v)^* u}{1+x^2}\, dx,
\end{aligned}$$

provided a is real. If so, $\mathcal{L} = \mathcal{L}^\dagger$ and $\mathcal{L}$ is Hermitian.

We now make a change of variable to $\theta = 2\tan^{-1} x$ with $-\pi/2 \le \theta \le \pi/2$, $f(x) = u(\theta)$ and

$$\frac{d\theta}{dx} = \frac{2}{1+x^2} = 2\cos^2(\theta/2).$$

The expression for $\mathcal{L}u$ becomes

$$\begin{aligned}
\mathcal{L}u &= \frac{1}{4}\sec^4(\theta/2)\, 2\cos^2(\theta/2)\, \frac{d}{d\theta}\left(2\cos^2(\theta/2)\, \frac{du}{d\theta} \right) + \ \frac{1}{2}\tan(\theta/2)\, 2\frac{du}{d\theta} + au \\
&= \sec^2(\theta/2) \left[-\cos(\theta/2)\sin(\theta/2)\frac{du}{d\theta} + \cos^2(\theta/2)\frac{d^2u}{d\theta^2} \right] + \tan(\theta/2)\, \frac{du}{d\theta} + au \\
&= \frac{d^2u}{d\theta^2} + au
\end{aligned}$$

Thus, we have to solve

$$\frac{d^2u}{d\theta^2} + au = \lambda u \text{ with } u(-\tfrac{1}{2}\pi) = u(\tfrac{1}{2}\pi) = 0 \text{ and } u(-\theta) = u(\theta).$$

In view of the boundary conditions we need solutions of the form

$$u(\theta) = A\cos(\sqrt{a-\lambda}\,\theta) \quad \text{with} \quad \sqrt{a-\lambda} = 2n+1.$$

(i) $n = 0$ and $\lambda = a - 1$.

$$f_1(x) = u(\theta) = A\cos\theta = A\frac{1-x^2}{1+x^2}.$$

(ii) $n = 1$ and $\lambda = a - 9$.

$$\begin{aligned}
f_2(x) = u(\theta) = B\cos 3\theta &= B(4\cos^3\theta - 3\cos\theta) \\
&= 4B\left(\frac{1-x^2}{1+x^2} \right)^3 - 3B\frac{1-x^2}{1+x^2}.
\end{aligned}$$

Both of these functions are functions of x^2 and therefore clearly even functions of x.

17.14 *Express the solution of Poisson's equation in electrostatics,*

$$\nabla^2\phi(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon_0,$$

where ρ is the non-zero charge density over a finite part of space, in the form of an integral and hence identify the Green's function for the ∇^2 operator.

Consider the (infinitesimal) potential $d\phi(\mathbf{r})$ due to a small element of charge $dq = \rho(\mathbf{r}')\,dv'$ situated at the position $\mathbf{r}'$. This is clearly

$$d\phi(\mathbf{r}) = \frac{\rho(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}.$$

Since Poisson's equation is linear, we may apply superposition and so obtain the total potential at position $\mathbf{r}$. This same potential must also be expressible in terms of the Green's function associated with Poisson's equation. Thus,

$$-\int G(\mathbf{r},\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon_0}\,dv' \equiv \phi(\mathbf{r}) = \int d\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}.$$

Hence, by inspection,

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

18

Special functions

> ***18.2*** *Express the function*
>
> $$f(\theta, \phi) = \sin\theta[\sin^2(\theta/2)\cos\phi + i\cos^2(\theta/2)\sin\phi] + \sin^2(\theta/2)$$
>
> *as a sum of spherical harmonics.*

Since every spherical harmonic can only contain ϕ as a multiplicative factor of the form $e^{\pm im\phi}$, we must decompose the given expression into a sum of terms containing such factors. Further, as the spherical harmonics are expressed in term of θ, (rather than of $\theta/2$) we also express the given function in these terms.

$$\begin{aligned}
f(\theta, \phi) &= \sin\theta\left[\sin^2\frac{\theta}{2}\cos\phi + i\cos^2\frac{\theta}{2}\sin\phi\right] + \sin^2\frac{\theta}{2} \\
&= \sin\theta\left[\frac{1}{2}(1-\cos\theta)\cos\phi + \frac{i}{2}(1+\cos\theta)\sin\phi\right] + \frac{1}{2}(1-\cos\theta) \\
&= \frac{1}{2}(1-\cos\theta) + \frac{1}{4}(1-\cos\theta)\sin\theta\,(e^{i\phi}+e^{-i\phi}) \\
&\quad + \frac{1}{4}(1+\cos\theta)\sin\theta(e^{i\phi}-e^{-i\phi}) \\
&= \frac{1}{2} - \frac{1}{2}\cos\theta + \frac{1}{2}\sin\theta\, e^{i\phi} - \frac{1}{2}\cos\theta\sin\theta\, e^{-i\phi} \\
&= \frac{1}{2}\sqrt{4\pi}\,Y_0^0 - \frac{1}{2}\sqrt{\frac{4\pi}{3}}\,Y_1^0 - \frac{1}{2}\sqrt{\frac{8\pi}{3}}\,Y_1^1 - \frac{1}{2}\sqrt{\frac{8\pi}{15}}\,Y_2^{-1} \\
&= \sqrt{\pi}\left(Y_0^0 - \sqrt{\frac{1}{3}}Y_1^0 - \sqrt{\frac{2}{3}}\,Y_1^1 - \sqrt{\frac{2}{15}}\,Y_2^{-1}\right).
\end{aligned}$$

18.4 *Carry through the following procedure as a proof of the result*

$$I_n = \int_{-1}^{1} P_n(z)P_n(z)\,dz = \frac{2}{2n+1}.$$

(a) *Square both sides of the generating-function definition of the Legendre polynomials,*

$$(1-2zh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z)h^n.$$

(b) *Express the RHS as a sum of powers of h, obtaining expressions for the coefficients.*
(c) *Integrate the RHS from -1 to 1 and use the orthogonality property of the Legendre polynomials.*
(d) *Similarly integrate the LHS and expand the result in powers of h.*
(e) *Compare coefficients.*

We are required to evaluate

$$I_n = \int_{-1}^{1} P_n(z)P_n(z)\,dz.$$

We start with the generating function and apply the steps indicated:

$$\begin{aligned}
(1-2zh+h^2)^{-1/2} &= \sum_{n=0}^{\infty} P_n(z)h^n,\\
\frac{1}{1-2zh+h^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(z)P_m(z)\,h^{m+n},\\
\int_{-1}^{1}\frac{dz}{1-2zh+h^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-1}^{1} P_n(z)P_m(z)\,dz\,h^{m+n},\\
-\frac{1}{2h}\left[\ln(1-2zh+h^2)\right]_{-1}^{1} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} I_m\,\delta_{mn}\,h^{m+n},
\end{aligned}$$

using the orthogonality property. Thus,

$$\begin{aligned}
\sum_{m=0}^{\infty} I_m\,h^{2m} &= -\frac{1}{2h}\ln\frac{(1-h)^2}{(1+h)^2}\\
&= \frac{1}{h}\ln\frac{1+h}{1-h}\\
&= \frac{1}{h}\left(\sum_{n=0}^{\infty}\frac{(-1)^n h^{n+1}}{n+1} - \sum_{n=0}^{\infty}\frac{(-1)h^{n+1}}{n+1}\right) = \sum_{n\ \text{even}}^{\infty}\frac{2h^n}{n+1}.
\end{aligned}$$

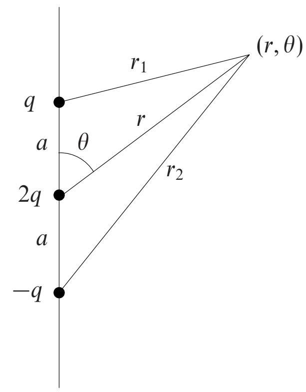

Figure 18.1 The arrangement of charges and notation for exercise 18.6.

Hence, from equating the coefficients of h^{2m} (i.e. setting $n = 2m$), we have

$$I_m = \frac{2}{2m+1},$$

as stated in the question.

> ***18.6*** *A charge $+2q$ is situated at the origin and charges of $-q$ are situated at distances $\pm a$ from it along the polar axis. By relating it to the generating function for the Legendre polynomials, show that the electrostatic potential Φ at a point (r, θ, ϕ) with $r > a$ is given by*
>
> $$\Phi(r, \theta, \phi) = \frac{2q}{4\pi\epsilon_0 r} \sum_{s=1}^{\infty} \left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\theta).$$

The situation is shown in figure 18.1. We superimpose the potentials due to the individual charges. That due to the charge $2q$ is simply $2q/(4\pi\epsilon_0 r)$. To obtain the distances r_1 and r_2 of the point (r, θ) from the negative charges we use the cosine rule:

$$r_1^2 = r^2 + a^2 - 2ar\cos\theta,$$

$$\frac{1}{r_1} = \frac{1}{r}\left[1 - \frac{2a}{r}\cos\theta + \left(\frac{a}{r}\right)^2\right]^{-1/2}.$$

This gives Φ_1 as

$$\Phi_1 = -\frac{q}{4\pi\epsilon_0 r}(1 - 2h\cos\theta + h^2)^{-1/2} = -\frac{q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)h^n,$$

where we have written $a/r = h$ and, having done so, identified the resulting expression as the generating function for Legendre polynomials. Similarly,

$$\Phi_2 = -\frac{q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty} P_n(-\cos\theta)h^n.$$

Since $P_n(-\cos\theta) = (-1)^n P_n(\cos\theta)$, when all three terms are added together we obtain

$$\begin{aligned}\Phi &= -\frac{2q}{4\pi\epsilon_0 r}\sum_{n \text{ even}}^{\infty} P_n(\cos\theta)h^n + \frac{2q}{4\pi\epsilon_0 r}\\ &= -\frac{2q}{4\pi\epsilon_0 r}\sum_{n \text{ even} \neq 0}^{\infty} P_n(\cos\theta)h^n\\ &= -\frac{2q}{4\pi\epsilon_0 r}\sum_{s=1}^{\infty}\left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\theta),\end{aligned}$$

as stated in the question.

18.8 *The quantum mechanical wavefunction for a one-dimensional simple harmonic oscillator in its nth energy level is of the form*

$$\psi(x) = \exp(-x^2/2)H_n(x),$$

where $H_n(x)$ is the nth Hermite polynomial. The generating function for the polynomials is

$$G(x,h) = e^{2hx-h^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n.$$

(a) *Find $H_i(x)$ for $i = 1, 2, 3, 4$.*
(b) *Evaluate by direct calculation*

$$\int_{-\infty}^{\infty} e^{-x^2} H_p(x)H_q(x)\,dx,$$

(i) *for $p = 2$, $q = 3$;* (ii) *for $p = 2$, $q = 4$;* (iii) *for $p = q = 3$. Check your answers against the expected values $2^p\, p!\sqrt{\pi}\,\delta_{pq}$.*

[*You will find it convenient to use*

$$\int_{-\infty}^{\infty} x^{2n}e^{-x^2}\,dx = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}$$

for integer $n \geq 0$.]

(a) The generating function is

$$G(x,h) = \exp(2hx - h^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n,$$

from which it follows that

$$H_n(x) = \frac{\partial^n}{\partial h^n} \left[\exp(2hx - h^2)\right]\big|_{h=0}.$$

We therefore calculate these derivatives:

$$\begin{aligned}
\frac{\partial G}{\partial h} &= (2x-2h)G \quad \Rightarrow \quad H_1(x) = 2x,\\
\frac{\partial^2 G}{\partial h^2} &= (2x-2h)^2 G - 2G \quad \Rightarrow \quad H_2(x) = 4x^2 - 2,\\
\frac{\partial^3 G}{\partial h^3} &= (2x-2h)^3 G + 2(-2)(2x-2h)G - 2(2x-2h)G\\
&= (2x-2h)^3 G - 6(2x-2h)G \quad \Rightarrow \quad H_3(x) = 8x^3 - 12x,\\
\frac{\partial^4 G}{\partial h^4} &= (2x-2h)^4 G + 3(-2)(2x-2h)^2 G - 6(2x-2h)^2 G + 12G\\
&= (2x-2h)^4 G - 12(2x-2h)^2 G + 12G\\
&\Rightarrow \quad H_4(x) = 16x^4 - 48x^2 + 12.
\end{aligned}$$

(b) Denote by J_n the integral

$$J_n = \int_{-\infty}^{\infty} x^n e^{-x^2}\, dx \text{ with } J_{2r} = \frac{(2r)!\,\sqrt{\pi}}{2^{2r}\, r!} \text{ and } J_{2r+1} = 0.$$

Further, define I_{pq} as

$$I_{pq} = \int_{-\infty}^{\infty} e^{-x^2} H_p(x) H_q(x)\, dx.$$

Then, for case (i)

$$\begin{aligned}
I_{23} &= \int_{-\infty}^{\infty} e^{-x^2}(4x^2-2)(8x^3-12)\, dx\\
&= 32J_5 - 16J_3 - 48J_3 + 24J_1 = 0, \quad \text{as all subscripts are odd.}
\end{aligned}$$

For case (ii)

$$\begin{aligned}
I_{24} &= \int_{-\infty}^{\infty} e^{-x^2}(4x^2-2)(16x^4-48x^2+12)\, dx\\
&= 64J_6 - 192J_4 + 48J_2 - 32J_4 + 96J_2 - 24J_0\\
&= 64\,\frac{6!\,\sqrt{\pi}}{2^6\,3!} - 224\,\frac{4!\,\sqrt{\pi}}{2^4\,2!} + 144\,\frac{2!\,\sqrt{\pi}}{2^2\,1!} - 24\,\frac{0!\,\sqrt{\pi}}{2^0\,0!}\\
&= \sqrt{\pi}(120 - 168 + 72 - 24) = 0.
\end{aligned}$$

Finally, for case (iii)

$$\begin{aligned} I_{33} &= \int_{-\infty}^{\infty} e^{-x^2}(8x^3 - 12x)^2\,dx \\ &= 64J_6 - 192J_4 + 144J_2 \\ &= 64\,\frac{6!\,\sqrt{\pi}}{2^6\,3!} - 192\,\frac{4!\,\sqrt{\pi}}{2^4\,2!} + 144\,\frac{2!\,\sqrt{\pi}}{2^2\,1!} \\ &= \sqrt{\pi}(120 - 144 + 72) = 48\sqrt{\pi}. \end{aligned}$$

The expected values are

$$\int_{-\infty}^{\infty} e^{-x^2} H_p(x) H_q(x)\,dx = 2^p p\,!\sqrt{\pi}\delta_{pq}.$$

This is equal to zero for $p \neq q$ and equal to $2^3\,3!\,\sqrt{\pi} = 48\sqrt{\pi}$ for $p = q = 3$. All three results agree with this.

18.10 *By choosing a suitable form for h in their generating function,*

$$G(z, h) = \exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) h^n,$$

show that integral repesentations of the Bessel functions of the first kind are given, for integral m, by

$$\begin{aligned} J_{2m}(z) &= \frac{(-1)^m}{\pi} \int_0^{2\pi} \cos(z\cos\theta)\,\cos 2m\theta\,d\theta, \qquad & m \geq 1, \\ J_{2m+1}(z) &= \frac{(-1)^{m+1}}{\pi} \int_0^{2\pi} \cos(z\cos\theta)\,\sin(2m+1)\theta\,d\theta, \qquad & m \geq 0. \end{aligned}$$

In the generating function equation,

$$G(z, h) = \exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) h^n,$$

we set $h = ie^{i\theta}$ and obtain

$$\begin{aligned} \exp\left[\frac{z}{2}\left(ie^{i\theta} + ie^{-i\theta}\right)\right] &= \sum_{n=-\infty}^{\infty} J_n(z) i^n e^{in\theta}, \\ \exp[\,iz\cos\theta\,] &= \sum_{n=-\infty}^{\infty} J_n(z) i^n (\cos n\theta + i\sin n\theta). \end{aligned}$$

Our choice for h was prompted by the presence in the quoted answer of a sinusoidal function with a sinusoidal arguement – or, equivalently, for complex variables, an exponential function with an exponential argument. Equating the real parts of both sides of the equality gives

$$\cos(z\cos\theta) = \sum_{m=-\infty}^{\infty} (-1)^m J_{2m} \cos 2m\theta + \sum_{m=-\infty}^{\infty} (-1)^{m+1} J_{2m+1} \sin(2m+1)\theta.$$

Now multiplying both sides of this equation by $\cos 2r\theta$ and integrating over θ from 0 to 2π gives (because of the mutual orthogonality of the sinusoidal functions) that

$$\int_0^{2\pi} \cos(z\cos\theta)\, \cos(2r\theta)\, d\theta = (-1)^r \pi\, J_{2r}(z) \text{ for } r \geq 1.$$

Hence the first result stated.

Similarly, multiplying through by $\sin(2r+1)\theta$ and integrating produces the second result.

18.12 *By making the substitution $z = (1-x)/2$ and suitable choices for a, b and c, convert the hypergeometric equation,*

$$z(1-z)\frac{d^2u}{dz^2} + [\,c - (a+b+1)z\,]\frac{du}{dz} - abu = 0,$$

into the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0.$$

Hence, using the hypergeometric series, generate the Legendre polynomials $P_\ell(x)$ for the integer values $\ell = 0, 1, 2, 3$. Comment on their normalisations.

From the substitution $z = (1-x)/2$, it follows that $1 - z = (1+x)/2$ and $d\,/dz = -2d\,/dx$. If $u(z) = y(x)$ then the hypergeometric equation becomes

$$\frac{1-x}{2}\,\frac{1+x}{2}\,(-2)^2\,\frac{d^2y}{dx^2} + \left[c - \frac{(a+b+1)(1-x)}{2}\right](-2)\,\frac{dy}{dx} - ab\,y = 0. \quad (*)$$

We now compare this with the Legendre equation. From the undifferentiated term we must have that the product $-ab = \ell(\ell+1)$, whilst, from the coefficient of dy/dx, we see that the sum $a+b$ must not depend upon ℓ. The only possibilities are $a = \ell$ with $b = -(\ell+1)$ and $a = -\ell$ with $b = \ell+1$. Noting that $a+b+1$ has the value 0 in the former case and 2 in the latter, we choose the second possibility, since we require a term containing x in the coefficient of dy/dx.

The equation (∗) now becomes

$$(1-x^2)\frac{d^2y}{dx^2} - 2[c-(1-x)]\frac{dy}{dx} + \ell(\ell+1)y = 0.$$

All that remains to reproduce the Legendre equation is to choose $c = 1$. Thus, apart from a possible normalisation factor,

$$P_\ell(x) = F\left(-\ell, \ell+1, 1, \frac{1-x}{2}\right).$$

The corresponding hypergeometric function is therefore

$$1 + \frac{(-\ell)(\ell+1)}{1!\,1!}\left(\frac{1-x}{2}\right) + \frac{(-\ell)(-\ell+1)(\ell+1)(\ell+2)}{2!\,(1+1)!}\left(\frac{1-x}{2}\right)^2 + \cdots.$$

Because of the factor $-\ell + n$ in the numerator of the $(n+2)$th term, each series terminates after $\ell + 1$ terms. For the specific values of ℓ:

$$\begin{aligned}
\ell = 0,\quad P_0(x) &= 1,\\
\ell = 1,\quad P_1(x) &= 1 + \frac{(-1)(2)}{1!\,1!}\left(\frac{1-x}{2}\right) = x,\\
\ell = 2,\quad P_2(x) &= 1 + \frac{(-2)(3)}{1!\,1!}\left(\frac{1-x}{2}\right) + \frac{(-2)(-1)(3)(4)}{2!\,(1+1)!}\left(\frac{1-x}{2}\right)^2\\
&= 1 - 3(1-x) + \tfrac{3}{2}(1-2x+x^2)\\
&= -\tfrac{1}{2} + \tfrac{3}{2}x^2 = \tfrac{1}{2}(3x^2-1),\\
\ell = 3,\quad P_3(x) &= 1 + \frac{(-3)(4)}{1!\,1!}\left(\frac{1-x}{2}\right) + \frac{(-3)(-2)(4)(5)}{2!\,(1+1)!}\left(\frac{1-x}{2}\right)^2\\
&\qquad + \frac{(-3)(-2)(-1)(4)(5)(6)}{3!\,(1+2)!}\left(\frac{1-x}{2}\right)^3\\
&= 1 - 6(1-x) + \tfrac{15}{2}(1-2x+x^2)\\
&\qquad - \tfrac{5}{2}(1-3x+3x^2-x^3)\\
&= (1 - 6 + \tfrac{15}{2} - \tfrac{5}{2}) + (6 - 15 + \tfrac{15}{2})x\\
&\qquad + (\tfrac{15}{2} - \tfrac{15}{2})x^2 + \tfrac{5}{2}x^3\\
&= -\tfrac{3}{2}x + \tfrac{5}{2}x^3 = \tfrac{1}{2}(5x^3 - 3x).
\end{aligned}$$

These are the first four Legendre polynomials — usually found by other means. That they are all correctly normalised is the result of the arbitrary, but standard, requirement that $P_\ell(1) = 1$ (rather than, say, $\int_{-1}^{1} P_\ell^2\,dx = 1$). This requirement is automatically satisfied by the hypergeometric series since when $x = 1$ we have $z = 0$ and $F(a, b, c; 0) = 1$ for all a and b, and for all c, except possibly when c is a negative integer; here $c = 1$.

> ***18.14*** *Prove that, if m and n are both greater than −1, then*
>
> $$I = \int_0^\infty \frac{u^m}{(au^2+b)^{(m+n+2)/2}}\,du = \frac{\Gamma[\,\frac{1}{2}(m+1)\,]\,\Gamma[\,\frac{1}{2}(n+1)\,]}{2a^{(m+1)/2}\,b^{(n+1)/2}\,\Gamma[\,\frac{1}{2}(m+n+2)\,]}.$$
>
> *Deduce the value of*
>
> $$J = \int_0^\infty \frac{(u+2)^2}{(u^2+4)^{5/2}}\,du.$$

Since the quoted answer strongly resembles a beta function and this is most easily connected to integrals over the range 0 to 1, we first take a factor au^2 out of the parentheses in the denominator and then make the change of variable $1 + \dfrac{b}{au^2} = \dfrac{1}{x}$. With this change,

$$u = \left[\frac{b}{a}\,\frac{x}{1-x}\right]^{1/2}, \quad du = \frac{1}{2}\left(\frac{b}{a}\right)^{1/2}\frac{dx}{x^{1/2}\,(1-x)^{3/2}}$$

and the integration limits (originally 0 and ∞) are 0 and 1. Thus,

$$\begin{aligned}
I &= \int_0^\infty \frac{u^m}{(au^2+b)^{(m+n+2)/2}}\,du \\
&= \int_0^\infty \frac{u^m}{a^{(m+n+2)/2}(1+\frac{b}{au^2})^{(m+n+2)/2}u^{m+n+2}}\,du \\
&= \int_0^1 \frac{x^{(m+n+2)/2}\;a^{(n+2)/2}\;(1-x)^{(n+2)/2}\;b^{1/2}}{a^{(m+n+2)/2}\;b^{(n+2)/2}\;x^{(n+2)/2}\;2a^{1/2}\;x^{1/2}(1-x)^{3/2}}\,dx \\
&= \int_0^1 \frac{x^{(m-1)/2}\;(1-x)^{(n-1)/2}}{2a^{(m+1)/2}\;b^{(n+1)/2}}\,dx.
\end{aligned}$$

This integral is a multiple of the beta function $B\left(\dfrac{m+1}{2}, \dfrac{n+1}{2}\right)$ and can therefore be expressed in terms of gamma functions as

$$I = \frac{\Gamma[\,\frac{1}{2}(m+1)\,]\,\Gamma[\,\frac{1}{2}(n+1)\,]}{2a^{(m+1)/2}\,b^{(n+1)/2}\,\Gamma[\,\frac{1}{2}(m+n+2)\,]}.$$

In the notation used above, we have for this given specific case that $a = 1$ and $b = 4$. J can be expressed as the sum of three integrals of the form considered there by expanding its numerator:

$$J = \int_0^\infty \frac{u^2}{(u^2+4)^{5/2}}\,du + \int_0^\infty \frac{4u}{(u^2+4)^{5/2}}\,du + \int_0^\infty \frac{4}{(u^2+4)^{5/2}}\,du.$$

The corresponding pairs of values of m and n are $m = 2,\ n = 1$ for the first term,

$m = 1,\ n = 2$ for the second and $m = 0,\ n = 3$ for the third. Thus the value of J is

$$J = \frac{1}{2\ 1\ 4}\,\frac{\Gamma(\frac{3}{2})\,\Gamma(1)}{\Gamma(\frac{5}{2})} + \frac{4}{2\ 1\ 4^{3/2}}\,\frac{\Gamma(1)\,\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} + \frac{4}{2\ 1\ 4^{2}}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(2)}{\Gamma(\frac{5}{2})}$$

$$= \frac{3\Gamma(\frac{3}{2})\,\Gamma(1) + \Gamma(\frac{1}{2})\,\Gamma(2)}{8\ \Gamma(\frac{5}{2})}$$

$$= \frac{3\ \frac{1}{2}\sqrt{\pi}\ 1 + \sqrt{\pi}\ 1}{8\ \frac{3}{4}\ \sqrt{\pi}} = \frac{5}{12}.$$

> ***18.16*** *For $-1 < \mathrm{Re}\, z < 1$, use the definition and value of the beta function to show that*
>
> $$z!\,(-z)! = \int_0^\infty \frac{u^z}{(1+u)^2}\,du.$$
>
> *Contour integration gives the value of the integral on the RHS of the above equation as $\pi z \operatorname{cosec} \pi z$. Use this to deduce the value of $(-\frac{1}{2})!$.*

From the expression for the beta function in terms of gamma functions and the relationship between the gamma and factorial functions for $\mathrm{Re}\, z_i > -1$, we have

$$\frac{z_1!\,z_2!}{(z_1+z_2+1)!} = B(z_1+1, z_2+1) = \int_0^1 t^{z_1}(1-t)^{z_2}\,dt.$$

Since $-1 < \mathrm{Re}\, z < 1$, $-z$ is not a negative integer and so $(-z)!$ is defined and finite. Setting $z_1 = z$ and $z_2 = -z$, we obtain

$$z!\,(-z)! = (z - z + 1)! \int_0^1 t^z (1-t)^{-z}\,dt.$$

Making the change of integration variable

$$t = \frac{u}{1+u}, \text{ with } 1 - t = \frac{1}{1+u} \text{ and } dt = \frac{1}{(1+u)^2}\,du,$$

gives

$$z!\,(-z)! = 1! \int_0^\infty \frac{u^z}{(1+u)^z}\,\frac{(1+u)^z}{1^z}\,\frac{1}{(1+u)^2}\,du$$

$$= \int_0^\infty \frac{u^z}{(1+u)^2}\,du = \frac{\pi z}{\sin \pi z}, \qquad \text{(given).}$$

Now, setting $z = -\frac{1}{2}$ and using the general result $(z+1)! = (z+1)z!$, we have

$$\tfrac{1}{2}!\,(-\tfrac{1}{2})! = (-\tfrac{1}{2} + 1)\,(-\tfrac{1}{2})!\,(-\tfrac{1}{2})!.$$

Since $(-\frac{1}{2}\pi)/\sin(-\frac{1}{2}\pi) = \pi/2$, it follows that $[(-\frac{1}{2})!]^2 = \pi$.

Now, $\frac{1}{2}! = \int_0^\infty u^{-1/2}e^{-u}\,du$ and is clearly positive, since the integrand is positive everywhere. Further, since $\frac{1}{2}! = \frac{1}{2}(-\frac{1}{2})!$ it follows that $(-\frac{1}{2})!$ has the same sign as $\frac{1}{2}!$, i.e. $(-\frac{1}{2})!$ is positive. Therefore $(-\frac{1}{2})! = \sqrt{\pi}$.

18.18 *Consider two series expansions of the error function as follows:*

(a) *Obtain a series expansion of the error function* erf(x) *in ascending powers of x. How many terms are needed to give a value correct to four significant figures for* erf(1)?

(b) *Obtain an asymptotic expansion that can be used to estimate* erfc(x) *for large $x(>0)$ in the form of a series*

$$\text{erfc}(x) = R(x) = e^{-x^2}\sum_{n=0}^{\infty}\frac{a_n}{x^n}.$$

Consider what bounds can be put on the estimate and at what point the infinite series should be terminated in a practical estimate. In particular, estimate erfc(1) *and test the answer for compatibility with that in part* (a).

(a) This series can be determined straightforwardly by expanding the integrand in a series of its own and then integrating term-by-term.

$$\begin{aligned}\text{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du \\ &= \frac{2}{\sqrt{\pi}}\int_0^x \left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots\right) du \\ &= \frac{2}{\sqrt{\pi}}\left(x - \frac{x^2}{3} + \frac{x^5}{2!\,5} - \frac{x^7}{3!\,7} + \cdots\right).\end{aligned}$$

From tables, either directly or by setting $x = \sqrt{2}$ in the relationship $\frac{1}{2}\text{erf}(x/\sqrt{2}) = \Phi(x) - \frac{1}{2}$ where $\Phi(x)$ is the (cumulative) Gaussian distribution function, we find that $\text{erf}(1) = 0.8427$.

From the calculated series, the successive partial sums corresponding to $1, 2, 3, \ldots$ terms are 1.1284, 0.7523, 0.8651, 0.8382, 0.8434, 0.8426, 0.8427, 0.8427, ... Thus seven terms are needed to obtain the desired accuracy.

(b) We start with

$$\text{erfc}(x) = R(x) = e^{-x^2}\sum_{n=0}^{\infty}\frac{a_n}{x^n}.$$

Now,

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du \quad \Rightarrow \quad \frac{dR}{dx} = -\frac{2}{\sqrt{\pi}} e^{-x^2}.$$

Substituting for $R(x)$ and differentiating gives, as the equation to be satisfied,

$$-2xe^{-x^2} \sum_{n=0}^{\infty} \frac{a_n}{x^n} + e^{-x^2} \sum_{n=1}^{\infty} \frac{(-n)a_n}{x^{n+1}} = -\frac{2}{\sqrt{\pi}} e^{-x^2}.$$

Equating the coefficients of x and the constant terms gives $a_0 = 0$ and $a_1 = 1/\sqrt{\pi}$, whilst equating inverse powers of x^{n-1} yields the recurrence relation

$$-2a_n + [\,-(n-2)\,]a_{n-2} = 0.$$

Thus, only the terms with n odd are present and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k-1)}{2} a_{2k-1} = \cdots = \frac{(-1)^k (2k-1)(2k-2)\cdots 1}{2^k} a_1 \\ &= \frac{(-1)^k (2k-1)!!}{2^k \sqrt{\pi}}, \end{aligned}$$

where $(2k-1)!!$ denotes the product $1 \times 3 \times 5 \times \cdots \times (2k-1)$.

The explicit form of $R(x)$ is therefore

$$R(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \left(\frac{1}{x} + \sum_{k=1}^{\infty} \frac{(-1)^k\,(2k-1)!!}{2^k\, x^{2k+1}} \right).$$

Examination, as $k \to \infty$, of the modulus of the ratio of successive terms in the sum, which is $(2k+1)/2x^2$, shows that the series does not converge for any finite fixed x. However, if the series in truncated at $k = K$ with value $R(x, K)$ then,

$$\begin{aligned} R(x, K-1) &< \text{erfc}(x) < R(x, K) \qquad \text{if } K \text{ is even,} \\ R(x, K-1) &> \text{erfc}(x) > R(x, K) \qquad \text{if } K \text{ is odd.} \end{aligned}$$

Thus successive pairs of values of the partial sum bracket the true value of erfc(x), but with the bracketting range ultimately getting larger (rather than smaller). Which value of K gives the tightest bounds on erfc(x) depends upon the value of x; the best pair of values for K are probably the two integers that bracket $x^2 - \frac{1}{2}$.

For erfc(1) we have the series

$$\text{erfc}(1) \approx \frac{e^{-1}}{\sqrt{\pi}} \left(\frac{1}{1} - \frac{1}{2} + \frac{3}{4} - \frac{3 \times 5}{8} + \cdots \right).$$

The partial sums for 1, 2, … terms are 0.2076, 0.1038, 0.2594, −0.1627, … . The correct answer [see part (a)] is $1.000 - 0.8427 = 0.1573$. This behaviour of the partial sums is as expected, with the correct value always lying between any successive pair and the tightest bounds given by $K = 0$ (i.e. just the first term) and $K = 1$; in fact it lies roughly mid-way between the two. Later terms cause the partial sum to swing with increasing amplitude on either side of the correct value.

***18.20** The Bessel function $J_\nu(z)$ can be considered as a special case of the solution $M(a, c; z)$ of the confluent hypergeometric equation, the connection being*

$$\lim_{a\to\infty} \frac{M(a, \nu+1; -z/a)}{\Gamma(\nu+1)} = z^{-\nu/2} J_\nu(2\sqrt{z}).$$

Prove this equality by writing each side in terms of an infinite series and showing that the series are the same.

The hypergeometric series can be written more compactly by introducing the notation

$$(c)_n \equiv c(c+1)(c+2)\cdots(c+n-1) \quad \text{with} \quad (c)_0 = 1,$$

for the n-factor product. We note that $\lim_{c\to\infty}(c)_n/c^n = 1$ and that

$$\Gamma(\nu+1)(\nu+1)_n = \Gamma(\nu+n+1).$$

On the one hand, with this notation,

$$\begin{aligned}
\lim_{a\to\infty} \frac{M(a, \nu+1; -z/a)}{\Gamma(\nu+1)} &= \frac{1}{\Gamma(\nu+1)} \lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{n!\,(\nu+1)_n} \left(-\frac{z}{a}\right)^n \\
&= \frac{1}{\Gamma(\nu+1)} \lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\,(\nu+1)_n} \left[\frac{(a)_n}{a^n}\right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\,\Gamma(\nu+n+1)}.
\end{aligned}$$

But, on the other hand, from the standard series for the Bessel function of order ν,

$$\begin{aligned}
z^{-\nu/2} J_\nu(2\sqrt{z}) &= z^{-\nu/2} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}\, 2\sqrt{z})^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{z})^{-\nu+\nu+2n}}{n!\,\Gamma(\nu+n+1)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (z)^n}{n!\,\Gamma(\nu+n+1)}.
\end{aligned}$$

Thus the two series expressions are the same and the equality is established.

***18.22** Show from its definition that the Bessel function of the second kind of integral order ν can be written as*

$$Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial \mu} - (-1)^\nu \frac{\partial J_{-\mu}(z)}{\partial \mu}\right]_{\mu=\nu}.$$

Using the explicit series expression for $J_\mu(z)$, show that $\partial J_\mu(z)/\partial\mu$ can be written as

$$J_\nu(z)\ln\left(\frac{z}{2}\right) + g(\nu, z),$$

and deduce that $Y_\nu(z)$ can be expressed as

$$Y_\nu(z) = \frac{2}{\pi}J_\nu(z)\ln\left(\frac{z}{2}\right) + h(\nu, z),$$

where $h(\nu, z)$, like $g(\nu, z)$, is a power series in z.

Using the fact that, for integer ν, $J_{-\nu}(z) = (-1)^\nu J_\nu(z)$, direct substitution in the definition of $Y_\nu(z)$ produces the indeterminate equation

$$Y_\nu(z) \equiv \lim_{\mu\to\nu}\left[\frac{J_\mu(z)\cos\mu\pi - J_{-\mu}(z)}{\sin\mu\pi}\right] = \left[\frac{J_\nu(z)(-1)^\nu - (-1)^\nu J_\nu(z)}{\sin\nu\pi}\right] = \frac{0}{0}.$$

We therefore employ l'Hôpital's rule:

$$\begin{aligned} Y_\nu(z) &\equiv \lim_{\mu\to\nu}\left[\frac{J_\mu(z)\cos\mu\pi - J_{-\mu}(z)}{\sin\mu\pi}\right] \\ &= \lim_{\mu\to\nu}\left[\frac{-\pi\sin(\mu\pi)J_\mu(z) + \cos\mu\pi\dot{J}_\mu(z) - \dot{J}_{-\mu}(z)}{\pi\cos\mu\pi}\right], \end{aligned}$$

where $\dot{J}_\mu(z) = \dfrac{\partial J_\mu(z)}{\partial\mu}$. Thus,

$$Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial \mu} - (-1)^\nu \frac{\partial J_{-\mu}(z)}{\partial \mu}\right]_{\mu=\nu}. \quad (*)$$

Now, we have as an explicit series represntation of $J_\mu(z)$

$$J_\mu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\mu+n+1)}\left(\frac{z}{2}\right)^{\mu+2n}.$$

We need the partial derivative of this with respect to μ and since μ appears in each term as an exponent, as well as part of a multiplicative factor, each term in the series will generate two terms in the derivative, one of which will contain a logarithm. This is a particular example of the general result that the derivative of x^μ with respect to μ is $x^\mu \ln x$. Carrying this through, the derivative with respect to μ is given by

$$\begin{aligned}\frac{\partial J_\mu}{\partial \mu} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{z}{2}\right)^{2n} \left[\frac{1}{\Gamma(\mu+n+1)} \frac{\partial}{\partial \mu} \left(\frac{z}{2}\right)^{\mu} \right. \\ &\qquad \left. + \left(\frac{z}{2}\right)^{\mu} \frac{\partial}{\partial \mu} \left(\frac{1}{\Gamma(\mu+n+1)} \right) \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{z}{2}\right)^{2n} \left[\frac{1}{\Gamma(\mu+n+1)} \left(\frac{z}{2}\right)^{\mu} \ln\left(\frac{z}{2}\right) \right. \\ &\qquad \left. + \left(\frac{z}{2}\right)^{\mu} \frac{\partial}{\partial \mu} \left(\frac{1}{\Gamma(\mu+n+1)} \right) \right].\end{aligned}$$

Hence, to obtain the second solution we set $\mu = \nu$:

$$\left(\frac{\partial J_\mu}{\partial \mu}\right)_{\mu=\nu} = J_\nu(z) \ln\left(\frac{z}{2}\right) + g(\nu, z),$$

where $g(\nu, z)$ is a power series in z. The coefficients in the power series are complicated, but well defined, functions of ν and n involving Γ-functions and their derivatives.

Setting $\mu = -\nu$ produces the corresponding result

$$\begin{aligned}\left(\frac{\partial J_\mu}{\partial \mu}\right)_{\mu=-\nu} &= -J_{-\nu}(z) \ln\left(\frac{z}{2}\right) + g(-\nu, z) \\ &= -(-1)^\nu J_\nu(z) \ln\left(\frac{z}{2}\right) + g(-\nu, z).\end{aligned}$$

We note that the early terms in $g(-\nu, z)$ will be absent because of the presence of (the infinite quantities) $\Gamma(n+1-\nu)$ and their derivatives in the denominator.

Finally, on substituting in (*), we note that the two logarithmic terms contribute in the same sense (as opposed to cancelling) and we have

$$Y_\nu(z) = \frac{2}{\pi} J_\nu(z) \ln\left(\frac{z}{2}\right) + h(\nu, z),$$

where $h(\nu, z)$ is a power series in z.

18.24 *The solutions $y(x,a)$ of the equation*

$$\frac{d^2y}{dx^2} - (\tfrac{1}{4}x^2 + a)y = 0 \qquad (*)$$

are known as parabolic cylinder functions.

(a) *If $y(x,a)$ is a solution of $(*)$, determine which of the following are also solutions:* (i) $y(a,-x)$, (ii) $y(-a,x)$, (iii) $y(a,ix)$ *and* (iv) $y(-a,ix)$.

(b) *Show that one solution of $(*)$, even in x, is*

$$y_1(x,a) = e^{-x^2/4}M(\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}x^2),$$

where $M(\alpha, c, z)$ is the confluent hypergeometric function satisfying

$$z\frac{d^2M}{dz^2} + (c-z)\frac{dM}{dz} - \alpha M = 0.$$

You may assume (or prove) that a second solution, odd in x, is given by $y_2(x,a) = xe^{-x^2/4}M(\tfrac{1}{2}a + \tfrac{3}{4}, \tfrac{3}{2}, \tfrac{1}{2}x^2)$.

(c) *Find, as an infinite series, an explicit expression for $e^{x^2/4}y_1(x,a)$.*

(d) *Using the results from part* (a) *show that $y_1(x,a)$ can also be written as*

$$y_1(x,a) = e^{x^2/4}M(-\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, -\tfrac{1}{2}x^2)$$

(e) *By making a suitable choice for a deduce that*

$$1 + \sum_{n=1}^{\infty} \frac{b_n\, x^{2n}}{(2n)!} = e^{x^2/2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, b_n\, x^{2n}}{(2n)!}\right),$$

where $b_n = \prod_{r=1}^{n}(2r - \tfrac{3}{2})$.

(a) When changing x to μx the second derivative of y is multiplied by μ^{-2} and the factor x^2 by μ^2. Thus

(i) The equation becomes $(-1)^{-2}y'' - (\tfrac{1}{4}(-1)^2x^2 + a)y = 0$, i.e is unaltered. Thus $y(a,-x)$ is also a solution.

(ii) The equation becomes $y'' - (\tfrac{1}{4}x^2 - a)y = 0$, i.e is a different equation. Thus $y(-a,x)$ is not a solution of $(*)$.

(iii) The equation becomes $(i^{-2})y'' - (\tfrac{1}{4}(i^2)x^2 + a)y = 0$. This is the same equation as in part (ii). Thus $y(a,ix)$ is not a solution of $(*)$.

(iv) The equation becomes $(i^{-2})y'' - (\tfrac{1}{4}(i^2)x^2 - a)y = 0$, i.e is unaltered. Thus $y(-a,ix)$ is a second solution of $(*)$.

(b) We first write $y_1(x,a)$ as $y_1(x,a) = e^{-x^2/4}u(x)$ and determine the equation $u(x)$

must satisfy. The function and derivatives needed are

$$\begin{aligned}
y_1 &= e^{-x^2/4}u, \\
y_1' &= -\frac{x}{2}e^{-x^2/4}u + e^{-x^2/4}u', \\
y_1'' &= -\frac{1}{2}e^{-x^2/4}u + \frac{x^2}{4}\,e^{-x^2/4}u - 2\frac{x}{2}\,e^{-x^2/4}u' + e^{-x^2/4}u''.
\end{aligned}$$

Thus, (cancelling a factor $e^{-x^2/4}$) substitution in (∗) yields

$$\begin{aligned}
-\frac{1}{2}u + \frac{x^2}{4}\,u - xu' + u'' - \frac{x^2}{4}\,u - au &= 0, \\
\Rightarrow \quad u'' - xu' - (a + \tfrac{1}{2})u &= 0. \qquad (**)
\end{aligned}$$

Now, the equation satisfied by $M(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}, z)$ is

$$z\frac{d^2M}{dz^2} + (\tfrac{1}{2} - z)\frac{dM}{dz} - (\tfrac{1}{2}a + \tfrac{1}{4})M = 0.$$

In this we set $z = \frac{1}{2}x^2$, with $d/dz = x^{-1}d/dx$, and write $M(z) = N(x)$, obtaining

$$\begin{aligned}
\frac{x^2}{2}\,\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{dN}{dx}\right) + \left(\frac{1}{2} - \frac{x^2}{2}\right)\frac{1}{x}\,\frac{dN}{dx} - \left(\frac{1}{2}a + \frac{1}{4}\right)N &= 0, \\
\frac{x}{2}\left(-\frac{1}{x^2}\,N' + \frac{1}{x}\,N''\right) + \frac{1}{2x}N' - \frac{x}{2}N' - \left(\frac{1}{2}a + \frac{1}{4}\right)N &= 0, \\
N'' - xN' - (a + \tfrac{1}{2})N &= 0.
\end{aligned}$$

This is the same equation as (∗∗) thus establishing that $y_1(a, x)$ can be written as

$$y_1(x, a) = e^{-x^2/4}M(\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}x^2).$$

Since the confluent hypergeometric function is a polynomial function of its third argument the solution is clearly even in x.

(c) With the result established in part (b) we need only evaluate a typical term of the hypergeometric series for $M(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}x^2)$. The zeroth term is 1 and the nth term ($n > 0$) is

$$\begin{aligned}
t_n &= \frac{(\frac{1}{2}a + \frac{1}{4})(\frac{1}{2}a + \frac{5}{4})\cdots(\frac{1}{2}a + \frac{1}{4} + n - 1)}{(\frac{1}{2})(\frac{3}{2})\cdots(\frac{2n-1}{2})}\,\frac{1}{n!}\,\left(\frac{x^2}{2}\right)^n \\
&= \frac{(a + \frac{1}{2})(a + \frac{5}{2})\cdots(a + 2n - \frac{3}{2})}{(1)(3)\cdots(2n-1)\;(2^n\,n!)}\,x^{2n}.
\end{aligned}$$

Writing the numerator as a product and noting that $2^n\,n! = (2)(4)\cdots(2n)$, we can now write the whole series as

$$e^{x^2/4}y_1(x, a) = 1 + \sum_{n=1}^{\infty}\frac{\prod_{r=1}^{n}(a + 2r - \frac{3}{2})}{(2n)!}\,x^{2n}.$$

(d) In part (a)(iv) we showed that since $y_1(x,a)$ is a solution of $(*)$ so is $y_1(-a,ix)$, i.e

$$\begin{aligned} y_3(x,a) \equiv y_1(-a,ix) &= e^{-i^2x^2/4}M(-\tfrac{1}{2}a+\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{2}i^2x^2) \\ &= e^{x^2/4}M(-\tfrac{1}{2}a+\tfrac{1}{4},\tfrac{1}{2},-\tfrac{1}{2}x^2) \end{aligned}$$

must also be a solution of $(*)$.

Since we already have two linearly independent solutions of $(*)$, namely y_1 and y_2, and $(*)$ is only a 2nd-order equation, y_3 must be linearly dependent on y_1 and y_2. However, it is clearly an even function of x and so it must be a multiple λ of y_1. Further, since $M(\alpha,c,0)=1$ for all α and c, and $\exp(\pm 0^2/4)=1$, we conclude from setting $x=0$ that $\lambda=1$ and consequently that $y_3(x,a)=y_1(x,a)$.

(e) Expressing this last result in term of the series representions of the parabolic cylinder functions gives the equality

$$\begin{aligned} y_1(x,a) &= e^{-x^2/4}\left[1+\sum_{n=1}^{\infty}\frac{\prod_{r=1}^{n}(a+2r-\frac{3}{2})}{(2n)!}\,x^{2n}\right] \\ = y_3(x,a) &= e^{x^2/4}\left[1+\sum_{n=1}^{\infty}\frac{\prod_{r=1}^{n}(-a+2r-\frac{3}{2})}{(2n)!}\,(-x^2)^n\right] \\ &= e^{x^2/4}\left[1+\sum_{n=1}^{\infty}\frac{\prod_{r=1}^{n}(a+\frac{3}{2}-2r)}{(2n)!}\,x^{2n}\right]. \end{aligned}$$

Now, choosing $a=0$ and writing $\prod_{r=1}^{n}(2r-\frac{3}{2})$ as b_n reduces the equality of the first and third lines to

$$e^{-x^2/4}\left[1+\sum_{n=1}^{\infty}\frac{b_n}{(2n)!}\,x^{2n}\right] = e^{x^2/4}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\,b_n}{(2n)!}\,x^{2n}\right]$$

from which the stated result follows immediately.

19

Quantum operators

> ***19.2*** *By expressing the operator L_z, corresponding to the z-component of angular momentum, in spherical polar coordinates (r, θ, ϕ), show that the angular momentum of a particle about the polar axis cannot be known at the same time as its azimuthal position around that axis.*

The expression for L_z in Cartesian coordinates is

$$L_z = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right),$$

the connections with spherical polar coordinates being

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$
$$r^2 = x^2 + y^2 + z^2, \quad \theta = \tan^{-1}\frac{(x^2+y^2)^{1/2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x}.$$

Using the chain rule, we have

$$\begin{aligned}\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \\ &= \frac{y}{r}\frac{\partial}{\partial r} + \frac{zy}{r^2(x^2+y^2)^{1/2}}\frac{\partial}{\partial \theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \phi}.\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial x} = \frac{x}{r}\frac{\partial}{\partial r} + \frac{zx}{r^2(x^2+y^2)^{1/2}}\frac{\partial}{\partial \theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \phi}.$$

Thus,

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{xy - yx}{r}\frac{\partial}{\partial r} + \frac{z(xy-yx)}{r^2(x^2+y^2)^{1/2}}\frac{\partial}{\partial \theta} + \frac{x^2+y^2}{x^2+y^2}\frac{\partial}{\partial \phi} = \frac{\partial}{\partial \phi}.$$

Thus, expressed in spherical polar coordinates, $L_z = -i\hbar\partial/\partial\phi$. To establish a relationship between the uncertainties, ΔL_z and $\Delta\phi$, in the z-component of the angular momentum and angular position about the z-axis, we need to evaluate the commutator of L_z and ϕ. This is done by considering

$$[\,L_z, \phi\,]\,|\psi\rangle = -i\hbar\frac{\partial}{\partial\phi}(\phi|\psi\rangle) + i\hbar\phi\frac{\partial}{\partial\phi}|\psi\rangle = -i\hbar|\psi\rangle,$$

i.e. $[\,L_z, \phi\,] = -i\hbar$. Since the commutator is a non-zero constant, comparison with the case of $[\,p_x, x\,]$ shows that $\Delta L_z \times \Delta\phi \geq \frac{1}{2}\hbar$, whatever the state $|\psi\rangle$ of the system. That is, if the value of the z-component of the angular momentum is known ($\Delta L_z = 0$), the value of ϕ (considered as a multivalued function) is completely unknown.

[The transformation of the Cartesian expression for L_z is very much simpler if cylindrical, rather than spherical, polar coordinates are used, as the reader may wish to verify. The result is the same, as it must be, since ϕ has the same meaning in both systems of coordinates.]

19.4 *Show that the Pauli matrices*

$$\mathsf{S}_x = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{S}_y = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathsf{S}_z = \tfrac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which are used as the operators corresponding to intrinsic spin of $\frac{1}{2}\hbar$ *in non-relativistic quantum mechanics, satisfy* $\mathsf{S}_x^2 = \mathsf{S}_y^2 = \mathsf{S}_z^2 = \frac{1}{4}\hbar^2\mathsf{I}$ *and have the same commutation properties as the components of orbital angular momentum. Deduce that any state* $|\psi\rangle$ *represented by the column vector* $(a,\ b)^{\mathrm{T}}$ *is an eigenstate of* S^2 *with eigenvalue* $3\hbar^2/4$.

We note that all the S_j are Hermitian and evaluate their various possible products.

$$\mathsf{S}_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4}\,\mathsf{I},$$

$$\mathsf{S}_x\mathsf{S}_y = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{i\hbar}{2}\,\mathsf{S}_z,$$

$$\mathsf{S}_y\mathsf{S}_x = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\frac{i\hbar}{2}\,\mathsf{S}_z.$$

Similarly, $\mathsf{S}_y^2 = \mathsf{S}_z^2 = \frac{1}{4}\hbar^2\mathsf{I}$ and

$$\mathsf{S}_y\mathsf{S}_z = \frac{i\hbar}{2}\mathsf{S}_x = -\mathsf{S}_z\mathsf{S}_y,$$

$$\mathsf{S}_z\mathsf{S}_x = \frac{i\hbar}{2}\mathsf{S}_y = -\mathsf{S}_x\mathsf{S}_z.$$

Thus,

$$\left[\mathsf{S}_x, \mathsf{S}_y \right] = \mathsf{S}_x\mathsf{S}_y - \mathsf{S}_y\mathsf{S}_x = \frac{i\hbar}{2}\,\mathsf{S}_z - \left(-\frac{i\hbar}{2}\,\mathsf{S}_z\right) = i\hbar\mathsf{S}_z,$$

and similarly

$$\left[\mathsf{S}_y, \mathsf{S}_z \right] = i\hbar\mathsf{S}_x \quad \text{and} \quad \left[\mathsf{S}_z, \mathsf{S}_x \right] = i\hbar\mathsf{S}_y.$$

Thus the commutators have the same structure as those for L_x, L_y and L_z in equation (19.27).

Since $\mathsf{S}_x^2 = \mathsf{S}_y^2 = \mathsf{S}_z^2 = \frac{1}{4}\hbar^2\mathsf{I}$, and $\mathsf{S}^2 = \mathsf{S}_x^2 + \mathsf{S}_y^2 + \mathsf{S}_z^2 = \frac{3}{4}\hbar^2\mathsf{I}$,

$$\mathsf{S}^2 \begin{pmatrix} a \\ b \end{pmatrix} = \tfrac{3}{4}\hbar^2\mathsf{I} \begin{pmatrix} a \\ b \end{pmatrix} = \tfrac{3}{4}\hbar^2 \begin{pmatrix} a \\ b \end{pmatrix}$$

for any a and b,i.e. any such state is an eigenstate of S^2 with eigenvalue $3\hbar^2/4$.

19.6 *Operators A and B anti-commute. Evaluate $(A+B)^{2n}$ for a few values of n and hence propose an expression for c_{nr} in the expansion*

$$(A+B)^{2n} = \sum_{r=0}^{n} c_{nr}\, A^{2n-2r}\, B^{2r}.$$

Prove your proposed formula for general values of n, using the method of induction.

Show that

$$\cos(A+B) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} d_{nr}\, A^{2n-2r} B^{2r},$$

where the d_{nr} are constants whose values you should determine.

By taking as A the matrix $\mathsf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, confirm that your answer is consistent with that obtained in exercise 19.5.

First a few trials, noting that $B^pA^q = (-1)^pAB^pA^{q-1} = \cdots = (-1)^{pq}A^qB^p$.

$n = 1$ $\quad (A+B)(A+B) = A^2+AB+BA+B^2 = A^2+B^2,$ since $AB = -BA$.

$n = 2$

$$\begin{aligned}(A+B)^4 &= (A^2+B^2)^2 \\ &= A^4 + A^2B^2 + B^2A^2 + B^4 \\ &= A^4 + A^2B^2 + (-1)^4A^2B^2 + B^4 \\ &= A^4 + 2A^2B^2 + B^4.\end{aligned}$$

$$n = 3 \qquad (A+B)^6 = (A^2+B^2)(A^4+2A^2B^2+B^4)$$
$$= A^6 + 2A^4B^2 + A^2B^4 + B^2A^4 + 2B^2A^2B^2 + B^6$$
$$= A^6 + 2A^4B^2 + A^2B^4 + (-1)^8A^4B^2 + 2(-1)^4A^2B^4 + B^6$$
$$= A^6 + 3A^4B^2 + 3A^2B^4 + B^6.$$

The obvious indication is that

$$(A+B)^{2n} = \sum_{r=0}^{n} {}^nC_r\, A^{2n-2r}\, B^{2r} \qquad (*).$$

To prove this result for general n, we assume that it is true for a particular value of n and consider $(A+B)^{2n+2}$

$$= (A^2+B^2)\sum_{r=0}^{n} {}^nC_r\, A^{2n-2r}\, B^{2r}$$
$$= \sum_{r=0}^{n} {}^nC_r\, A^{2n+2-2r}\, B^{2r} + \sum_{r=0}^{n} {}^nC_r\,(-1)^{4n-4r}\, A^{2n-2r}\, B^{2r+2}$$
$$= \sum_{r=0}^{n} {}^nC_r\, A^{2n+2-2r}\, B^{2r} + \sum_{s=1}^{n+1} {}^nC_{s-1}\, A^{2n-2s+2}\, B^{2s}, \text{ with } s = r+1,$$
$$= {}^nC_0A^{2n+2}B^0 + \sum_{r=1}^{n}\left[({}^nC_r + {}^nC_{r-1})\, A^{2n+2-2r}\, B^{2r}\right] + {}^nC_nA^0B^{2n+2}.$$

Now

$${}^nC_r + {}^nC_{r-1} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$
$$= \frac{(n+1)!}{(n+1-r)!\,r!} = {}^{n+1}C_r$$

whilst

$${}^nC_0 = 1 = {}^{n+1}C_0 \quad \text{and} \quad {}^nC_n = 1 = {}^{n+1}C_{n+1}.$$

Thus,

$$(A+B)^{2n+2} = {}^{n+1}C_0A^{2n+2}B^0 + \sum_{r=1}^{n} {}^{n+1}C_r\, A^{2n+2-2r}\, B^{2r} + {}^{n+1}C_{n+1}A^0B^{2n+2}$$
$$= \sum_{r=0}^{n+1} {}^{n+1}C_r\, A^{2n+2-2r}\, B^{2r},$$

i.e. the same form as $(*)$ but with $n \to n+1$, thus proving the form for general n, since it has already been shown to be valid for $n = 1$.

We calculate the cosine function from its defining series

$$\cos(A+B) = \sum_{n=0}^{\infty}(-1)^n \frac{(A+B)^{2n}}{(2n)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{(2n)!} \sum_{r=0}^{n} \frac{A^{2n-2r}\, B^{2r}}{(n-r)!\, r!}.$$

Thus

$$d_{nr} = \frac{(-1)^n\, n!}{(2n)!\,(n-r)!\,r!}.$$

Since in exercise 19.5 $\mathsf{C} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, in order to use the given form of A we must take

$$\mathsf{B} = \mathsf{C} - \mathsf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now,

$$\mathsf{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathsf{BA};$$

so they do anticommute, and we can apply our previous result. But

$$\mathsf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\text{and} \quad \mathsf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that

$$\cos \mathsf{C} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^n\, n!}{(2n)!\,(n-r)!\,r!}\, \mathsf{I}^{n-r}\, \mathsf{I}^r.$$

But

$$\sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} = (1+1)^n = 2^n,$$

and so

$$\cos \mathsf{C} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n}{(2n)!}\, \mathsf{I} = (\cos\sqrt{2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

as in exercise 19.5.

19.8 *For a system of N electrons in their ground state $|0\rangle$, the Hamiltonian is*

$$H = \sum_{n=1}^{N} \frac{p_{x_n}^2 + p_{y_n}^2 + p_{z_n}^2}{2m} + \sum_{n=1}^{N} V(x_n, y_n, z_n).$$

Show that $\left[p_{x_n}^2, x_n \right] = -2i\hbar p_{x_n}$, and hence that the expectation value of the double commutator $[\,[\,x, H\,]\,, x\,]$, where $x = \sum_{n=1}^{N} x_n$ is given by

$$\langle 0 \,|\, [\,[\,x, H\,]\,, x\,] \,|\, 0 \rangle = \frac{N\hbar^2}{m}.$$

Now evaluate the expectation value using the eigenvalue properties of H, namely $H|r\rangle = E_r|r\rangle$, and deduce the sum rule for oscillation strengths*,*

$$\sum_{r=0}^{\infty} (E_r - E_0) |\, \langle r \,|\, x \,|\, 0 \rangle \,|^2 = \frac{N\hbar^2}{2m}.$$

First we evaluate the commutator

$$\begin{aligned} \left[p_{x_n}^2, x_n \right] &= p_{x_n} \left[p_{x_n}, x \right] + \left[p_{x_n}, x \right] p_{x_n} \\ &= p_{x_n}(-i\hbar) + (-i\hbar) p_{x_n} = -2i\hbar p_{x_n}. \end{aligned}$$

Now, all variables with differing values of n, or referring to different coordinate directions even if n is the same, commute with each other whilst each x_m commutes with $V(x_n, y_n, z_n)$. Consequently the only non-zero terms in the commutator $[\,x, H\,]$ are terms like $\left[x_n, p_{x_n}^2/2m \right]$ which, as shown above, have the values $i\hbar p_{x_n}/m$. Thus,

$$D \equiv \langle 0 \,|\, [\,[\,x, H\,]\,, x\,] \,|\, 0 \rangle = \langle 0 \,|\, \sum_{n=1}^{N} \frac{i\hbar}{m} \left[p_{x_n}, x_n \right] \,|\, 0 \rangle = \frac{i\hbar}{m} (-i\hbar) N = \frac{N\hbar^2}{m}.$$

We now evaluate D in a different way, making use of result (19.11):

$$\begin{aligned} D &= \langle 0 \,|\, (xH - Hx)x \,|\, 0 \rangle - \langle 0 \,|\, x(xH - Hx) \,|\, 0 \rangle \\ &= \sum_{r=0}^{\infty} \langle 0 \,|\, (xH - Hx) \,|\, r \rangle \langle r \,|\, x \,|\, 0 \rangle - \sum_{r=0}^{\infty} \langle 0 \,|\, x \,|\, r \rangle \langle r \,|\, (xH - Hx) \,|\, 0 \rangle \\ &= \sum_{r=0}^{\infty} \langle 0 \,|\, (xE_r - E_0 x) \,|\, r \rangle \langle r \,|\, x \,|\, 0 \rangle - \sum_{r=0}^{\infty} \langle 0 \,|\, x \,|\, r \rangle \langle r \,|\, (xE_0 - E_r x) \,|\, 0 \rangle \\ &= 2 \sum_{r=0}^{\infty} (E_r - E_0) \langle 0 \,|\, x \,|\, r \rangle \langle r \,|\, x \,|\, 0 \rangle = 2 \sum_{r=0}^{\infty} (E_r - E_0) |\langle r \,|\, x \,|\, 0 \rangle|^2 \end{aligned}$$

Equating the two expressions for D gives the stated result.

19.10 *For a system containing more than one particle, the total angular momentum* J *and its components are represented by operators that have completely analogous commutation relations to those for the operators for a single particle, i.e.* J^2 *has eigenvalue* $j(j+1)\hbar^2$ *and* J_z *has eigenvalue* $m_j\hbar$ *for the state* $|j, m_j\rangle$. *The usual orthonormality relationship* $\langle j', m'_j \,|\, j, m_j\rangle = \delta_{j'j}\,\delta_{m'_j m_j}$ *is also valid.*

A system consists of two (distinguishable) particles A *and* B. *Particle* A *is in an* $\ell = 3$ *state and can have state functions of the form* $|A, 3, m_A\rangle$ *whilst* B *is in an* $\ell = 2$ *state with possible state functions* $|B, 2, m_B\rangle$. *The range of possible values for* j *is* $|3-2| \le j \le |3+2|$, *i.e.* $1 \le j \le 5$, *and the overall state function can be written as*

$$|j, m_j\rangle = \sum_{m_A+m_B=m_j} C^{j\,m_j}_{m_A\,m_B}\,|\,A, 3, m_A\rangle\,|\,B, 2, m_B\rangle.$$

The numerical coefficients $C^{j\,m_j}_{m_A\,m_B}$ *are known as* Clebsch–Gordon *coefficients.*

Assume (as can be shown) that the ladder operators $U(AB)$ *and* $D(AB)$ *for the system can be written as* $U(A)+U(B)$ *and* $D(A)+D(B)$ *respectively and that they lead to relationships equivalent to (19.34) and (19.35) with* ℓ *replaced by* j *and* m *by* m_j.

(a) *Apply the operators to the (obvious) relationship*

$$|AB, 5, 5\rangle = |A, 3, 3\rangle\,|B, 2, 2\rangle$$

to show that

$$|AB, 5, 4\rangle = \sqrt{\tfrac{6}{10}}\,|A, 3, 2\rangle\,|B, 2, 2\rangle + \sqrt{\tfrac{4}{10}}\,|A, 3, 3\rangle\,|B, 2, 1\rangle.$$

(b) *Find, to within an overall sign, the real coefficients* c *and* d *in the expansion*

$$|AB, 4, 4\rangle = c|A, 3, 2\rangle\,|B, 2, 2\rangle + d|A, 3, 3\rangle\,|B, 2, 1\rangle$$

by requiring it to be orthogonal to $|AB, 5, 4\rangle$. *Check your answer by considering* $U(AB)|AB, 4, 4\rangle$.

(c) *Find, to within an overall sign and as efficiently as possible, an expression for* $|AB, 4, -3\rangle$ *as a sum of products of the form* $|A, 3, m_A\rangle\,|B, 2, m_B\rangle$.

(a) We start with $|AB, 5, 5\rangle = |A, 3, 3\rangle\,|B, 2, 2\rangle$ and apply $D(AB) = D(A) + D(B)$ to both sides, yielding

$$\begin{aligned}\sqrt{(5)(6)-(5)(4)}\,|AB, 5, 4\rangle &= \sqrt{(3)(4)-(3)(2)}\,|A, 3, 2\rangle\,|B, 2, 2\rangle \\ &\quad + \sqrt{(2)(3)-(2)(1)}\,|A, 3, 3\rangle\,|B, 2, 1\rangle \\ |AB, 5, 4\rangle &= \sqrt{\tfrac{6}{10}}\,|A, 3, 2\rangle\,|B, 2, 2\rangle + \sqrt{\tfrac{4}{10}}\,|A, 3, 3\rangle\,|B, 2, 1\rangle.\end{aligned}$$

(b) Since

$$|AB,4,4\rangle = c|A,3,2\rangle\,|B,2,2\rangle + d|A,3,3\rangle\,|B,2,1\rangle$$

must be orthogonal to $|AB,5,4\rangle$, we have (remembering the orthonormality relation $\langle j', m_j' \,|\, j, m_j\rangle = \delta_{j'j}\,\delta_{m_j' m_j}$)

$$\begin{aligned} 0 &= \langle AB,5,4\,|\,AB,4,4\rangle \\ &= \sqrt{\tfrac{6}{10}}\,c(1)(1) + \sqrt{\tfrac{6}{10}}\,d(0)(0) + \sqrt{\tfrac{4}{10}}\,c(0)(0) + \sqrt{\tfrac{4}{10}}\,d(1)(1). \end{aligned}$$

It must also be normalised, and so

$$1 = \langle AB,4,4\,|\,AB,4,4\rangle = c^2(1)(1) + cd(0)(0) + dc(0)(0) + d^2(1)(1).$$

Thus $c = \pm(4/10)^{1/2}$ and $d = \mp(6/10)^{1/2}$.

As a check, consider

$$\begin{aligned} &U(AB)|AB,4,4\rangle \\ &= [\,U(A) + U(B)\,]\left[\sqrt{\tfrac{4}{10}}\,|A,3,2\rangle\,|B,2,2\rangle - \sqrt{\tfrac{6}{10}}\,|A,3,3\rangle\,|B,2,1\rangle\right] \\ &= \sqrt{\tfrac{4}{10}}\left(\sqrt{(3)(4)-(2)(3)}|A,3,3\rangle\,|B,2,2\rangle + |\emptyset\rangle\right) \\ &\qquad - \sqrt{\tfrac{6}{10}}\left(|\emptyset\rangle + \sqrt{(2)(3)-(1)(2)}|A,3,3\rangle\,|B,2,2\rangle\right) \\ &= \left(\sqrt{\tfrac{4}{10}}\,\sqrt{6} - \sqrt{\tfrac{6}{10}}\,\sqrt{4}\right)|A,3,3\rangle\,|B,2,2\rangle \\ &= |\emptyset\rangle, \text{ as it should.} \end{aligned}$$

(c) We abbreviate our notation from $|A,3,m_A\rangle\,|B,2,m_B\rangle$ to $|m_A\rangle\,|m_B\rangle$, in that order.
We start with the known relationship that is 'closest' to $|AB,4,-3\rangle$, namely

$$|AB,5,-5\rangle = |-3\rangle\,|-2\rangle,$$

and apply $U(AB)$ to both sides, obtaining

$$\begin{aligned} \sqrt{(5)(6)-(-5)(-4)}|AB,5,-4\rangle &= \sqrt{(3)(4)-(-3)(-2)}\,|-2\rangle\,|-2\rangle \\ &\quad + \sqrt{(2)(3)-(-2)(-1)}\,|-3\rangle\,|-1\rangle, \\ |AB,5,-4\rangle &= \sqrt{\tfrac{6}{10}}\,|-2\rangle\,|-2\rangle + \sqrt{\tfrac{4}{10}}\,|-3\rangle\,|-1\rangle. \end{aligned}$$

The state $|AB,4,4\rangle$ must be orthogonal to this, but consist of a different linear combination of the same two states. As it must be normalised it can only be

$$|AB,4,-4\rangle = \sqrt{\tfrac{4}{10}}\,|-2\rangle\,|-2\rangle - \sqrt{\tfrac{6}{10}}\,|-3\rangle\,|-1\rangle.$$

Now use $U(AB)$ again on both sides:

$$\begin{aligned}
&\sqrt{(4)(5)-(-4)(-3)}\,|AB,4,-3\rangle \\
&= \sqrt{\tfrac{4}{10}}\left(\sqrt{(3)(4)-(-2)(-1)}\,|-1\rangle\,|-2\rangle\right. \\
&\qquad\left. + \sqrt{(2)(3)-(-2)(-1)}\,|-2\rangle\,|-1\rangle\right) \\
&\quad - \sqrt{\tfrac{6}{10}}\left(\sqrt{(3)(4)-(-3)(-2)}\,|-2\rangle\,|-1\rangle\right. \\
&\qquad\left. + \sqrt{(2)(3)-(-1)(0)}\,|-3\rangle\,|0\rangle\right).
\end{aligned}$$

Simplifying the RHS of this equation and dividing through by $\sqrt{8}$ then gives

$$\begin{aligned}
|AB,4,-3\rangle &= \sqrt{\tfrac{1}{2}}\;|-1\rangle\,|-2\rangle + \left(\sqrt{\tfrac{1}{5}} - \sqrt{\tfrac{9}{20}}\right)|-2\rangle\,|-1\rangle \\
&\qquad - \sqrt{\tfrac{9}{20}}\;|-3\rangle\,|0\rangle \\
&= \sqrt{\tfrac{1}{2}}\;|-1\rangle\,|-2\rangle - \sqrt{\tfrac{1}{20}}\;|-2\rangle\,|-1\rangle - \sqrt{\tfrac{9}{20}}\;|-3\rangle\,|0\rangle.
\end{aligned}$$

This is the required expansion and, as expected, it is automatically normalised:

$$\left(\sqrt{\tfrac{1}{2}}\right)^2 + \left(-\sqrt{\tfrac{1}{20}}\right)^2 + \left(-\sqrt{\tfrac{9}{20}}\right)^2 = 1.$$

20

Partial differential equations: general and particular solutions

20.2 *Find partial differential equations satisfied by the following functions $u(x, y)$ for all arbitrary functions f and all arbitrary constants a and b:*

(a) $u(x, y) = f(x^2 - y^2)$;
(b) $u(x, y) = (x - a)^2 + (y - b)^2$;
(c) $u(x, y) = y^n f(y/x)$;
(d) $u(x, y) = f(x + ay)$.

In each case we need to generate enough partial derivatives of u that the arbitrary functions and constants can be eliminated by re-substitution.

(a) $\dfrac{\partial u}{\partial x} = 2xf', \quad \dfrac{\partial u}{\partial y} = -2yf' \quad \Rightarrow \quad y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0.$

(b) $\dfrac{\partial u}{\partial x} = 2(x - a), \quad \dfrac{\partial u}{\partial y} = 2(y - b) \quad \Rightarrow \quad \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = 4u.$

(c)

$$\begin{aligned} u(x, y) &= y^n f\left(\frac{y}{x}\right), \\ \frac{\partial u}{\partial x} &= y^n f' \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y^{n+1}}{x^2} f', \\ \frac{\partial u}{\partial y} &= ny^{n-1} f + y^n f' \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = ny^{n-1} f + \frac{y^n}{x} f'. \end{aligned}$$

Substituting for f and f' from the first two equations into the third one gives

$$\frac{\partial u}{\partial y} = n\frac{u}{y} - \frac{x}{y}\frac{\partial u}{\partial x},$$

which can be rearranged as

$$\Rightarrow \quad y\frac{\partial u}{\partial y} + x\frac{\partial u}{\partial x} = nu.$$

(d) Since both the constant a and the form of the function f are to be eliminated, second partial derivatives will be required.

$$\frac{\partial u}{\partial x} = f' \quad \text{and} \quad \frac{\partial u}{\partial y} = af' \quad \Rightarrow \quad \frac{\partial u}{\partial y} = a\frac{\partial u}{\partial x}.$$

Differentiating again with respect to x, say, and then eliminating a between the two PDEs gives

$$\frac{\partial^2 u}{\partial x \partial y} = a\frac{\partial^2 u}{\partial x^2} \text{ and hence } \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y}$$

If the second partial derivative is taken with respect to y (rather than x) the equivalent result is

$$\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}.$$

20.4 *Find the most general solutions $u(x, y)$ of the following equations, consistent with the boundary conditions stated:*

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0, \quad u(x, 0) = 1 + \sin x;$

(b) $i\dfrac{\partial u}{\partial x} = 3\dfrac{\partial u}{\partial y}, \quad u = (4 + 3i)x^2$ *on the line* $x = y;$

(c) $\sin x \sin y\dfrac{\partial u}{\partial x} + \cos x \cos y\dfrac{\partial u}{\partial y} = 0, \quad u = \cos 2y$ *on* $x + y = \pi/2;$

(d) $\dfrac{\partial u}{\partial x} + 2x\dfrac{\partial u}{\partial y} = 0, \quad u = 2$ *on the parabola* $y = x^2.$

In each case, we need to determine a $p = p(x, y)$ such that the solution for general x and y is $u(x, y) = f(p)$. The form of p will be determined by the PDE and that of f by the (given) form that u takes on the relevant boundary.

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad \dfrac{dx}{y} = -\dfrac{dy}{x} \quad \Rightarrow \quad x^2 + y^2 = p.$

The given boundary is $y = 0$ and on this line the expression for p is $p = x^2$. For this to match the given form, $1 + \sin x$, the form of $f(p)$ must be

$$f(p) = 1 + \sin p^{1/2}.$$

This then determines the form of $u(x, y) = f(p)$ for all x and y, not just for $y = 0$ and general x:

$$u(x, y) = 1 + \sin[(x^2 + y^2)^{1/2}].$$

The remaining parts of this exercise are tackled in an analogous way and are given with little commentary.

(b) $i\dfrac{\partial u}{\partial x} = 3\dfrac{\partial u}{\partial y} \quad \Rightarrow \quad \dfrac{dx}{i} = -\dfrac{dy}{3} \quad \Rightarrow \quad 3x+iy = p.$

On $x = y$, $p = (3 + i)x$ and

$$\begin{aligned} u(x, x) = (4 + 3i)x^2 \quad &\Rightarrow \quad f(p) = \alpha p^2, \\ \text{where } \alpha(3 + i)^2 \quad &= \quad 4 + 3i, \\ \alpha[(9 - 1) + 6i] \quad &= \quad 4 + 3i \quad \Rightarrow \quad \alpha = \tfrac{1}{2}, \\ \Rightarrow \quad u(x, y) \quad &= \quad \tfrac{1}{2}p^2 = \tfrac{1}{2}(9x^2 + 6ixy - y^2). \end{aligned}$$

(c) For $\sin x \sin y\dfrac{\partial u}{\partial x} + \cos x \cos y\dfrac{\partial u}{\partial y} = 0, \quad u = \cos 2y$ on $x + y = \pi/2$,

$$\begin{aligned} \frac{dx}{\sin x \sin y} &= \frac{dy}{\cos x \cos y}, \\ \frac{\cos x\, dx}{\sin x} &= \frac{\sin y\, dy}{\cos y}, \\ \ln(\sin x) &= -\ln(\cos y) + k, \\ \sin x \cos y &= p. \end{aligned}$$

On $x + y = \frac{1}{2}\pi$, $u(x, y) = \cos 2y$ and

$$\begin{aligned} p &= \sin(\tfrac{1}{2}\pi - y)\cos y = \cos^2 y, \\ f(p) &= \cos 2y = 2\cos^2 y - 1 = 2p - 1, \\ u(x, y) &= 2p - 1 = 2\sin x \cos y - 1. \end{aligned}$$

(d) For $\dfrac{\partial u}{\partial x} + 2x\dfrac{\partial u}{\partial y} = 0$ with $u(x, y) = 2$ on $y = x^2$,

$$\begin{aligned} \frac{dx}{1} &= \frac{dy}{2x}, \\ x^2 - y &= p. \end{aligned}$$

On $y = x^2$, $p = 0$ and $f(p) = g(p) + 2$, where $g(p)$ is *any* function for which $g(0) = 0$. The general solution is $u(x, y) = g(y - x^2) + 2$; this indeterminacy is related to the boundary curve being a characteristic of the PDE.

20.6 *Find the most general solutions $u(x, y)$ of the following equations consistent with the boundary conditions stated:*

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u = x^2$ *on the line* $y = 0$;

(b) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u(1, 0) = 2$;

(c) $y^2\dfrac{\partial u}{\partial x} + x^2\dfrac{\partial u}{\partial y} = x^2y^2(x^3 + y^3), \quad$ *no boundary conditions.*

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u = x^2$ on the line $y = 0$.

The CF is given by

$$\frac{dx}{y} = -\frac{dy}{x} \quad \Rightarrow \quad x^2 + y^2 = p.$$

An obvious PI is $u(x, y) = -3y$ and, as *any* valid PI will do, the general solution is

$$u(x, y) = f(x^2 + y^2) - 3y.$$

On $y = 0$, $p = x^2$ and

$$x^2 = u(x, 0) = p \quad \Rightarrow \quad f(p) = p \quad \Rightarrow \quad u(x, y) = p - 3y = x^2 + y^2 - 3y.$$

(b) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u(1, 0) = 2$.

As in part (a), the general solution is $u(x, y) = f(x^2 + y^2) - 3y$.

At $(1, 0)$, $p = 1$ and we require $2 = u(1, 0) = f(1) - 0$. Thus,

$$f(x^2 + y^2) = 2 + g(x^2 + y^2) \text{ where } g(1) = 0.$$

Thus the most general solution consistent with the (one-point) boundary condition is

$$\begin{aligned} u(x, y) &= 2 - 3y + g(x^2 + y^2) \\ &\text{or } 2 - 3y + (x^2 + y^2 - 1) + h(x^2 + y^2) \\ &\text{or } 2 - 3y + \sin[(x^2 + y^2)\pi] + j(x^2 + y^2) \\ &\text{or } \ldots, \end{aligned}$$

where any arbitrary function not written explicitly has value 0 when its argument has value 1.

(c) For $y^2\dfrac{\partial u}{\partial x} + x^2\dfrac{\partial u}{\partial y} = x^2y^2(x^3 + y^3)$ with no boundary conditions.

The CF is found from

$$\frac{dx}{y^2} = \frac{dy}{x^2} \quad \Rightarrow \quad x^3 - y^3 = p.$$

From the symmetry between x and y in the equation, we are led to try $u(x, y) = \alpha(x^n + y^n)$ for some n and α as a possible PI. Substituting this trial solution:

$$\alpha n y^2 x^{n-1} + \alpha n x^2 y^{n-1} = y^2 x^5 + x^2 y^5.$$

This is satisfied if $n = 6$ and $\alpha = 1/6$. Thus

$$u(x, y) = \tfrac{1}{6}(x^6 + y^6) + f(x^3 - y^3),$$

where $f(p)$ is any function of p. As there are no boundary conditions to be satisfied, there are no restrictions on the form of f, provided it is differentiable.

20.8 *A function $u(x, y)$ satisfies*

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 10,$$

and takes the value 3 on the line $y = 4x$. Evaluate $u(2, 4)$.

To find the CF we set

$$\frac{dx}{2} = \frac{dy}{3} \quad \Rightarrow \quad 3x - 2y = p.$$

An elementary PI, obvious from inspection, is $u = 5x$. Consequently the general solution is $u(x, y) = f(p) + 5x$.

On the line $y = 4x$, we have $p = 3x - 2(4x) = -5x$ and so

$$3 = u(x, 4x) = f(-5x) + 5x = f(-5x) - (-5x) \quad \Rightarrow \quad f(p) = p + 3.$$

This gives the form of $f(p) = u(x, y)$ everywhere, not just on the line $y = 4x$, and so re-expressing it in terms of x and y shows that

$$u(x, y) = f(3x - 2y) + 5x = 3x - 2y + 3 + 5x = 8x - 2y + 3.$$

We can now compute $u(2, 4)$ as

$$u(2, 4) = 16 - 8 + 3 = 11.$$

20.10 *Consider the partial differential equation*

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0. \qquad (*)$$

(a) *Find the function $u(x, y)$ that satisfies $(*)$ and the boundary condition $u = \partial u/\partial y = 1$ when $y = 0$ for all x. Evaluate $u(0, 1)$.*

(b) *In which region of the xy-plane would u be determined if the boundary condition were $u = \partial u/\partial y = 1$ when $y = 0$ for all $x > 0$?*

(a) For solutions of the form $u(x, y) = f(x + \lambda y)$, λ must satisfy

$$1 - 3\lambda + 2\lambda^2 = 0 \quad \Rightarrow \quad \lambda = \tfrac{1}{2},\ 1.$$

Thus the general solution is $u(x, y) = g(x + \frac{1}{2}y) + f(x + y) \equiv g(p_1) + f(p_2)$.

With the boundary conditions $u = \dfrac{\partial u}{\partial y} = 1$ for $y = 0$ and all x, $p_1 = p_2 = x$ on the boundary and

$$1 = u(x, 0) = g(x) + f(x), \qquad (*)$$

$$1 = \frac{\partial u}{\partial y}(x, 0) = \tfrac{1}{2}g'(x) + f'(x).$$

From $(*)$, $\quad 0 = g'(x) + f'(x).$

Subtracting, $\quad 1 = -\frac{1}{2}g'(x).$

Integrating, $\quad g(x) = -2x + k \quad \Rightarrow \quad f(x) = 2x - k + 1$, from $(*)$.

Hence,

$$u(x, y) = -2(x + \tfrac{1}{2}y) + k + 2(x + y) - k + 1$$
$$= y + 1$$
$$\Rightarrow \quad u(0, 1) = 2.$$

(b) For $u = \dfrac{\partial u}{\partial y} = 1$ for $y = 0$ and $x > 0$, the validity of the solution obtained in part (a) is restricted to the region whose characteristic curves intersect the *positive* x-axis (as opposed to the *whole* x-axis). The characteristic curves in this case are the families of straight lines

$$x + \tfrac{1}{2}y = p_1 \qquad \text{and} \qquad x + y = p_2.$$

For both families, the lowest value of p_i for which the curve cuts the positive x-axis is 0. [For negative values of p_i the curves cut the negative x-axis.] The common slope of the first family is -2 and for the second family it is -1. The two lines with these slopes that pass through the origin determine the limit of the region of validity of the solution (both constraints must be satisfied). In terms of the conventional angle θ measured from the positive x-axis, $-\frac{1}{4}\pi < \theta < \frac{1}{2}\pi + \phi$, where $\tan\phi = 2$. A rough sketch of typical characteristics will probably be found helpful.

20.12 *Solve*

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 14,$$

subject to $u = 2x + 1$ *and* $\partial u/\partial y = 4 - 6x$, *both on the line* $y = 0$.

For solutions of the form $u(x, y) = f(x + \lambda y)$ we require

$$6 - 5\lambda + \lambda^2 = 0 \quad \Rightarrow \quad \lambda = 2,\ 3.$$

One possible (trivial) PI is $u(x, y) = 7y^2$, making the general solution

$$u(x, y) = f(x + 2y) + g(x + 3y) + 7y^2.$$

Imposing the given boundary conditions

$$2x + 1 = u(x, 0) = f(x) + g(x), \qquad (*)$$
$$4 - 6x = \frac{\partial u}{\partial y}(x, 0) = 2f'(x) + 3g'(x),$$

Differentiating $(*)$ gives

$$2 = f' + g',$$

Eliminating f' from these two equations yields

$$-6x = g'(x),$$

from which it follows that

$$g(x) = -3x^2 + k$$
$$\Rightarrow \quad f(x) = 2x + 1 + 3x^2 - k.$$

Thus, the solution for general x and y is

$$u(x, y) = 2(x + 2y) + 1 + 3(x + 2y)^2 - k - 3(x + 3y)^2 + k + 7y^2$$
$$= -8y^2 - 6xy + 2x + 4y + 1.$$

It can be verified by re-substitution into the initial equation and checking the boundary conditions.

20.14 *Solve*

$$\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = x(2y + 3x).$$

For the homogeneous equation to have solutions of the form $u(x, y) = f(x + \lambda y)$ we require

$$\lambda + 3\lambda^2 = 0 \quad \Rightarrow \quad \lambda = 0, -3 \quad \Rightarrow \quad u(x, y) = f(x - 3y) + g(x).$$

This is the CF part of the solution.

For a PI we try $u(x, y) = Ax^m y^n$:

$$Amnx^{m-1}y^{n-1} + 3An(n-1)x^m y^{n-2} = 2xy + 3x^2.$$

Such an equation is not guaranteed to have a consistent solution for m and n, but in this case it has; it is satisfied by $m = 2$, $n = 2$ and $A = \frac{1}{2}$. The general solution is, therefore,

$$u(x, y) = f(x - 3y) + g(x) + \tfrac{1}{2}x^2 y^2.$$

20.16 *An infinitely long string on which waves travel at speed c has an initial displacement*

$$y(x) = \begin{cases} \sin(\pi x/a), & -a \le x \le a, \\ 0, & |x| > a. \end{cases}$$

It is released from rest at time $t = 0$, and its subsequent displacement is described by $y(x, t)$.

By expressing the initial displacement as one explicit function incorporating Heaviside step functions, find an expression for $y(x, t)$ at a general time $t > 0$. In particular, determine the displacement as a function of time (a) *at $x = 0$,* (b) *at $x = a$, and* (c) *at $x = a/2$.*

The solution of the wave equation at a general time t can be expressed in terms of the initial displacement $\phi(x)$ by making the substitution

$$\phi(x) \to \tfrac{1}{2}[\,\phi(x - ct) + \phi(x + ct)\,]$$

and adding an integral of the initial velocity profile. In the present case there is no initial velocity and the integral contributes nothing.

The initial displacement profile, described piece-wise in the question, can be written as a single function of x by incorporating Heaviside functions as follows:

$$\phi(x) = \sin\left(\frac{\pi x}{a}\right)[\,H(x + a) - H(x - a)\,].$$

Crudely speaking, this formalism 'turns on' the sine function at $x = -a$ and turns it off again at $x = a$. It is only when x is between these limits that the expression in square brackets is non-zero.

Now making the substitution described above we obtain for a general time t that

$$\begin{aligned} y(x,t) &= \frac{1}{2}\sin\left(\frac{\pi(x-ct)}{a}\right)[H(x-ct+a)-H(x-ct-a)] \\ &\quad + \frac{1}{2}\sin\left(\frac{\pi(x+ct)}{a}\right)[H(x+ct+a)-H(x+ct-a)] \\ &= \frac{1}{2}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi ct}{a}\right)[H(x-ct+a)-H(x-ct-a) \\ &\qquad\qquad + H(x+ct+a)-H(x+ct-a)] \\ &\quad + \frac{1}{2}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi ct}{a}\right)[H(x+ct+a)-H(x+ct-a) \\ &\qquad\qquad - H(x-ct+a)+H(x-ct-a)]. \end{aligned}$$

Although this final expression is lengthy, its evaluation is fairly straightforward.

(a) At $x = 0$ the first term is zero for all t and the second contains the factor

$$[H(ct+a)-H(ct-a)-H(-ct+a)+H(-ct-a)] = [1-H(ct-a)-H(-ct+a)+0].$$

Whatever the sign of $ct - a$, one of the middle two terms in this bracket is -1 and the other is 0. Thus the bracket has total value 0 and the displacement is zero at all times.

(b) At $x = a$ the first term in $y(x,t)$ is zero for all t and the second contains the factor

$$[H(2a+ct)-H(ct)-H(2a-ct)+H(-ct)].$$

The first term in this is $+1$, the second -1 and the last is 0; the result therefore depends solely on whether or not $ct > 2a$. If it is, there is no displacement. If $0 \le ct \le 2a$ then the displacement is

$$\frac{1}{2}\cos\pi\sin\left(\frac{\pi ct}{a}\right)(-1) = \frac{1}{2}\sin\left(\frac{\pi ct}{a}\right).$$

(c) At $x = \frac{1}{2}a$ the second term in $y(x,t)$ is zero for all t; the first term contains the factor

$$[H(\frac{3}{2}a-ct)-H(-ct-\frac{1}{2}a)+H(\frac{3}{2}a+ct)-H(ct-\frac{1}{2}a)].$$

For $0 < 2ct < a$ this factor has the value $1-0+1-0=2$.
For $a < 2ct < 3a$ it has the value $1-0+1-1=1$.
For $3a < 2ct < \infty$ the bracket has the value $0-0+1-1=0$.

In summary the displacement at this value of x is $\cos(\pi ct/a)$ for $0 \le t \le a/2c$, $\frac{1}{2}\cos(\pi ct/a)$ for $a/2c \le t \le 3a/2c$, and 0 otherwise.

20.18 *Like the Schrödinger equation, the equation describing the transverse vibrations of a rod,*

$$a^4 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0,$$

has different orders of derivatives in its various terms. Show, however, that it has solutions of exponential form $u(x, t) = A \exp(\lambda x + i\omega t)$ *provided that the relation* $a^4\lambda^4 = \omega^2$ *is satisfied.*

Use a linear combination of such allowed solutions, expressed as the sum of sinusoids and hyperbolic sinusoids of λx*, to describe the transverse vibrations of a rod of length L clamped at both ends. At a clamped point both u and* $\partial u/\partial x$ *must vanish; show that this implies that* $\cos(\lambda L)\cosh(\lambda L) = 1$*, thus determining the frequencies* ω *at which the rod can vibrate.*

Direct substitution of $u(x, t) = A \exp(\lambda x + i\omega t)$ yields immediately that

$$a^4\lambda^4 u(x, t) + (i\omega)^2 u(x, t) = 0 \quad \Rightarrow \quad a^4\lambda^4 - \omega^2 = 0.$$

This gives $\pm\sqrt{\omega}/a$ and $\pm i\sqrt{\omega}/a$ as the four possible values of λ corresponding to any particular frequency ω. The four solutions were obtained as exponential functions, but we may work with any four independent linear combinations of them; for our purposes the four sinusoidal and hyperbolic sinusoids form a convenient set.

At each of the clamped ends, we apply both of the stated boundary conditions to a general expression for the (maximum) transverse displacement of the form

$$u(x) = A \sin \lambda x + B \cos \lambda x + C \sinh \lambda x + D \cosh \lambda x,$$

with

$$u'(x) = \lambda(A \cos \lambda x - B \sin \lambda x + C \cosh \lambda x + D \sinh \lambda x).$$

The four conditions will be enough to determine the four initially unknown constants, A, B, C and D.

At the $x = 0$ end of the rod:

$$u(0) = 0 \quad \Rightarrow \quad D = -B,$$

$$u'(0) = 0 \quad \Rightarrow \quad C = -A.$$

Hence,

$$u(x) = A(\sin \lambda x - \sinh \lambda x) + B(\cos \lambda x - \cosh \lambda x),$$

$$u'(x) = \lambda A(\cos \lambda x - \cosh \lambda x) + \lambda B(-\sin \lambda x - \sinh \lambda x).$$

Now, writing $\lambda L = \theta$, we have from the conditions at the other end $x = L$ that

$$A(\sin\theta - \sinh\theta) + B(\cos\theta - \cosh\theta) = 0,$$
$$\lambda A(\cos\theta - \cosh\theta) + \lambda B(-\sin\theta - \sinh\theta) = 0.$$

For consistency, $$(\cos\theta - \cosh\theta)^2 + (\sin^2\theta - \sinh^2\theta) = 0,$$
$$\cos^2\theta - 2\cos\theta\cosh\theta + \cosh^2\theta + \sin^2\theta - \sinh^2\theta = 0,$$
$$2 - 2\cos\theta\cosh\theta = 0,$$

i.e. $\cos(\lambda L)\cosh(\lambda L) = 1$. For a given value of L this gives the spectrum of values of λ, and hence of ω, at which the rod can undergo free vibrations.

20.20 *A sheet of material of thickness w, specific heat capacity c and thermal conductivity k is isolated in a vacuum, but its two sides are exposed to fluxes of radiant heat of strengths J_1 and J_2. Ignoring short-term transients, show that the temperature difference between its two surfaces is steady at $(J_2 - J_1)w/2k$, whilst their average temperature increases at a rate $(J_2 + J_1)/cw$.*

As the short-term transients are being ignored, we need a solution of the diffusion equation, i.e. a solution of $k\partial^2 u/\partial x^2 = c\partial u/\partial t$, that does not involve time-dependent decaying exponentials. The required solution has the form

$$u(x,t) = \frac{\alpha c}{2k}x^2 + gx + \alpha t + \beta,$$

where $x = 0$ is one of the surfaces and g, α and β are constants to be determined. At each surface, the rate at which heat arrives must equal that at which it is carried into the material by the temperature gradient there. So

$$\text{at } x = 0,\ -k\frac{\partial u}{\partial x} = J_1 \quad \Rightarrow \quad -\alpha c\, 0 - kg = J_1,$$
$$\text{at } x = w,\ -k\frac{\partial u}{\partial x} = -J_2 \quad \Rightarrow \quad -\alpha c\, w - kg = -J_2,$$

leading to $\alpha cw = J_1 + J_2$ and $g = -J_1/k$. Thus

$$u(x,t) = \frac{J_1 + J_2}{2kw}x^2 - \frac{J_1}{k}x + \frac{J_1 + J_2}{cw}t + \beta.$$

The temperature difference between the surfaces and the rate at which the average temperature rises are therefore given by

$$u(w,t) - u(0,t) = \frac{J_1 + J_2}{2kw}w^2 - \frac{J_1}{k}w = \frac{(J_2 - J_1)w}{2k},$$
$$\bar{u} = \frac{u(w,t) + u(0,t)}{2} = \frac{(J_1 + J_2)w}{4k} - \frac{J_1 w}{2k} + \frac{J_1 + J_2}{cw}t + \beta,$$
$$\frac{\partial \bar{u}}{\partial t} = 0 + \frac{J_1 + J_2}{cw} + 0.$$

20.22 *The daily and annual variations of temperature at the surface of the earth may be represented by sine-wave oscillations, with equal amplitudes and periods of 1 day and 365 days, respectively. Assume that for (angular) frequency* ω *the temperature at depth* x *in the earth is given by* $u(x,t) = A\sin(\omega t + \mu x)\exp(-\lambda x)$*, where* λ *and* μ *are constants.*

(a) *Use the diffusion equation to find the values of* λ *and* μ*.*
(b) *Find the ratio of the depths below the surface at which the two amplitudes have dropped to* $1/20$ *of their surface values.*
(c) *At what time of year is the soil coldest at the greater of these depths, assuming that the smoothed annual variation in temperature at the surface has a minimum on February 1st?*

(a) Substituting the form $u(x,t) = A\sin(\omega t + \mu x)\exp(-\lambda x)$ into the diffusion equation,

$$\kappa \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t},$$

gives

$$A\kappa\left[-\mu^2\sin(\omega t+\mu x)e^{-\lambda x} + 2\mu(-\lambda)\cos(\omega t+\mu x)e^{-\lambda x} + \lambda^2\sin(\omega t+\mu x)e^{-\lambda x}\right] = A\omega\cos(\omega t+\mu x)e^{-\lambda x}.$$

From comparing coefficients it is clear that we need

$$\lambda^2 = \mu^2, \text{ and } 2\mu(-\lambda)\kappa = \omega \quad \Rightarrow \quad \lambda = -\mu = \left(\frac{\omega}{2\kappa}\right)^{1/2}.$$

(b) For the two sinusoids to be attenuated by the same factor they must have equal values of λx. Thus

$$\frac{x_{\rm d}}{x_{\rm y}} = \frac{\lambda_{\rm y}}{\lambda_{\rm d}} = \left(\frac{\omega_{\rm y}}{\omega_{\rm d}}\right)^{1/2} = \left(\frac{1}{365}\right)^{1/2}.$$

(c) At the greater depth $x_{\rm y}$ only the yearly variation is significant and its phase relative to that on the surface is $\mu_{\rm y}x_{\rm y}$. This is equal to $-\lambda_{\rm y}x_{\rm y}$ which, in turn, is equal to $-\ln 20$ (from the way $x_{\rm y}$ was defined). Thus, the temperature at this depth is $\ln 20/2\pi$ of a year behind that at the surface, i.e. it is at its coldest on 1 February + (0.477×365) days, about 23 July.

20.24 *This example gives a formal demonstration that the type of a second-order PDE (elliptic, parabolic or hyperbolic) cannot be changed by a new choice of independent variable. The algebra is somewhat lengthy, but straightforward.*

If a change of variable $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ is made in

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = R(x, y),$$

so that it reads

$$A'\frac{\partial^2 u}{\partial \xi^2} + B'\frac{\partial^2 u}{\partial \xi \partial \eta} + C'\frac{\partial^2 u}{\partial \eta^2} + D'\frac{\partial u}{\partial \xi} + E'\frac{\partial u}{\partial \eta} + F'u = R'(\xi, \eta),$$

show that

$$B'^2 - 4A'C' = (B^2 - 4AC)\left[\frac{\partial(\xi, \eta)}{\partial(x, y)}\right]^2.$$

Hence deduce the conclusion stated above.

To save space, we denote $\dfrac{\partial \xi}{\partial x}$ by ξ_x, etc.

By the chain rule, the differential operators with respect to x and y take the following forms when expressed in terms of the new variables:

$$\frac{\partial}{\partial x} = \xi_x\frac{\partial}{\partial \xi} + \eta_x\frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = \xi_y\frac{\partial}{\partial \xi} + \eta_y\frac{\partial}{\partial \eta}.$$

Then, with $u(x, y) = v(\xi, \eta)$, the second derivative of u with respect to x becomes

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \left(\xi_x\frac{\partial}{\partial \xi} + \eta_x\frac{\partial}{\partial \eta}\right)\left(\xi_x\frac{\partial v}{\partial \xi} + \eta_x\frac{\partial v}{\partial \eta}\right) \\
&= \xi_x^2\frac{\partial^2 v}{\partial \xi^2} + 2\xi_x\eta_x\frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_x^2\frac{\partial^2 v}{\partial \eta^2} \\
&\qquad + \xi_x\frac{\partial \xi_x}{\partial \xi}\frac{\partial v}{\partial \xi} + \eta_x\frac{\partial \xi_x}{\partial \eta}\frac{\partial v}{\partial \xi} + \xi_x\frac{\partial \eta_x}{\partial \xi}\frac{\partial v}{\partial \eta} + \eta_x\frac{\partial \eta_x}{\partial \eta}\frac{\partial v}{\partial \eta}.
\end{aligned}$$

There are similar expressions for $\dfrac{\partial^2 u}{\partial x \partial y}$ and $\dfrac{\partial^2 u}{\partial y^2}$.

Since the nature of a second-order PDE is determined purely by the sign of $B^2 - 4AC$, for the purposes of this exercise we need only consider terms containing second derivatives (mixed or otherwise) of v, i.e. only three terms in each of $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial x \partial y}$ and $\dfrac{\partial^2 u}{\partial y^2}$ and no terms at all for $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ and u.

The relevant terms and their origins are thus

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}:&\qquad \xi_x^2\frac{\partial^2 v}{\partial \xi^2}+2\xi_x\eta_x\frac{\partial^2 v}{\partial\xi\partial\eta}+\eta_x^2\frac{\partial^2 v}{\partial \eta^2},\\
\frac{\partial^2 u}{\partial x\partial y}:&\qquad \xi_x\xi_y\frac{\partial^2 v}{\partial \xi^2}+(\xi_x\eta_y+\xi_y\eta_x)\frac{\partial^2 v}{\partial\xi\partial\eta}+\eta_x\eta_y\frac{\partial^2 v}{\partial \eta^2},\\
\frac{\partial^2 u}{\partial y^2}:&\qquad \xi_y^2\frac{\partial^2 v}{\partial \xi^2}+2\xi_y\eta_y\frac{\partial^2 v}{\partial\xi\partial\eta}+\eta_y^2\frac{\partial^2 v}{\partial \eta^2}.
\end{aligned}$$

The coefficients A', B' and C' of the transformed equations are therefore

$$\begin{aligned}
A' &= A\xi_x^2+B\xi_x\xi_y+C\xi_y^2,\\
B' &= 2A\xi_x\eta_x+B(\eta_x\xi_y+\eta_y\xi_x)+2C\xi_y\eta_y,\\
C' &= A\eta_x^2+B\eta_x\eta_y+C\eta_y^2.
\end{aligned}$$

We now face the (messy) task of evaluating $D'=B'^2-4A'C'$.

$$\begin{aligned}
D' &= 4A^2\xi_x^2\eta_x^2+4C^2\xi_y^2\eta_y^2+B^2(\eta_x\xi_y+\eta_y\xi_x)^2\\
&\quad+4AB\xi_x\eta_x(\eta_x\xi_y+\eta_y\xi_x)+4BC\xi_y\eta_y(\eta_x\xi_y+\eta_y\xi_x)\\
&\quad+8AC\xi_x\eta_x\xi_y\eta_y-4A^2\xi_x^2\eta_x^2-4AB\xi_x\xi_y\eta_x^2-4AC\xi_y^2\eta_x^2\\
&\quad-4AB\xi_x^2\eta_x\eta_y-4B^2\xi_x\xi_y\eta_x\eta_y-4CB\xi_y^2\eta_x\eta_y\\
&\quad-4AC\xi_x^2\eta_y^2-4BC\xi_x\xi_y\eta_y^2-4C^2\xi_y^2\eta_y^2\\
&= B^2(\eta_x\xi_y+\eta_y\xi_x)^2+8AC\xi_x\eta_x\xi_y\eta_y\\
&\quad-4AC\xi_y^2\eta_x^2-4B^2\xi_x\xi_y\eta_x\eta_y-4AC\xi_x^2\eta_y^2\\
&= B^2(\eta_x\xi_y-\eta_y\xi_x)^2-4AC(\eta_x^2\xi_y^2-2\xi_x\eta_x\xi_y\eta_y+\xi_x^2\eta_y^2)\\
&= (B^2-4AC)(\eta_x\xi_y-\eta_y\xi_x)^2\\
&= (B^2-4AC)\left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right]^2.
\end{aligned}$$

Since the square of the Jacobian is positive, D' has the same sign as B^2-4AC, showing that the equation type is not altered by the change of independent variables.

21

Partial differential equations: separation of variables and other methods

21.2 *A cube, made of material whose conductivity is k, has as its six faces the planes $x = \pm a$, $y = \pm a$ and $z = \pm a$, and contains no internal heat sources. Verify that the temperature distribution*

$$u(x, y, z, t) = A \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right)$$

obeys the appropriate diffusion equation. Across which faces is there heat flow? What is the direction and rate of heat flow at the point $(3a/4, a/4, a)$ at time $t = a^2/(\kappa\pi^2)$?

The diffusion equation is

$$\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial u}{\partial t}.$$

Substituting the given expression

$$u(x, y, z, t) = A \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right)$$

into the equation gives

$$\kappa\left(-\frac{\pi^2}{a^2} + 0 - \frac{\pi^2}{a^2}\right) u = -\frac{2\kappa\pi^2}{a^2}\, u,$$

which is satisfied, i.e. the given temperature distribution obeys the relevant diffusion equation.

Heat will flow across a face if, at any point on it, the temperature gradient $\partial u/\partial n$, is not equal to zero; here $\mathbf{n}$ is the (local) outward normal to the face.

For the faces $x = \pm a$:

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial x} = \pm\left(-\frac{\pi}{a}\right) A \sin\frac{\pi x}{a}\sin\frac{\pi z}{a}\exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right) = 0 \text{ at } x = \pm a.$$

Thus, although in general there is some heat flow in the x- direction, at the surfaces of the cube the rate of flow is zero.

Since u does not depend upon y, all derivatives with respect to y are zero. This means that there is no heat flow in the y-direction, not even in the body of the cube. In particular, for the faces $y = \pm a$:

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial y} = 0 \text{ for all } x \text{ and } z\ ,$$

and no heat flows across any part of these two faces.

For the faces $z = \pm a$:

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial z} = \pm\left(\frac{\pi}{a}\right) A \cos\frac{\pi x}{a}\cos\frac{\pi z}{a}\exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right) \neq 0 \text{ for general } x.$$

In summary, on the surface of the cube there is heat flow only across the faces $z = \pm a$.

For the point $(x, y, z) = \frac{1}{4}(3a, a, 4a)$, which lies in the face $z = a$, at time $t = a^2/(\kappa\pi^2)$,

$$\frac{\partial u}{\partial n} = +\frac{\partial u}{\partial z} = \left(\frac{\pi}{a}\right) A\left(\cos\frac{3\pi}{4}\right)(\cos\pi)\exp\left(-\frac{2\kappa\pi^2}{a^2}\,\frac{a^2}{\kappa\pi^2}\right)$$

and the heat flow is

$$-k\frac{\partial u}{\partial n} = -\frac{kA\pi}{a}\left(-\frac{1}{\sqrt{2}}\right)(-1)e^{-2}.$$

The heat flux *into* the cube is therefore $\dfrac{kA\pi e^{-2}}{\sqrt{2}a}$.

Note that k and κ are related by $\kappa = k/c$ where c is the specific heat (thermal capacity) of the material from which the cube is made.

21.4 *Schrödinger's equation for a non-relativistic particle in a constant potential region can be taken as*

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=i\hbar\frac{\partial u}{\partial t}.$$

(a) *Find a solution, separable in the four independent variables, that can be written in the form of a plane wave,*

$$\psi(x,y,z,t)=A\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)].$$

Using the relationships associated with de Broglie ($\mathbf{p}=\hbar\mathbf{k}$) and Einstein ($E=\hbar\omega$), show that the separation constants must be such that

$$p_x^2+p_y^2+p_z^2=2mE.$$

(b) *Obtain a different separable solution describing a particle confined to a box of side a (ψ must vanish at the walls of the box). Show that the energy of the particle can only take the quantised values*

$$E=\frac{\hbar^2\pi^2}{2ma^2}(n_x^2+n_y^2+n_z^2),$$

where n_x, n_y and n_z are integers.

(a) Take $u(x,y,z,t)=X(x)Y(y)Z(z)T(t)$. After substituting and dividing through by u, we obtain

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}\right)=i\hbar\frac{T'}{T}\quad (*).$$

For a solution that can be written both as a plane wave and in the separable form (∗) we must have

$$\begin{aligned}\psi(x,y,z,t)&=A\exp[\,i(\mathbf{k}\cdot\mathbf{r}-\omega t)\,]\\&=Ae^{ik_x x}e^{ik_y y}e^{ik_z z}\,e^{-i\omega t}.\end{aligned}$$

This then implies that the separation constants k_i and ω satisfy

$$\begin{aligned}-\frac{\hbar^2}{2m}(-k_x^2-k_y^2-k_z^2)&=i\hbar(-i\omega),\\ \Rightarrow\quad \frac{1}{2m}(p_x^2+p_y^2+p_z^2)&=\hbar\omega=E,\end{aligned}$$

where we have used the de Broglie ($\mathbf{p}=\hbar\mathbf{k}$) and Einstein ($E=\hbar\omega$) relationships.

(b) For solutions that vanish on any of the walls of the box $x=0$, $x=a$, etc., we must have a product of sine waves of the form

$$u(x,y,z,t)=A\sin\left(\frac{n_x\pi x}{a}\right)\sin\left(\frac{n_y\pi y}{a}\right)\sin\left(\frac{n_z\pi z}{a}\right)\,e^{-i\omega t},$$

where the n_i are integers.

For this solution of $(*)$ it follows that

$$-\frac{\hbar^2}{2m}\left[-\left(\frac{n_x\pi}{a}\right)^2-\left(\frac{n_y\pi}{a}\right)^2-\left(\frac{n_z\pi}{a}\right)^2\right]=i\hbar(-i\omega),$$
$$\frac{\hbar^2\pi^2}{2ma^2}(n_x^2+n_y^2+n_z^2)=\hbar\omega=E.$$

This shows that the possible values of E are quantised, since n_x, n_y and n_z can only take discrete integer values.

21.6 *Prove that the expression*

$$P_\ell^m(\mu)=(1-\mu^2)^{|m|/2}\frac{d^{|m|}}{d\mu^{|m|}}P_\ell(\mu),\qquad (*)$$

for the associated Legendre function $P_\ell^m(\mu)$ *satisfies the appropriate equation,*

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-\mu^2}\right]M=0,\qquad (**)$$

as follows.

(a) *Evaluate* $dP_\ell^m(\mu)/d\mu$ *and* $d^2P_\ell^m(\mu)/d\mu^2$ *using the form given in* $(*)$ *and substitute them into* $(**)$.
(b) *Differentiate Legendre's equation m times using Leibnitz' theorem.*
(c) *Show that the equations obtained in* (a) *and* (b) *are multiples of each other, and hence that the validity of* (b) *implies that of* (a).

To save space (and clutter) we will omit all references for (ordinary) Legendre functions to the fixed subscript ℓ and denote $\dfrac{d^m[P_\ell(\mu)]}{d\mu^m}$ by d^mP. Further, we will take $m>0$.

(a) From the given definition

$$\begin{aligned}
P_\ell^m(\mu)&=(1-\mu^2)^{m/2}d^mP,\\
(P_\ell^m)'&=-m\mu(1-\mu^2)^{(m/2)-1}d^mP+(1-\mu^2)^{m/2}d^{m+1}P\\
(P_\ell^m)''&=m(m-2)\mu^2(1-\mu^2)^{(m/2)-2}d^mP-m(1-\mu^2)^{(m/2)-1}d^mP\\
&\quad-2m\mu(1-\mu^2)^{(m/2)-1}d^{m+1}P+(1-\mu^2)^{m/2}d^{m+2}P.
\end{aligned}$$

We now substitute these forms into the associated Legendre equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-\mu^2}\right]M=0,$$

divide through by a factor $(1-\mu^2)^{m/2}$ and collect together the terms involving each particular derivative d^nP.

$$\begin{aligned}\big[m(m-2)\mu^2(1-\mu^2)^{-1} - m + 2m\mu^2(1-\mu^2)^{-1}\\ + \ell(\ell+1) - m^2(1-\mu^2)^{-1}\big]\, d^mP\\ + (-2m\mu - 2\mu)d^{m+1}P + (1-\mu^2)d^{m+2}P &= 0,\end{aligned}$$

$$\begin{aligned}\left[\frac{m^2\mu^2 - 2m\mu^2 + 2m\mu^2 - m^2}{1-\mu^2} - m + \ell(\ell+1)\right] d^mP\\ - 2\mu(m+1)d^{m+1}P + (1-\mu^2)d^{m+2}P &= 0,\end{aligned}$$

$$(1-\mu^2)d^{m+2}P - 2\mu(m+1)d^{m+1}P + [\,\ell(\ell+1) - m(m+1)\,]d^mP = 0.$$

This is an equation that must be valid if the given prescription generates associated Legendre functions, the latter being defined as being the solutions to the associated Legendre equation.

We now proceed to show that it is valid, taking Legendre's equation,

$$(1-\mu^2)P'' - 2\mu P' + \ell(\ell+1) = 0.$$

as our starting point.

(b) Using Leibnitz' theorem, we differentiate Legendre's equation m times and obtain

$$\begin{aligned}(1-\mu^2)d^{m+2}P + m(-2\mu)d^{m+1}P + \tfrac{1}{2}m(m-1)(-2)d^mP\\ -2\mu d^{m+1}P - 2md^mP + \ell(\ell+1)d^mP &= 0,\\ (1-\mu^2)d^{m+2}P - 2\mu(m+1)d^{m+1}P + [\,\ell(\ell+1) - m(m+)\,]d^mP &= 0.\end{aligned}$$

(c) We now note that the final equation obtained in part (b) is the same as the putative one obtained in part (a) and so, from the line of reasoning given in (a), we conclude that

$$P_\ell^m(\mu) = (1-\mu^2)^{|m|/2}\frac{d^{|m|}}{d\mu^{|m|}}P_\ell(\mu),$$

does indeed generate associated Legendre functions.

The solutions for negative m have the same forms as those for positive m but their signs and normalisations are defined by convention. As the equation is homogeneous they are still solutions of it.

21.8 *The motion of a very viscous fluid in the two-dimensional (wedge) region $-\alpha < \phi < \alpha$ can be described in (ρ, ϕ) coordinates by the (biharmonic) equation*

$$\nabla^2\nabla^2\psi \equiv \nabla^4\psi = 0,$$

together with the boundary conditions $\partial\psi/\partial\phi = 0$ at $\phi = \pm\alpha$, which represent the fact that there is no radial fluid velocity close to either of the bounding walls because of the viscosity, and $\partial\psi/\partial\rho = \pm\rho$ at $\phi = \pm\alpha$, which impose the condition that azimuthal flow increases linearly with r along any radial line. Assuming a solution in separated-variable form, show that the full expression for ψ is

$$\psi(\rho, \phi) = \frac{\rho^2}{2}\,\frac{\sin 2\phi - 2\phi\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$$

The conditions to be met are

$$\begin{aligned}
&\nabla^4\psi = 0 && \text{with } \psi(r, \theta) = R(\rho)\Phi(\phi),\\
&\frac{\partial\psi}{\partial\phi} = 0, \quad \frac{\partial\psi}{\partial\rho} = -\rho, && \text{at } \phi = -\alpha,\\
&\frac{\partial\psi}{\partial\phi} = 0, \quad \frac{\partial\psi}{\partial\rho} = \rho, && \text{at } \phi = \alpha.
\end{aligned}$$

Since $\dfrac{\partial\psi}{\partial\rho} \propto \rho$, we need $R(\rho) \propto \rho^2$; let $R(\rho) = \rho^2$, with any multiplicative constant being absorbed into $\Phi(\phi)$.

With this choice of $R(\rho)$, $\nabla^2\psi$ takes the form

$$\begin{aligned}
\nabla^2\psi &= \Phi\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\; 2\rho) + \frac{\rho^2}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}\\
&= 4\Phi + \frac{d^2\Phi}{d\phi^2},
\end{aligned}$$

and $\nabla^4\psi = 0$ is

$$0 = \nabla^4\psi = 0 + \frac{1}{\rho^2}\frac{d^2}{d\phi^2}(4\Phi + \Phi'').$$

After this equation has been integrated twice we obtain

$$\Phi'' + 4\Phi = k\phi + c,$$

which has a CF of $C\cos 2\phi + D\sin 2\phi$ and a PI of $\frac{1}{4}(k\phi + c)$.

The general solution for Φ is therefore

$$\Phi = C\cos 2\phi + D\sin 2\phi + A\phi + B.$$

The boundary condition $\dfrac{\partial\psi}{\partial\phi} = 0$ requires Φ' to be zero at $\phi = \pm\alpha$, i.e.

$$-2C\sin 2\phi + 2D\cos 2\phi + A = 0 \text{ at both } \phi = \alpha \text{ and } \phi = -\alpha.$$

These two conditions jointly imply that

$$C = 0 \text{ and } A = -2D\cos 2\alpha.$$

Correspondingly, after substituting for A, the boundary condition on $\dfrac{\partial\psi}{\partial\rho}$ requires that both the equations

$$2\rho(D\sin 2\phi - 2D\cos 2\alpha\,\phi + B) = \pm\rho \text{ at } \phi = \pm\alpha,$$

are satisfied and so determines B and D as

$$B = 0 \text{ and } D = \frac{1}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}.$$

Thus, finally,

$$\psi(\rho,\phi) = \frac{\rho^2}{2}\,\frac{\sin 2\phi - 2\phi\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$$

21.10 *Consider possible solutions of Laplace's equation inside a circular domain, as follows:*

(a) *Find the solution in plane polar coordinates ρ, ϕ that takes the value $+1$ for $0 < \phi < \pi$ and the value -1 for $-\pi < \phi < 0$, when $\rho = a$.*

(b) *For a point (x, y) on or inside the circle $x^2 + y^2 = a^2$, identify the angles α and β defined by*

$$\alpha = \tan^{-1}\frac{y}{a+x} \qquad \textit{and} \qquad \beta = \tan^{-1}\frac{y}{a-x}.$$

Show that $u(x, y) = (2/\pi)(\alpha + \beta)$ is a solution of Laplace's equation that satisfies the boundary conditions given in (a).

(c) *Deduce a Fourier series expansion for the function*

$$\tan^{-1}\frac{\sin\phi}{1+\cos\phi} + \tan^{-1}\frac{\sin\phi}{1-\cos\phi}.$$

(a) The prescribed boundary values give an antisymmetric square-wave function for $-\pi < \phi \le \pi$. The sinusoidal terms in the general solution of the Laplace equation in plane polars are those used in a Fourier expansion. The required

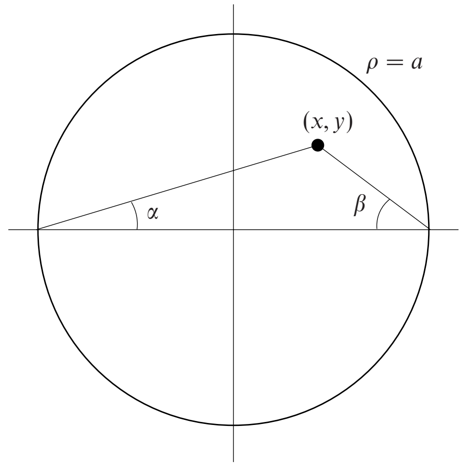

Figure 21.1 The angles α and β defined in exercise 21.10.

solution is thus one that becomes a Fourier sine series on the circle $\rho = a$. In anticipation of part (b) we take a solution valid *inside* the circle, namely

$$u(\rho, \phi) = \sum_{n=1}^{\infty} A_n \rho^n \sin n\phi.$$

The Fourier sine series for a square-wave,

$$u(\rho, \phi) = \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{1}{n} \sin n\phi,$$

can be found in almost any textbook and the calculation will not be repeated here.

For the presumed form of $u(\rho, \phi)$ to coincide with the Fourier series on $\rho = a$, it is necessary that

$$A_n = \frac{4}{\pi n a^n} \text{ for } n \text{ odd and } A_n = 0 \text{ for } n \text{ even.}$$

The required solution is thus

$$u(\rho, \phi) = \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\rho^n}{n a^n} \sin n\phi.$$

(b) As is clear from figure 21.1, the acute angles α and β are those made with the x-axis by the lines joining (x, y) to the extremes of the diameter of the circle that coincides with that axis. When (x, y) lies anywhere *on* the circle, $\alpha + \beta = \frac{1}{2}\pi$ (by the usual ‘angle in a semi-circle’ property) and

$$u(x, y) = \frac{2(\alpha + \beta)}{\pi} = \frac{2}{\pi}\,\frac{\pi}{2} = 1.$$

Further,

$$\frac{\partial \alpha}{\partial x} = \frac{1}{1 + \left(\dfrac{y}{a+x}\right)^2}\,\frac{-y}{(a+x)^2} = -\frac{y}{(a+x)^2 + y^2},$$

$$\frac{\partial^2 \alpha}{\partial x^2} = \frac{2(a+x)y}{[\,(a+x)^2 + y^2\,]^2}, \text{ and similarly } \frac{\partial^2 \beta}{\partial x^2} = \frac{2(a-x)y}{[\,(a-x)^2 + y^2\,]^2},$$

$$\frac{\partial \alpha}{\partial y} = \frac{1}{1 + \left(\dfrac{y}{a+x}\right)^2}\,\frac{1}{a+x} = \frac{a+x}{(a+x)^2 + y^2},$$

$$\frac{\partial^2 \alpha}{\partial y^2} = \frac{-(a+x)2y}{[\,(a+x)^2 + y^2\,]^2}, \text{ and similarly } \frac{\partial^2 \beta}{\partial y^2} = -\frac{(a-x)2y}{[\,(a-x)^2 + y^2\,]^2}.$$

Hence $\nabla^2 \alpha = 0$ and $\nabla^2 \beta = 0$, clearly showing that $\nabla^2 \left[\dfrac{2}{\pi}(\alpha + \beta)\right] = 0$. Thus $u(x, y)$ solves the Laplace equation and takes the values ± 1 on the upper and lower halves of the circle $\rho = a$, i.e. takes the boundary values given in part (a).

(c) Since the solution to Laplace’s equation with a given set of Dirichlet boundary values is unique, the answers to (a) and (b) must coincide. Hence,

$$\frac{2}{\pi}\left(\tan^{-1}\frac{\rho \sin\phi}{a + \rho\cos\phi} + \tan^{-1}\frac{\rho\sin\phi}{a - \rho\cos\phi}\right) = \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty} \frac{\rho^n}{n a^n}\sin n\phi.$$

Finally, setting $\rho = a$, we have

$$\tan^{-1}\frac{\sin\phi}{1 + \cos\phi} + \tan^{-1}\frac{\sin\phi}{1 - \cos\phi} = \sum_{n \text{ odd}}^{\infty} \frac{2}{n}\sin n\phi.$$

21.12 *A membrane is stretched between two concentric rings of radii a and b ($b > a$). If the smaller ring is transversely distorted from the planar configuration by an amount $c|\phi|$, $-\pi \le \phi \le \pi$, show that the membrane then has a shape given by*

$$u(\rho, \phi) = \frac{c\pi}{2}\,\frac{\ln(b/\rho)}{\ln(b/a)} - \frac{4c}{\pi}\sum_{m \text{ odd}} \frac{a^m}{m^2(b^{2m} - a^{2m})}\left(\frac{b^{2m}}{\rho^m} - \rho^m\right)\cos m\phi.$$

A stationary membrane obeys the wave equation with the time derivative term set to zero, i.e. it obeys Laplace's equation $\nabla^2 u = 0$.

The most general single-valued solution of $\nabla^2 u = 0$ in plane-polar coordinates is

$$T(\rho, \phi) = C \ln \rho + D + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi)(C_n \rho^n + D_n \rho^{-n}),$$

Since the given problem is symmetric in ϕ about $\phi = 0$, the solution will not contain any $\sin\phi$ terms. Further, on $\rho = b$ the average value of u is zero, implying that $0 = C \ln b + D$. In the same way, on $\rho = a$ the average value of u is $c\pi/2$, implying that $c\pi/2 = C \ln a + D$. Thus the first two terms of the solution can be written together as

$$u_1 = C \ln \rho - C \ln b = C \ln(\rho/b) = \frac{c\pi}{2(\ln a - \ln b)} \ln(\rho/b) = \frac{c\pi}{2} \frac{\ln(b/\rho)}{\ln(b/a)}.$$

The remainder of the solution is

$$u_2 = \sum_{n=1}^{\infty} \left(C_n \rho^n + D_n \rho^{-n} \right) \cos n\phi.$$

Using the mutual orthogonality of the $\cos m\phi$ functions for integer m, we may now obtain two equations linking C_m with D_m; one from $\rho = a$, the other from $\rho = b$.

$$\begin{aligned}
C_m a^m + D_m a^{-m} &= \frac{2}{2\pi} \int_{-\pi}^{\pi} c|\phi| \cos m\phi \, d\phi \\
&= \frac{2c}{\pi} \int_0^{\pi} \phi \cos m\phi \, d\phi \\
&= \frac{2c}{\pi} \left\{ \left[\frac{\phi \sin m\phi}{m} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin m\phi}{m} \, d\phi \right\} \\
&= \frac{2c}{\pi} \left[\frac{\cos m\phi}{m^2} \right]_0^{\pi} \\
&= -\frac{4c}{\pi m^2} \text{ for } m \text{ odd}, = 0 \text{ for } m \text{ even}, \\
C_m b^m + D_m b^{-m} &= \frac{2}{2\pi} \int_{-\pi}^{\pi} 0 \cos m\phi \, d\phi = 0.
\end{aligned}$$

Thus $D_m = -C_m b^{2m}$ and $C_m a^m - C_m b^{2m} a^{-m} = -\dfrac{4c}{\pi m^2}$ for m odd.
Hence

$$\begin{aligned}
u_2 &= \sum_{n \text{ odd}} \left(\frac{-4c}{\pi n^2} \right) \left(\frac{1}{a^n - b^{2n} a^{-n}} \right) \left(\rho^n - b^{2n} \rho^{-n} \right) \cos n\phi \\
&= -\frac{4c}{\pi} \sum_{n \text{ odd}} \frac{a^n}{n^2 (b^{2n} - a^{2n})} \left(\frac{b^{2n}}{\rho^n} - \rho^n \right) \cos n\phi,
\end{aligned}$$

and $u = u_1 + u_2$ is as given in the question.

21.14 *A conducting spherical shell of radius a is cut round its equator and the two halves connected to voltages of* $+V$ *and* $-V$*. Show that an expression for the potential at the point* (r, θ, ϕ) *anywhere inside the two hemispheres is*

$$u(r, \theta, \phi) = V \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! (4n+3)}{2^{2n+1} n! (n+1)!} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta).$$

This problem is almost identical to the last worked example in the subsection *Laplace's equation in polar coordinates*, page 735. The only difference is that the two halves of the sphere are at potentials V and $-V$, rather than v_0 and 0. We can therefore take over that result by the change $v_0 \to 2V$ and then subtracting V from the complete solution; this latter change has the effect of removing the constant term and leaving a sum that contains *only* odd Legendre polynomials.

Their expansion coefficients are evaluated using the result from exercise 18.3; thus,

$$A_{2n+1} a^{2n+1} = \frac{2(2n+1)+1}{2} \, 2V \, \frac{(-1)^n \, (2n)!}{2^{2n+1} \, n! \, (n+1)!},$$

giving

$$u(r, \theta, \phi) = V \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! (4n+3)}{2^{2n+1} n! (n+1)!} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta).$$

21.16 *A slice of biological material of thickness L is placed into a solution of a radioactive isotope of constant concentration* C_0 *at time* $t = 0$*. For a later time t find the concentration of radioactive ions at a depth x inside one of its surfaces if the diffusion constant is* κ*.*

The concentration is governed by the diffusion law

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

Ultimately the concentration will be C_0 everywhere; this is formally, but trivially, a solution of the equation. To this must be added time-dependent solutions of the diffusion equation that represent the (decaying) transients and $\to 0$ as $t \to \infty$.

Writing $u(x,t) = C_0 + X(x)T(t)$ we obtain the usual separated variable equations

$$\frac{X''}{X} = \frac{T'}{\kappa T} = -\frac{\mu}{\kappa}, \qquad \mu > 0,$$

with the sign of acceptable separation constants chosen so as to ensure solutions decaying with time.

The solution for the time variation is trivial, $T(t) = T(0)e^{-\mu t}$. That for the spatial variation is of the form

$$X(x) = A \sin \sqrt{\frac{\mu}{\kappa}}\, x + B \cos \sqrt{\frac{\mu}{\kappa}}\, x.$$

At all times $u(0,t) = u(L,t) = C_0$ and so $X(0) = X(L) = 0$. This determines that $B = 0$ (whatever the value of μ) and imposes the condition that $\sqrt{\mu}L = n\pi\sqrt{\kappa}$ where n is an integer.

At this stage all positive integer values of n are possible and thus the general solution is a linear superposition of them:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \exp\left(-\frac{n^2\pi^2\kappa}{L^2} t\right).$$

At $t = 0$, before any diffusion has taken place, $u(x,0) = 0$ and so

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = -C_0.$$

This is, in fact, a Fourier expansion and we determine the coefficients A_n in the usual way, using the mutual orthogonality of sinusoidal functions. Multiplying both sides by $\sin(m\pi x/L)$ and integrating from 0 to L gives

$$\begin{aligned} A_m \frac{1}{2} L &= \int_0^L (-C_0) \sin \frac{m\pi x}{L}\, dx \\ &= C_0 \left[\frac{L}{m\pi} \cos \frac{m\pi x}{L} \right]_0^L \\ &= \begin{cases} -\dfrac{2C_0 L}{m\pi} & \text{for } m \text{ odd}, \\ 0 & \text{for } m \text{ even}. \end{cases} \end{aligned}$$

Inserting these values yields

$$u(x,t) = C_0 - \frac{4C_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi x}{L} \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

so giving the concentration at a general place and time.

***21.18** A sphere of radius a and thermal conductivity k_1 is surrounded by an infinite medium of conductivity k_2 in which far away the temperature tends to T_∞. A distribution of heat sources $q(\theta)$ embedded in the sphere's surface establish steady temperature fields $T_1(r,\theta)$ inside the sphere and $T_2(r,\theta)$ outside it. It can be shown, by considering the heat flow through a small volume that includes part of the sphere's surface, that*

$$k_1\frac{\partial T_1}{\partial r} - k_2\frac{\partial T_2}{\partial r} = q(\theta) \quad on \quad r = a.$$

Given that

$$q(\theta) = \frac{1}{a}\sum_{n=0}^{\infty} q_n P_n(\cos\theta),$$

find complete expressions for $T_1(r,\theta)$ and $T_2(r,\theta)$. What is the temperature at the centre of the sphere?

The general azimuthally symmetric solution in spherical polar coordinates of the time-independent diffusion equation, i.e. of Laplace's equation, is

$$T(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta).$$

Since T_1 covers a region including the origin, it must contain no inverse powers of r. Likewise, since T_2 covers a region including $r \to \infty$, it must not contain any positive powers of r. Thus

$$T_1(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \text{ and } T_2(r,\theta) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta) + T_\infty P_0.$$

The boundary conditions on $r = a$ are

$$T_1 = T_2 \quad \text{and} \quad k_1\frac{\partial T_1}{\partial r} - k_2\frac{\partial T_2}{\partial r} = \frac{1}{a}\sum_{n=0}^{\infty} q_n P_n(\cos\theta),$$

the RHS of the second one representing a 'Legendre expansion' of the distribution of heat sources, analogous to a Fourier series.

Since the P_n are mutually orthogonal we may equate their coefficients on the two sides of an equation. The first boudary condition therefore yields

$$A_0 = \frac{B_0}{a} + T_\infty \quad \text{and} \quad A_n a^n = \frac{B_n}{a^{n+1}}.$$

The second condition (for $n \geq 0$) is

$$k_1 n A_n a^{n-1} + k_2\frac{(n+1)B_n}{a^{n+2}} = \frac{q_n}{a},$$

$$\Rightarrow \quad [k_1 n + k_2(n+1)]A_n a^n = q_n.$$

These simultaneous equations give all the A_n and B_n and the temperatures in the two regions as

$$T_1(r,\theta) = \sum_{n=0}^{\infty} \frac{q_n}{k_1 n + k_2(n+1)} \left(\frac{r}{a}\right)^n P_n(\cos\theta) + T_\infty,$$

$$T_2(r,\theta) = \sum_{n=0}^{\infty} \frac{q_n}{k_1 n + k_2(n+1)} \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) + T_\infty.$$

The temperature at the centre of the sphere is $T_1(0,\theta) = \dfrac{q_0}{k_2} + T_\infty$. Perhaps surprisingly, this depends only on q_0 and k_2, and not on k_1. However, since q_0 is the only component-source that has a net non-zero output of heat (averaged over all directions), it is this and the rate at which the heat it produces is conducted to infinity that determine the level to which the temperature at the centre rises; hence k_2 is the controlling factor. How quickly the equilibrium state would be established starting from (say) a uniform temperature of T_∞ *would* be affected by the value of k_1, but that is not asked for here.

21.20 *Working in* spherical *polar coordinates* $\mathbf{r} = (r, \theta, \phi)$*, but for a system that has azimuthal symmetry around the polar axis, consider the following gravitational problem.*

(a) *Show that the gravitational potential due to a uniform disc of radius a and mass M, centred at the origin, is given for* $r < a$ *by*

$$\frac{2GM}{a}\left[1 - \frac{r}{a}P_1(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{1}{8}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \cdots\right],$$

and for $r > a$ *by*

$$\frac{GM}{r}\left[1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) - \cdots\right],$$

where the polar axis is normal to the plane of the disc.

(b) *Reconcile the presence of a term* $P_1(\cos\theta)$*, which is odd under* $\theta \to \pi - \theta$*, with the symmetry with respect to the plane of the disc of the physical system.*

(c) *Deduce that the gravitational field near an infinite sheet of matter of constant density* ρ *per unit area is* $2\pi G\rho$.

We tackle this problem by first calculating directly the potential at a general point on the polar axis, i.e. on the central normal to the disc, and then choosing the constants in the general solution of Laplace's equation to make the two expressions match. Finally (though it is often not explicitly stated) we appeal to the uniqueness theorem to claim that the solution so found is the correct one.

(a) An annulus of the disc of radius ρ and width $d\rho$ produces a potential $du(z)$ at a point on the polar axis distant z from the disc given by

$$du = \frac{2\pi\rho\,\sigma G}{(\rho^2 + z^2)^{1/2}}\,d\rho,$$

where σ is the area density of the disc. The total potential at the point is thus

$$u(z) = \int_0^a \frac{2\pi\sigma G\,\rho}{(\rho^2 + z^2)^{1/2}}\,d\rho = 2\pi\sigma G\left[\,(\rho^2 + z^2)^{1/2}\,\right]_0^a$$
$$= 2\pi\frac{M}{\pi a^2}G[\,(a^2 + z^2)^{1/2} - z\,].$$

For $r < a$, we now expand this function in positive powers of z and then replace z^n by $r^n P_n(\cos\theta)$, these two expressions being the same for $\theta = 0$. The resulting expression $u(r, \theta)$ will be valid for all $r < a$ and all θ, not just on the axis:

$$u(z) = \frac{2MGa}{a^2}\left(1 + \frac{1}{2}\frac{z^2}{a^2} - \frac{1}{8}\frac{z^4}{a^4} + \cdots - \frac{z}{a}\right),$$
$$u(r, \theta) = \frac{2MG}{a}\left(1 + \frac{1}{2}\frac{r^2}{a^2}P_2(\cos\theta) - \frac{1}{8}\frac{r^4}{a^4}P_4(\cos\theta) + \cdots - \frac{r}{a}P_1(\cos\theta)\right).$$

For $r > a$, the function has to be expanded in negative powers of z and then z^{-n-1} has to be replaced by $r^{-n-1}P_n(\cos\theta)$, these two expressions being the same for $\theta = 0$. The resulting expression $u(r, \theta)$ will be valid for all $r > a$ and all θ, not just on the axis:

$$u(z) = \frac{2MG}{a^2}\left(z + \frac{1}{2}\frac{a^2}{z} - \frac{1}{8}\frac{a^4}{z^3} + \frac{1}{16}\frac{a^6}{z^5}\cdots - z\right),$$
$$u(r, \theta) = \frac{2MG}{a^2}\frac{a^2}{2r}\left(1 - \frac{1}{4}\frac{a^2}{r^2}P_2(\cos\theta) + \frac{1}{8}\frac{a^4}{r^4}P_4(\cos\theta) - \cdots\right).$$

(b) Under the change $\theta \to \pi - \theta$, z becomes negative and the appropriate form for $\left[\,(\rho^2 + z^2)^{1/2}\,\right]_0^a$ is $(a^2 + z^2)^{1/2} - (-z)$, i.e. $(a^2 + z^2)^{1/2} + z$. When this is expressed in polar coordinates, $P_1(\cos\theta)$ changes sign under the interchange but r does not; neither do any of the $P_n(\cos\theta)$ when n is even. The derived expression is therefore still valid when $\theta > \pi/2$.

(c) The gravitational field is given by $-\partial u/\partial r$ in the direction $\theta = 0$, where all $P_n = 1$. An infinite sheet of matter is equivalent to a finite r and $a \to \infty$. Clearly the $r < a$ form is needed and M must be expressed as $M = \pi a^2 \rho$:

$$\frac{\partial u}{\partial r} = \frac{2G\pi a^2 \rho}{a}\left(-\frac{1}{a}P_1(1) + \frac{r}{a^2}P_2(1) + \cdots\right).$$

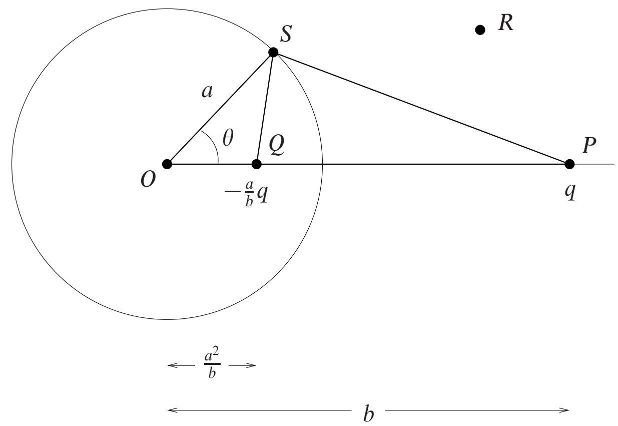

Figure 21.2 The charges as described in exercise 21.22 and the resulting spherical surface on which the potential is zero.

It then follows that

$$g = -\lim_{a\to\infty} \frac{\partial u}{\partial r} = 2\pi G\rho$$

is the gravitational field near an infinite sheet of constant-density matter.

> ***21.22*** *Point charges q and $-qa/b$ (with $a < b$) are placed, respectively, at a point P, a distance b from the origin O, and a point Q between O and P, a distance a^2/b from O. Show, by considering similar triangles QOS and SOP, where S is any point on the surface of the sphere centred at O and of radius a, that the net potential anywhere on the sphere due to the two charges is zero.*
>
> *Use this result (backed up by the uniqueness theorem) to find the force with which a point charge q placed a distance b from the centre of a spherical conductor of radius a $(< b)$ is attracted to the sphere* (i) *if the sphere is earthed, and* (ii) *if the sphere is uncharged and insulated.*

As can be seen from figure 21.2,

$$\phi_S = \frac{q}{4\pi\epsilon_0 |PS|} - \frac{(a/b)q}{4\pi\epsilon_0 |QS|}.$$

Now, the triangles QOS and SOP have sides in the ratio

$$\frac{OQ}{OS} = \frac{(a^2/b)}{a} = \frac{a}{b} = \frac{OS}{OP};$$

they also have the same included angle θ. This shows that they are similar triangles and therefore that the ratio QS/PS is also equal to a/b. It now follows that $\phi_S = 0$.

(i) The potential at a general point R outside the earthed sphere must be given by

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}_R - \mathbf{r}_P|} - \frac{a}{b|\mathbf{r}_R - \mathbf{r}_Q|}\right).$$

This is so because each term is a solution of Laplace's equation, $\phi(\mathbf{r}) = 0$ for $\mathbf{r}$ anywhere on the sphere and $\phi(\mathbf{r}) \to 0$ as $r \to \infty$; the uniqueness theorem shows that there is only one such solution and so this must be it.

A deduction from this is that a charge of $-aq/b$ placed at $\mathbf{r}_Q$ is the appropriate image charge for this situation. Since it produces the same potential distribution outside the sphere as the actual induced charge on the sphere does, it will also produce the same electrostatic field there. Consequently, the force with which the real charge at P will be attracted to the sphere is the same as that between the real charge and its image, i.e.

$$\begin{aligned} F_1 &= \frac{(a/b)q\,q}{4\pi\epsilon_0\,|\,PQ\,|^2} \\ &= \frac{aq^2}{4\pi\epsilon_0 b\left(b - \dfrac{a^2}{b}\right)^2} = \frac{abq^2}{4\pi\epsilon_0(b^2 - a^2)^2}. \end{aligned}$$

(ii) With the sphere uncharged and insulated it must still be an equipotential surface (though not necessarily at zero potential). However, it must now also have the property that zero net charge is contained in any surface that surrounds it.

These two requirements can be met by adding a further image charge at O, equal in magnitude but opposite in sign to that at Q. The additional charge affects all parts of the sphere equally, leaving it as an equipotential surface, but increases its potential by $(a/b)q/(4\pi\epsilon_0 a)$. The new force of attraction will be

$$F_2 = F_1 - \frac{(a/b)q^2}{4\pi\epsilon_0 b^2} = F_1 - \frac{aq^2}{4\pi\epsilon_0 b^3}.$$

Although we have not explicitly said so, the uniqueness theorem has again been invoked in arriving at this result.

21.24 *Electrostatic charge is distributed in a sphere of radius R centred on the origin. Determine the form of the resultant potential $\phi(\mathbf{r})$ at distances much greater than R, as follows.*

(a) *Express in the form of an integral over all space the solution of*

$$\nabla^2\phi = -\frac{\rho(\mathbf{r})}{\epsilon_0}.$$

(b) *Show that, for $r \gg r'$,*

$$|\mathbf{r}-\mathbf{r}'| = r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} + \mathrm{O}\left(\frac{1}{r}\right).$$

(c) *Use results* (a) *and* (b) *to show that $\phi(\mathbf{r})$ has the form*

$$\phi(\mathbf{r}) = \frac{M}{r} + \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + \mathrm{O}\left(\frac{1}{r^3}\right).$$

Find expressions for M and $\mathbf{d}$, and identify them physically.

(a) The formal expression for the integral solution of Poisson's equation is

$$\phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\,\mathbf{r}-\mathbf{r}'\,|}\,d\mathbf{r}',$$

where V is the sphere $r' \le R$. (b) When $r \gg r'$ we may expand the expression for $|\mathbf{r}-\mathbf{r}'|$ in powers of r'/r using the binomial theorem:

$$\begin{aligned}
|\,\mathbf{r}-\mathbf{r}'\,| &= [\,(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')\,]^{1/2} \\
&= (r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2)^{1/2} \\
&= r\left(1 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right)^{1/2} \\
&= r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} + \mathrm{O}\left(\frac{1}{r}\right).
\end{aligned}$$

(c) Putting this into the integral in part (a) and again using the binomial theorem:

$$\begin{aligned}
\phi(\mathbf{r}) &= \int_V \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0\, r}\left[1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \mathrm{O}\left(\frac{1}{r^2}\right)\right]^{-1} d\mathbf{r}' \\
&= \frac{M}{r} + \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + \mathrm{O}\left(\frac{1}{r^3}\right),
\end{aligned}$$

where

$$M = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\,d\mathbf{r}' \text{ and } \mathbf{d} = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\mathbf{r}'\,d\mathbf{r}'.$$

The first term M is $(4\pi\epsilon_0)^{-1}$ times the total charge contained in the region of space bounded by the sphere of radius R, showing that from far enough away the charge appears like a point charge. The second term is similarly related to the dipole moment of the charge, and measures the grosser aspects of any deviation from spherical symmetry of the charge distribution.

21.26 *Find the Green's function for the three-dimensional Neumann problem*

$$\nabla^2\phi = 0 \quad \textit{for } z > 0 \qquad \textit{and} \qquad \frac{\partial\phi}{\partial z} = f(x, y) \quad \textit{on } z = 0.$$

Determine $\phi(x, y, z)$ *if*

$$f(x, y) = \begin{cases} \delta(y) & \textit{for } |x| < a, \\ 0 & \textit{for } |x| \geq a. \end{cases}$$

The general solution to the Poisson equation with Neumann boundary conditions is

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \langle u(\mathbf{r})\rangle_S - \int_S G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r})\, dS(\mathbf{r}),$$

where $\langle u(\mathbf{r})\rangle_S$ is the average of u over the surface S.

In the present case the charge density ρ is zero (Laplace, rather than the more general Poisson), except of course at $\mathbf{r}_0$. Further, as one of the bounding surfaces is the hemisphere at infinity in the region $z > 0$ and $u = 0$ there, as well as on the plane $z = 0$, $\langle u(\mathbf{r})\rangle_S = 0$. Thus we are left with only the third term on the RHS and

$$u(\mathbf{r}_0) = -\int_S G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r})\, dS(\mathbf{r}).$$

Now, guided by the solution to the corresponding Dirichlet problem, we are able place an image charge outside the region (i.e. an image charge with $z < 0$) and so write a suitable form for the Green's function. One fundamental difference, however, is that the image charge must have the *same* sign as the charge at $\mathbf{r}_0$; this is because it is the normal derivative of G (rather than G itself) that must vanish on S_1, the plane $z = 0$. The explicit form of the Green's functions reads

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}_0) &= -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} - \frac{1}{4\pi|\mathbf{r} + \mathbf{r}_0|} \qquad (*) \\ &= -\frac{1}{4\pi}\left\{\frac{1}{[\,(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\,]^{1/2}}\right. \\ &\qquad \left. + \frac{1}{[\,(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2\,]^{1/2}}\right\}. \end{aligned}$$

Differentiating with respect to z then gives

$$\frac{\partial G}{\partial z} = -\frac{1}{4\pi}\left\{-\frac{2(z-z_0)}{2[\ldots+(z-z_0)^2]^{3/2}} - \frac{2(z+z_0)}{2[\ldots+(z+z_0)^2]^{3/2}}\right\},$$

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = \frac{1}{4\pi}\left\{\frac{-z_0}{[\ldots+z_0^2]^{3/2}} + \frac{z_0}{[\ldots+z_0^2]^{3/2}}\right\} = 0.$$

To find the dominant term in the normal derivative of G over the hemisphere S_2 at infinity, we may neglect $\mathbf{r}_0$ when differentiating (∗) and obtain

$$\begin{aligned}\frac{\partial G}{\partial r} &= -\frac{1}{4\pi}\frac{\partial}{\partial r}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{1}{|\mathbf{r}+\mathbf{r}_0|}\right)_{r\to\infty} \\ &\simeq -\frac{1}{4\pi}\frac{\partial}{\partial r}\left(\frac{2}{|\mathbf{r}|}\right)_{r\to\infty} \\ &= \frac{1}{2\pi r^2}.\end{aligned}$$

It follows that the surface integral over the hemisphere of this normal derivative is simply

$$\int_{S_2}\frac{\partial G}{\partial r}\,dS = \frac{\partial G}{\partial r}\,2\pi r^2 = 1.$$

When this is added to the zero contribution arising from the integration of the zero derivative over S_1, a sum of unity is obtained, showing that the consistency condition for a Neumann Green's function is satisfied.

We now calculate $\phi(x,y,z)$ for the given distribution of $\partial\phi/\partial z$ on $z=0$.

$$\begin{aligned}\phi(x_0,y_0,z_0) &= \int_{-a}^{a}\int_{-\infty}^{\infty}\frac{1}{4\pi}\left\{\frac{\delta(y)}{[\,(x-x_0)^2+(y-y_0)^2+(0-z_0)^2\,]^{1/2}}\right. \\ &\qquad\left. + \frac{\delta(y)}{[\,(x-x_0)^2+(y-y_0)^2+(0+z_0)^2\,]^{1/2}}\right\}dy\,dx \\ &= \frac{2}{4\pi}\int_{-a}^{a}\frac{dx}{[\,(x-x_0)^2+y_0^2+z_0^2\,]^{1/2}}.\end{aligned}$$

Now transform the integral by setting $x-x_0=\sqrt{y_0^2+z_0^2}\sinh\theta \equiv \mu\sinh\theta$.

$$\phi(x_0,y_0,z_0) = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\frac{\mu\cosh\theta\,d\theta}{\mu(\sinh^2\theta+1)^{1/2}},$$

where $\mu \sinh\theta_1 = -a - x_0$ and $\mu \sinh\theta_2 = a - x_0$.

$$\begin{aligned}\phi(x_0, y_0, z_0) &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} d\theta \\ &= \frac{1}{2\pi}(\theta_2 - \theta_1) \\ &= \frac{1}{2\pi}\left(\sinh^{-1}\frac{a - x_0}{\sqrt{y_0^2 + z_0^2}} + \sinh^{-1}\frac{a + x_0}{\sqrt{y_0^2 + z_0^2}}\right).\end{aligned}$$

This could, of course, be equally well written in terms of unsubscripted variables x, y and z. The subscript 0 is an artefact of the notation used to indicate which quantities were to be held constant during the integration over the Green's function; unsubscripted variables were used for the integration.

21.28 *Consider the PDE $\mathcal{L}u(\mathbf{r}) = \rho(\mathbf{r})$, for which the differential operator $\mathcal{L}$ is given by*

$$\mathcal{L} = \nabla \cdot [\,p(\mathbf{r})\nabla\,] + q(\mathbf{r}),$$

where $p(\mathbf{r})$ and $q(\mathbf{r})$ are functions of position. By proving the generalised form of Green's theorem,

$$\int_V (\phi\mathcal{L}\psi - \psi\mathcal{L}\phi)\, dV = \oint_S p(\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{\mathbf{n}}\, dS,$$

show that the solution of the PDE is given by

$$\begin{aligned}u(\mathbf{r}_0) = &\int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) \\ &+ \oint_S p(\mathbf{r})\left[u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right] dS(\mathbf{r}),\end{aligned}$$

where $G(\mathbf{r}, \mathbf{r}_0)$ is the Green's function satisfying $\mathcal{L}G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$.

First, consider the divergence of the quantity in parentheses appearing in the integrand on the RHS of the supposed generalised form of Green's theorem:

$$\begin{aligned}\text{divergence} &= \nabla\cdot(p\,\phi\nabla\psi - p\,\psi\nabla\phi) \\ &= p\,\phi\nabla^2\psi + \nabla\psi\cdot\nabla(p\,\phi) - p\,\psi\nabla^2\phi - \nabla\phi\cdot\nabla(p\,\psi) \\ &= p\,\phi\nabla^2\psi + p\,\nabla\psi\cdot\nabla\phi + \phi(\nabla\psi\cdot\nabla p\,) \\ &\quad - p\,\psi\nabla^2\phi - p\,\nabla\phi\cdot\nabla\psi - \psi(\nabla\phi\cdot\nabla p\,) \\ &= p\,\phi\nabla^2\psi + \phi(\nabla\psi\cdot\nabla p\,) - p\,\psi\nabla^2\phi - \psi(\nabla\phi\cdot\nabla p\,).\end{aligned}$$

From the divergence theorem it follows that the surface integral on the RHS must be equal to the volume integral of the expression in the last line of this equation.

Next, consider the LHS of the supposed equation:

$$\begin{aligned}
\text{LHS} &= \int_V (\phi\mathcal{L}\psi - \psi\mathcal{L}\phi)\, dV \\
&= \int_V \{\phi(\nabla\cdot[\,p\,\nabla\,] + q\,)\psi - \psi(\nabla\cdot[\,p\,\nabla\,] + q\,)\phi\}\, dV \\
&= \int_V \{\phi(\nabla\cdot[\,p\,\nabla\psi\,] - \psi\nabla\cdot[\,p\,\nabla\phi\,]\}\, dV \\
&= \int_V \{\phi(\nabla p\,\cdot\nabla\psi) + \phi p\,\nabla^2\psi - \psi(\nabla p\,\cdot\nabla\phi) - \psi p\,\nabla^2\phi\}\, dV.
\end{aligned}$$

Comparison of this result with that of the previous paragraph establishes the generalised form of Green's theorem.

Now, taking $\phi(\mathbf{r}) = u(\mathbf{r})$ with $\mathcal{L}u(\mathbf{r}) = \rho(\mathbf{r})$ and $\psi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}_0)$ with $\mathcal{L}G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$, we have

$$\begin{aligned}
&\int_V u(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0)\, dV - \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) \\
&\qquad\qquad = \oint_S p(\mathbf{r}) \left[u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n} \right] dS(\mathbf{r}),
\end{aligned}$$

from which the stated result,

$$\begin{aligned}
u(\mathbf{r}_0) &= \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) \\
&\qquad + \oint_S p(\mathbf{r}) \left[u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n} \right] dS(\mathbf{r}),
\end{aligned}$$

follows immediately.

22

Calculus of variations

22.2 *Show that the lowest value of the integral*

$$\int_A^B \frac{(1+y'^2)^{1/2}}{y}\,dx,$$

where A is $(-1,1)$ *and B is* $(1,1)$, *is* $2\ln(1+\sqrt{2})$. *Assume that the Euler–Lagrange equation gives a minimising curve.*

If the integrand $F(y', y, x)$ does not contain x explicitly then a first integral of the E–L equation is that $F - y'\dfrac{\partial F}{\partial y'}$ is equal to a constant. Here,

$$\frac{(1+y'^2)^{1/2}}{y} - y'\left[\frac{y'}{y(1+y'^2)^{1/2}}\right] = C,$$

$$\frac{1}{y(1+y'^2)^{1/2}} = C.$$

On rearrangement, this gives

$$\frac{dy}{dx} = \pm\frac{(1-C^2y^2)^{1/2}}{Cy},$$

which can now be integrated:

$$\int \frac{y\,dy}{\sqrt{1-C^2y^2}} = \pm\int\frac{dx}{C},$$

$$\frac{\sqrt{1-C^2y^2}}{C^2} = \mp\frac{x}{C} + D.$$

Since the curve must pass through $(-1,1)$ and $(1,1)$, $D=0$ and

$$\sqrt{1-C^2\,1} = \mp C \quad\Rightarrow\quad C = \pm\frac{1}{\sqrt{2}}.$$

Re-substituting these values and squaring both sides of the final equation shows that $x^2 + y^2 = 2$ is the minimising curve, with $2x + 2yy' = 0$ and hence that $y' = -x/y$.

The minimal integral has the value V given by

$$
\begin{aligned}
V &= \int_{-1}^{1} \frac{[\,1 + (x/y)^2\,]^{1/2}}{y}\, dx \\
&= \int_{-1}^{1} \frac{(x^2 + y^2)^{1/2}}{y^2}\, dx \\
&= \int_{-1}^{1} \frac{\sqrt{2}}{2 - x^2}\, dx \\
&= \frac{1}{2} \int_{-1}^{1} \left(\frac{1}{\sqrt{2} - x} + \frac{1}{\sqrt{2} + x} \right) dx \\
&= \frac{1}{2} \left[\ln \frac{\sqrt{2} + x}{\sqrt{2} - x} \right]_{-1}^{1} \\
&= 2 \ln(\sqrt{2} + 1),
\end{aligned}
$$

as stated in the question.

> ***22.4*** *The Lagrangian for a π-meson is given by*
>
> $$L(\mathbf{x}, t) = \tfrac{1}{2}(\dot{\phi}^2 - |\nabla\phi|^2 - \mu^2\phi^2),$$
>
> *where μ is the meson mass and $\phi(\mathbf{x}, t)$ is its wavefunction. Assuming Hamilton's principle find the wave equation satisfied by ϕ.*

This is a situation in which there are four independent variables, x, y, z and t and so we apply the E–L equation

$$\frac{\partial L}{\partial \phi} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \phi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \phi_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \phi_z}\right) + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \phi_t}\right),$$

where $\phi_x = \dfrac{\partial \phi}{\partial x}$, etc. and $\phi_t = \dot{\phi}$.

With

$$L = \frac{1}{2}\left[\dot{\phi}^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 - \left(\frac{\partial \phi}{\partial y}\right)^2 - \left(\frac{\partial \phi}{\partial z}\right)^2 - \mu^2\phi^2 \right],$$

the function that makes $\int L\,dx\,dy\,dz\,dt$ stationary satisfies

$$-\mu^2\phi = -\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) + \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\right)$$

$$\mu^2\phi = \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2}.$$

This is the equation satisfied by the meson's wavefunction. It is known as the Klein–Gordon equation; here it is expressed in units in which $\hbar = c = 1$, where $\hbar$ is the Planck constant and c is the speed of light *in vacuo*.

22.6 *For a system specified by the coordinates q and t, show that the equation of motion is unchanged if the Lagrangian $L(q, \dot{q}, t)$ is replaced by*

$$L_1 = L + \frac{d\phi(q,t)}{dt},$$

where ϕ is an arbitrary function. Deduce that the equation of motion of a particle that moves in one dimension subject to a force $-dV(x)/dx$ (x being measured from a point O) is unchanged if O is forced to move with a constant velocity v (x still being measured from O).

We start with the Lagrangian $L(q, \dot{q}, t)$ giving an equation of motion

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right). \qquad (*)$$

Now consider

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{d\phi}{dt}\right)\right]\\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{\partial\phi}{\partial q}\,\dot{q} + \frac{\partial\phi}{\partial t}\right)\right]\\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left[\frac{\partial\phi}{\partial q} + 0 + 0\right], \qquad (**)\end{aligned}$$

since ϕ and $\dfrac{\partial\phi}{\partial t}$ do not contain $\dot{q}$.

Now, since $\phi = \phi(q, t)$,

$$\frac{d}{dt}\left(\frac{\partial\phi}{\partial q}\right) = \frac{\partial^2\phi}{\partial q^2}\,\dot{q} + \frac{\partial^2\phi}{\partial t\partial q} = \frac{\partial^2\phi}{\partial q^2}\,\dot{q} + \frac{\partial^2\phi}{\partial q\partial t} = \frac{\partial}{\partial q}\left(\frac{d\phi}{dt}\right).$$

Consequently, using $(*)$ to replace the first term on the RHS of $(**)$,

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\left(\frac{d\phi}{dt}\right) = \frac{\partial L_1}{\partial q}.$$

Thus, the equation of motion is unchanged by the addition to the Lagrangian.

When the point O is moved with constant velocity v, the potential function $V(x)$ is unchanged (as x is still measured from O) but the kinetic energy term, T, in the Lagrangian becomes

$$\begin{aligned} T_1 &= \tfrac{1}{2}m(\dot{x}+v)^2 \\ &= \tfrac{1}{2}m\dot{x}^2 + mv\dot{x} + \tfrac{1}{2}mv^2 \\ &= T + \frac{d}{dt}\,(mvx + \tfrac{1}{2}mv^2 t). \end{aligned}$$

And so the new Lagrangian is

$$L_1 = L + \frac{d}{dt}\,(mvx + \tfrac{1}{2}mv^2 t).$$

The additional term is of the form $d\phi(x,t)/dt$ previously considered and therefore the equations of motion are not changed.

> ***22.8*** *Derive the differential equations for the plane-polar coordinates r, ϕ of a particle of unit mass moving in a field of potential $V(r)$. Find the form of V if the path of the particle is given by $r = a\sin\phi$.*

In plane polar coordinates the kinetic energy of a particle of unit mass is $T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$ and so the Lagrangian is

$$L = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r).$$

Hamilton's principle, that the integral of L with respect to time t (the independent variable) is stationary, gives the E–L equations for the two dependent variables r and ϕ in their usual form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i},$$

which in this case implies that

$$\frac{d}{dt}(\dot{r}) = r\dot{\phi}^2 - \frac{\partial V}{\partial r} \quad \text{and} \quad \frac{d}{dt}(r^2\dot{\phi}) = 0.$$

The second of these expresses angular momentum conservation as $r^2\dot{\phi} = k$, whilst the first can be interpreted physically as saying that the actual radial acceleration is the difference between the outward centripetal acceleration and the inward one due to the potential field.

If the actual path is $r = a\sin\phi$, then this must be a solution of these coupled equations. The path in this form does not give the time dependence of either variable and so we must aim to eliminate time and differentiations with respect to it. In particular, we need an expression for $\ddot{r}$ that does not contain time.

$$\dot{r} = a\cos\phi\,\dot{\phi} = \frac{ak}{r^2}\cos\phi,$$

$$\begin{aligned}\ddot{r} &= -\frac{2ak\cos\phi}{r^3}\,\dot{r} + \frac{ak(-\sin\phi)}{r^2}\,\dot{\phi}\\ &= -\frac{2a^2k^2\cos^2\phi}{r^5} - \frac{kr}{r^2}\,\frac{k}{r^2}\\ &= -\frac{2(a^2-r^2)k^2}{r^5} - \frac{k^2}{r^3}.\end{aligned}$$

Substitution in the radial equation now gives

$$-\frac{2(a^2-r^2)k^2}{r^5} - \frac{k^2}{r^3} = r\frac{k^2}{r^4} - \frac{\partial V}{\partial r} \quad \Rightarrow \quad \frac{\partial V}{\partial r} = \frac{2k^2a^2}{r^5}.$$

From this we conclude that

$$V(r) = -\frac{k^2a^2}{2r^4} + c$$

and that the potential is an inverse fourth-power law. This (admittedly unphysical) situation is of interest because the particle's orbit passes through the centre of force, and with infinite speed, in theory. This raises the question of relativistic effects ... .

22.10 *Extend to the case of several dependent variables $y_i(x)$, the standard result about the first integral of the E–L equation when x does not appear explicitly in the general integrand $F(y_i', y_i, x)$. In particular, show that the first integral is*

$$F - \sum_{i=1}^{n} y_i'\frac{\partial F}{\partial y_i'} = \text{constant}.$$

For each of the dependent variables y_i, $i = 1, 2, \ldots, n$, we have

$$\frac{\partial F}{\partial y_i} = \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right).$$

These n equations can be manipulated as follows.

$$y_i'\frac{\partial F}{\partial y_i} = y_i'\frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = \frac{d}{dx}\left(y_i'\frac{\partial F}{\partial y_i'}\right) - y_i''\frac{\partial F}{\partial y_i'},$$

$$y_i''\frac{\partial F}{\partial y_i'} + y_i'\frac{\partial F}{\partial y_i} = \frac{d}{dx}\left(y_i'\frac{\partial F}{\partial y_i'}\right),$$

$$\sum_{i=1}^{n}\left(y_i''\frac{\partial F}{\partial y_i'} + y_i'\frac{\partial F}{\partial y_i}\right) = \frac{d}{dx}\left[\sum_{i=1}^{n}\left(y_i'\frac{\partial F}{\partial y_i'}\right)\right].$$

But, since $F \neq F(x)$, the LHS is the total derivative of F, i.e. $\dfrac{dF}{dx}$. Thus

$$\frac{dF}{dx} = \frac{d}{dx}\left[\sum_{i=1}^{n}\left(y_i'\frac{\partial F}{\partial y_i'}\right)\right],$$

$$\Rightarrow \quad F - \sum_{i=1}^{n}\left(y_i'\frac{\partial F}{\partial y_i'}\right) = k,$$

is the first integral of the E–L equations.

> ***22.12*** *Light travels in the vertical xz-plane through a slab of material which lies between the planes $z = z_0$ and $z = 2z_0$ and in which the speed of light $v(z) = c_0 z/z_0$. Using Fermat's principle in the form that the travel time is minimised, show that the ray paths are arcs of circles.*
>
> *Deduce that, if a ray enters the material at $(0, z_0)$ at an angle to the vertical, $\pi/2 - \theta$, of more than 30° then it does not reach the far side of the slab.*

We start with the three defining equations

$$v(z) = c_0\frac{z}{z_0}, \qquad (ds)^2 = (dx)^2 + (dz)^2$$

and

$$t = \int \frac{ds}{v} = \int \frac{(1+z'^2)^{1/2}}{v(z)}\,dx = \frac{z_0}{c_0}\int \frac{(1+z'^2)^{1/2}}{z}\,dx.$$

The independent variable x is not present in the integrand F and so a first integral of the E–L equation is $F - z'\dfrac{\partial F}{\partial z'} = k$:

$$\frac{(1+z'^2)^{1/2}}{z} - \frac{z'}{z}\frac{z'}{(1+z'^2)^{1/2}} = k,$$

$$\frac{1}{z(1+z'^2)^{1/2}} = k,$$

$$\frac{z\,dz}{(A-z^2)^{1/2}} = dx, \quad \text{where } A = k^{-1},$$

$$\Rightarrow \quad -(A-z^2)^{1/2} = x + B,$$

$$(x+B)^2 + z^2 = (\sqrt{A})^2.$$

This is a circle of radius $\sqrt{A}$ centred on $(-B, 0)$.

If the ray enters the slab at $(0, z_0)$ with $\dfrac{dz}{dx} = \tan\theta$, then

$$B^2 + z_0^2 = A \quad \text{and} \quad \frac{A - z_0^2}{z_0^2} = \left(\frac{dz}{dx}\right)^2_{x=0} = \tan^2\theta.$$

From these it follows that

$$\sqrt{A} = z_0 \sec\theta \quad \text{and} \quad B = z_0 \tan\theta.$$

The ray is horizontal when $z' = 0$, i.e. when $z^2 = A$, i.e. when $z = z_0 \sec\theta$. This will be below the top of the slab if $z_0 \sec\theta < 2z_0$, i.e. if $\cos\theta > \frac{1}{2}$. This requires $\theta < 60^\circ$ and $\pi/2 - \theta$ to be more than 30°. When this happens the ray will not reach the far side of the slab.

> ***22.14*** *In the brachistochrone problem of subsection 22.3.4 show that if the upper end-point can lie anywhere on the curve $h(x, y) = 0$ then the curve of quickest descent $y(x)$ meets $h(x, y) = 0$ at right angles.*

The slope m_h of the curve $h(x, y) = 0$ is given by

$$\frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy = 0 \quad \Rightarrow \quad m_h = \frac{dy}{dx} = -\left.\frac{\partial h}{\partial x}\right/\frac{\partial h}{\partial y}.$$

For the brachistochrone,

$$F(y, y', x) = F(y, y') = \sqrt{\frac{1 + y'^2}{y}},$$

giving

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y}}\,\frac{y'}{\sqrt{1 + y'^2}}$$

and

$$F - y'\frac{\partial F}{\partial y'} = \frac{\sqrt{1 + y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1 + y'^2}} = \frac{1}{\sqrt{y}\sqrt{1 + y'^2}}.$$

The end-point condition, equation (22.20), is

$$\left(F - y'\frac{\partial F}{\partial y'}\right)\frac{\partial h}{\partial y} - \frac{\partial F}{\partial y'}\,\frac{\partial h}{\partial x} = 0,$$

$$\frac{1}{\sqrt{y}\sqrt{1 + y'^2}}\,\frac{\partial h}{\partial y} - \frac{1}{\sqrt{y}}\,\frac{y'}{\sqrt{1 + y'^2}}\,\frac{\partial h}{\partial x} = 0,$$

$$\frac{\partial h}{\partial y} - y'\frac{\partial h}{\partial x} = 0,$$

$$y' = \left.\frac{\partial h}{\partial y}\right/\frac{\partial h}{\partial x} = -\frac{1}{m_h}.$$

The condition for curves of slopes m_1 and m_2 to meet at right angles is $m_1 m_2 = -1$. This condition is satisfied here and we conclude that the curves $y = y(x)$ and $h(x, y) = 0$ meet at right angles at the upper end point.

22.16 *Use the result*

$$\int_a^b (y_j' p y_i' - y_j q y_i)\,dx = \lambda_i \delta_{ij}$$

to evaluate

$$J = \int_{-1}^{1} (1 - x^2) P_m'(x) P_n'(x)\,dx,$$

where $P_m(x)$ is a Legendre polynomial of order m.

The result

$$\int_a^b (y_j' p y_i' - y_j q y_i)\,dx = \lambda_i \delta_{ij}, \qquad (*)$$

applies to normalised eigenfunctions of a Sturm–Liouville equation. Legendre's equation is such an equation, with $p(x) = (1 - x^2)$, $q(x) = 0$ and $\rho(x) = 1$. The limits are $a = -1$ and $b = 1$.

The normalised Legendre function corresponding to eigenvalue $m(m+1)$ is

$$y_m(x) = \sqrt{\frac{2m+1}{2}}\, P_m(x),$$

and so $(*)$ reads

$$\int_{-1}^{1} \left[\sqrt{\frac{2m+1}{2}}\, P_m'(x)(1 - x^2) \sqrt{\frac{2n+1}{2}}\, P_n'(x) - 0 \right] dx = m(m+1)\delta_{mn}.$$

From this it follows immediately that

$$J = \int_{-1}^{1} (1 - x^2) P_m'(x) P_n'(x)\,dx = \frac{2m(m+1)}{2m+1}\,\delta_{mn}.$$

22.18 *Show that $y'' - xy + \lambda x^2 y = 0$ has a solution for which $y(0) = y(1) = 0$ and $\lambda \leq 147/4$.*

The equation is already in S–L form with $p = 1$, $q = -x$ and $\rho = x^2$. The boundary conditions require $y(0) = y(1) = 0$. The simplest polynomial that satisfies these conditions is $y(x) = x(1 - x)$ and so we use this as a trial function. For *any* trial function the lowest eigenvalue λ_0 must satisfy

$$\lambda_0 \leq \frac{\int (p y'^2 - q y^2)\,dx}{\int \rho y^2\,dx}.$$

With the trial function we have chosen, this means that

$$\lambda_0 \le \frac{\int_0^1 [\,(1)(1-2x)^2 - (-x)x^2(1-x)^2\,]\,dx}{\int_0^1 x^2 x^2 (1-x)^2\,dx}$$
$$= \frac{\int_0^1 (x^5 - 2x^4 + x^3 + 4x^2 - 4x + 1)\,dx}{\int_0^1 (x^6 - 2x^5 + x^4)\,dx}$$
$$= \frac{\frac{1}{6} - \frac{2}{5} + \frac{1}{4} + \frac{4}{3} - 2 + 1}{\frac{1}{7} - \frac{2}{6} + \frac{1}{5}}$$
$$= \frac{10 - 24 + 15 + 80 - 120 + 60}{60} \frac{210}{30 - 70 + 42}$$
$$= \frac{21}{60}\frac{210}{2} = \frac{147}{4}.$$

Thus, there must be a solution of the differential equation for which $y(0) = y(1) = 0$ and $\lambda \le 147/4$. In fact, the inequality sign must hold since the trial function used is *not* a solution to the given equation, as can be easily verified by substitution.

> ***22.20*** *Estimate the lowest eigenvalue λ_0 of the equation*
>
> $$\frac{d^2y}{dx^2} - x^2 y + \lambda y = 0, \qquad y(-1) = y(1) = 0,$$
>
> *using a quadratic trial function.*

Following the normal procedure for an S–L equation with, in this case, $p = 1$, $q = -x^2$, $\rho = 1$ and a quadratic trial function $y(x) = 1 - x^2$ chosen to fit the boundary conditions, we obtain

$$\lambda_0 \le \frac{\int_{-1}^1 [\,4x^2 + x^2(1-x^2)^2\,]\,dx}{\int_{-1}^1 (1-x^2)^2\,dx}$$
$$= \frac{\int_{-1}^1 (5x^2 - 2x^4 + x^6)\,dx}{\int_{-1}^1 (1 - 2x^2 + x^4)\,dx}$$
$$= \frac{\frac{10}{3} - \frac{4}{5} + \frac{2}{7}}{2 - \frac{4}{3} + \frac{2}{5}}$$
$$= \frac{350 - 84 + 30}{105}\frac{15}{30 - 20 + 6}$$
$$= \frac{296}{105}\frac{15}{16} = \frac{37}{14}.$$

We also note that this problem can be recast to use the Rayleigh–Ritz principle by writing the integrand in the numerator as $y(x)\left(-\dfrac{d^2}{dx^2}+x^2\right)y(x)$. With the same trial function, the same upper bound is obtained.

22.22 *Consider the problem of finding the lowest eigenvalue λ_0 of the equation*

$$(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\lambda y=0, \qquad y(\pm 1)=0.$$

(a) *Recast the problem in variational form, and derive an approximation λ_1 to λ_0 by using the trial function $y_1(x)=1-x^2$.*
(b) *Show that an improved estimate λ_2 is obtained by using $y_2(x)=\cos(\pi x/2)$.*
(c) *Prove that the estimate $\lambda(\gamma)$ obtained by taking $y_1(x)+\gamma y_2(x)$ as the trial function is*

$$\lambda(\gamma)=\frac{64/15+64\gamma/\pi-384\gamma/\pi^3+(\pi^2/3+1/2)\gamma^2}{16/15+64\gamma/\pi^3+\gamma^2}.$$

Investigate $\lambda(\gamma)$ numerically as γ is varied, or, more simply, show that $\lambda(-1.80)=3.668$, an improvement on both λ_1 and λ_2.

The given equation is already in S–L form with $p(x)=1+x^2$, $q=0$ and $\rho=1$. We therefore take

$$I=\int_{-1}^{1}(1+x^2)y'^2\,dx \quad \text{and} \quad J=\int_{-1}^{1}y^2\,dx,$$

where $y(x)$ must satisfy $y(\pm 1)=0$, to estimate λ_0 as I/J.

(a) With trial function $y_1(x)=1-x^2$, we have as a first approximation

$$\lambda_1=\frac{\int_{-1}^{1}(1+x^2)4x^2\,dx}{\int_{-1}^{1}(1-2x^2+x^4)\,dx}
=\frac{4\left(\frac{2}{3}+\frac{2}{5}\right)}{2-\frac{4}{3}+\frac{2}{5}}=\frac{4(10+6)}{30-20+6}=4.$$

(b) The corresponding calculation for trial function $y_2(x)=\cos(\pi x/2)$ is

$$\lambda_2=\frac{\int_{-1}^{1}(1+x^2)\frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{2}\right)\,dx}{\int_{-1}^{1}\cos^2\left(\frac{\pi x}{2}\right)\,dx}
=\frac{\frac{\pi^2}{4}\left(1+\frac{1}{3}+\frac{2}{\pi^2}\right)}{1}=\frac{\pi^2}{3}+\frac{1}{2}=3.79, \text{ (an improvement).}$$

To evaluate the integral we used

$$\begin{aligned}\int_{-1}^{1} x^2 \sin^2\left(\frac{\pi x}{2}\right)\,dx &= \frac{1}{2}\int_{-1}^{1} x^2(1-\cos\pi x)\,dx \\ &= \frac{1}{3} - \frac{1}{2}\left[\frac{x^2\sin\pi x}{\pi}\right]_{-1}^{1} + \int_{-1}^{1}\frac{2x\sin\pi x}{2\pi}\,dx \\ &= \frac{1}{3} + 0 + \frac{1}{\pi}\left[\frac{-x\cos\pi x}{\pi}\right]_{-1}^{1} + \frac{1}{\pi^2}\int_{-1}^{1}\cos\pi x\,dx \\ &= \frac{1}{3} + \frac{2}{\pi^2} + 0.\end{aligned}$$

(c) Taking as a third trial function the linear combination of $y_3 = y_1 + \gamma y_2$, where γ is an adjustable parameter, we have

$$y_3(x) = 1 - x^2 + \gamma\cos\left(\frac{\pi x}{2}\right) \quad \text{and} \quad y_3' = -2x - \frac{\pi\gamma}{2}\sin\left(\frac{\pi x}{2}\right).$$

To evaluate the integrals I and J we will need the following additional results.

$$\begin{aligned}\int_{-1}^{1}\cos\left(\frac{\pi x}{2}\right)\,dx &= \left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} = \frac{4}{\pi}, \\ \int_{-1}^{1} x\sin\left(\frac{\pi x}{2}\right)\,dx &= \left[-\frac{2x}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} + \int_{-1}^{1}\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\,dx \\ &= 0 + \frac{2}{\pi}\,\frac{4}{\pi} = \frac{8}{\pi^2}, \\ \int_{-1}^{1} x^2\cos\left(\frac{\pi x}{2}\right) &= \left[\frac{2x^2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} - \int_{-1}^{1}\frac{4x}{\pi}\sin\left(\frac{\pi x}{2}\right)\,dx \\ &= \frac{4}{\pi} - \frac{4}{\pi}\,\frac{8}{\pi^2}, \\ \int_{-1}^{1} x^3\sin\left(\frac{\pi x}{2}\right) &= \left[-\frac{2x^3}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} + \int_{-1}^{1}\frac{6x^2}{\pi}\cos\left(\frac{\pi x}{2}\right)\,dx \\ &= 0 + \frac{6}{\pi}\left(\frac{4}{\pi} - \frac{32}{\pi^3}\right).\end{aligned}$$

We can now calculate I as

$$\begin{aligned}I &= \int_{-1}^{1}(1+x^2)\left[4x^2 + 2\pi\gamma x\sin\left(\frac{\pi x}{2}\right) + \frac{\pi^2\gamma^2}{4}\sin^2\left(\frac{\pi x}{2}\right)\right]\,dx \\ &= 4\left(\frac{2}{3}+\frac{2}{5}\right) + 2\pi\gamma\frac{8}{\pi^2} + 2\pi\gamma\,\frac{6}{\pi}\left(\frac{4}{\pi} - \frac{32}{\pi^3}\right) + \frac{\pi^2\gamma^2}{4}\left(1 + \frac{1}{3} + \frac{2}{\pi^2}\right) \\ &= \frac{64}{15} + \frac{64\gamma}{\pi}\left(1 - \frac{6}{\pi^2}\right) + \gamma^2\left(\frac{\pi^2}{3} + \frac{1}{2}\right).\end{aligned}$$

The corresponding calculation for J is

$$J = \int_{-1}^{1} \left[(1-x^2)^2 + 2\gamma(1-x^2)\cos\left(\frac{\pi x}{2}\right) + \gamma^2 \cos^2\left(\frac{\pi x}{2}\right) \right] dx$$
$$= 2 - \frac{4}{3} + \frac{2}{5} + 2\gamma\frac{4}{\pi} - 2\gamma\left(\frac{4}{\pi} - \frac{32}{\pi^3}\right) + \gamma^2$$
$$= \frac{16}{15} + \frac{64\gamma}{\pi^3} + \gamma^2.$$

Inserting numerical values, we find that the estimate of λ_0 is

$$\frac{I}{J} = \frac{4.2667 + 7.9872\gamma + 3.7898\gamma^2}{1.0667 + 2.0641\gamma + \gamma^2}.$$

This reproduces results (a) and (b) for $\gamma = 0$ and $\gamma \gg 1$ respectively, as expected. However some numerical experimentation shows that the ratio drops to 3.6653 when $\gamma = -1.694$, thus providing a better upper limit than either (a) or (b).

22.24 *This is an alternative approach to the example in section 22.8. Using the notation of that section, the expectation value of the energy of the state ψ is given by $\int \psi^* H\psi\, dv$. Denote the eigenfunctions of H by ψ_i, so that $H\psi_i = E_i\psi_i$, and, since H is Hermitian, $\int \psi_j^*\psi_i\, dv = \delta_{ij}$.*

(a) *By writing any function ψ as $\sum c_j\psi_j$ and following an argument similar to that in section 22.7, show that*

$$E = \frac{\int \psi^* H\psi\, dv}{\int \psi^*\psi\, dv} \geq E_0,$$

the energy of the lowest state. This is the Rayleigh–Ritz principle.

(b) *Using the same trial function as in section 22.8, $\psi = \exp(-\alpha x^2)$, show that the same result is obtained.*

In order to find the energy E_0 of the lowest state, we seek to minimise

$$\langle H \rangle = \int \psi^* H\psi\, dv \text{ subject to } \int \psi^*\psi\, dv = 1.$$

(a) We begin by writing the trial function ψ as a linear combination of the eigenfunctions ψ_i of the Hamiltonian H; they satisfy $H\psi_i = E_i\psi_i$ and $\int \psi_j^*\psi_i\, dv = \delta_{ij}$. Thus

$$\psi = \sum_j c_j\psi_j,$$

where the c_j, as well as the ψ_j, can be complex. This results in an expression for $\langle H \rangle$ that is a double summation:

$$\begin{aligned}\langle H \rangle &= \int \left(\sum_j c_j^* \psi_j^* \right) H \left(\sum_i c_i \psi_i \right) dv \\ &= \sum_{i\,j} \int c_j^* c_i \, \psi_j^* \, E_i \psi_i \, dv, \quad \text{using } H\psi_i = E_i \psi_i, \\ &= \sum_{i\,j} c_j^* c_i \, E_i \delta_{ij} \\ &= \sum_i |c_i|^2 E_i.\end{aligned}$$

We also have for the normalisation integral of ψ that

$$\begin{aligned}\int \psi^* \psi \, dv &= \int \left(\sum_j c_j^* \psi_j^* \right) \left(\sum_i c_i \psi_i \right) dv \\ &= \sum_{i\,j} \int c_j^* c_i \, \psi_j^* \psi_i \, dv, \\ &= \sum_{i\,j} c_j^* c_i \, \delta_{ij} \\ &= \sum_i |c_i|^2.\end{aligned}$$

Now, since E_0 is the energy of the lowest state, $E_i \geq E_0$ for all i. Consequently

$$E = \frac{\int \psi^* H \psi \, dv}{\int \psi^* \psi \, dv} = \frac{\sum_i |c_i|^2 E_i}{\sum_i |c_i|^2} \geq \frac{\sum_i |c_i|^2 E_0}{\sum_i |c_i|^2} = E_0.$$

(b) In section 22.8 the Hamiltonian operator has the form

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{kx^2}{2}.$$

Denote the integral $\int_{-\infty}^{\infty} x^2 \exp(-2\alpha x^2) \, dx$ by J. Then, for the trial function $\psi = \exp(-\alpha x^2)$,

$$\begin{aligned}\langle H \rangle &= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-\alpha x^2} \frac{d^2}{dx^2}(e^{-\alpha x^2}) \, dx + \frac{k}{2} \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2} \, dx \\ &= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-\alpha x^2} (4\alpha^2 x^2 - 2\alpha) e^{-\alpha x^2} \, dx + \frac{k}{2} J \\ &= -\frac{4\hbar^2 \alpha^2 J}{2m} + \frac{\hbar^2 \alpha}{m} \int_{-\infty}^{\infty} e^{-2\alpha x^2} \, dx + \frac{k}{2} J.\end{aligned}$$

Now, from integrating the definition of J by parts,

$$J = \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2}\, dx$$
$$= \left[-\frac{x}{4\alpha} e^{-2\alpha x^2} \right]_{-\infty}^{\infty} + \frac{1}{4\alpha} \int_{-\infty}^{\infty} e^{-2\alpha x^2}\, dx,$$
$$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-2\alpha x^2}\, dx = 4\alpha J.$$

We can now conclude that

$$E = \frac{\langle H \rangle}{\int \psi^* \psi\, dv} = \frac{1}{4\alpha J} \left(-\frac{2\hbar^2 \alpha^2 J}{m} + \frac{4\hbar^2 \alpha^2 J}{m} + \frac{kJ}{2} \right) = \frac{\hbar^2 \alpha}{2m} + \frac{k}{8\alpha}.$$

This is exactly the same expression for E as that obtained in equation (22.34) and so when it is minimised with respect to α it gives the same result, $\frac{1}{2}\hbar(k/m)^{1/2}$, for the upper limit on the ground state energy.

22.26 *The Hamiltonian H for the hydrogen atom is*

$$-\frac{\hbar^2}{2m}\nabla^2 - \frac{q^2}{4\pi\epsilon_0 r}.$$

For a spherically symmetric state, as may be assumed for the ground state, the only relevant part of ∇^2 *is that involving differentiation with respect to* r.

(a) *Define the integrals* J_n *by*

$$J_n = \int_0^{\infty} r^n e^{-2\beta r}\, dr$$

and show that, for a trial wavefunction of the form $\exp(-\beta r)$ *with* $\beta > 0$, $\int \psi^* H \psi\, dv$ *and* $\int \psi^* \psi\, dv$ *can be expressed as* $aJ_1 - bJ_2$ *and* cJ_2 *respectively, where* a, b, c *are factors which you should determine.*

(b) *Show that the Rayleigh–Ritz estimate of E is minimised when* β *takes the value* $mq^2/(4\pi\epsilon_0\hbar^2)$.

(c) *Hence find an upper limit for the ground-state energy of the hydrogen atom. In fact,* $\exp(-\beta r)$ *is the correct form for the wavefunction and the limit gives the actual value.*

Working in spherical polar coordinates, the expression for $H\psi$, where ψ is a spherically symmetric state, takes the form

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr} \right) - \frac{q^2 \psi}{4\pi\epsilon_0 r}.$$

If $\psi(r) = e^{-\beta r}$,

$$\begin{aligned}\langle H \rangle &= -\frac{\hbar^2}{2m}\int_0^\infty 4\pi r^2 \frac{e^{-\beta r}}{r^2}\frac{d}{dr}[\,r^2(-\beta)e^{-\beta r}\,]\,dr \\ &\qquad\qquad - \frac{q^2}{4\pi\epsilon_0}\int_0^\infty \frac{4\pi r^2 e^{-2\beta r}}{r}\,dr \\ &= \frac{2\pi\hbar^2\beta}{m}\int_0^\infty e^{-\beta r}(2r - r^2\beta)e^{-\beta r}\,dr - \frac{q^2}{\epsilon_0}J_1 \\ &= \frac{2\pi\hbar^2\beta}{m}(2J_1 - \beta J_2) - \frac{q^2}{\epsilon_0}J_1.\end{aligned}$$

Thus $a = \dfrac{4\pi\hbar^2\beta}{m} - \dfrac{q^2}{\epsilon_0}$ and $b = \dfrac{2\pi\hbar^2\beta^2}{m}$. Also

$$\int \psi^*\psi\,dv = \int_0^\infty 4\pi r^2 e^{-2\beta r}\,dr = 4\pi J_2 \quad\Rightarrow\quad c = 4\pi.$$

(b) The estimate of E is

$$\frac{\langle H\rangle}{\int \psi^*\psi\,dv} = \left(\frac{\hbar^2\beta}{m} - \frac{q^2}{4\pi\epsilon_0}\right)\frac{J_1}{J_2} - \frac{\hbar^2\beta^2}{2m}.$$

Now, integration by parts gives the relationship

$$J_2 = \int_0^\infty r^2 e^{-2\beta r}\,dr = \left[\frac{r^2 e^{-2\beta r}}{-2\beta}\right]_0^\infty - 2\int_0^\infty \frac{r e^{-2\beta r}}{-2\beta}\,dr = \frac{1}{\beta}J_1.$$

Hence,

$$E_{\text{estim}} = \frac{\hbar^2\beta^2}{m} - \frac{q^2\beta}{4\pi\epsilon_0} - \frac{\hbar^2\beta^2}{2m} = \frac{\hbar^2\beta^2}{2m} - \frac{q^2\beta}{4\pi\epsilon_0}.$$

This is minimised when the parameter β is chosen to satisfy

$$0 = \frac{\partial E_{\text{estim}}}{\partial\beta} = \frac{\hbar^2\beta}{m} - \frac{q^2}{4\pi\epsilon_0} \quad\Rightarrow\quad \beta = \frac{q^2 m}{4\pi\epsilon_0\hbar^2}.$$

(c) The upper limit on the ground-state energy of the hydrogen atom provided by this form of trial function is therefore

$$\frac{\hbar^2}{2m}\frac{q^4 m^2}{(4\pi\epsilon_0)^2\hbar^4} - \frac{q^4 m}{(4\pi\epsilon_0)^2\hbar^2} = -\frac{q^4 m}{2(4\pi\epsilon_0\hbar)^2}.$$

As noted in the question, the trial wavefunction happens to be of the correct form and the estimate obtained for the ground state energy is the actual one (within the limits of the model Hamiltonian used).

22.28 *A particle of mass m moves in a one-dimensional potential well of the form*

$$V(x) = -\mu \frac{\hbar^2 \alpha^2}{m} \text{sech}^2 \alpha x,$$

where μ and α are positive constants. The expectation value $\langle E \rangle$ of the energy of the system is $\int \psi^* H \psi \, dx$, where the self-adjoint operator $H = -(\hbar^2/2m)d^2/dx^2 + V(x)$. Using trial wavefunctions of the form $y = A \,\text{sech}\, \beta x$, show the following:

(a) *for $\mu = 1$ there is an exact eigenfunction of H, with a corresponding $\langle E \rangle$ of half of the maximum depth of the well;*
(b) *for $\mu = 6$ the 'binding energy' of the ground state is at least $10\hbar^2\alpha^2/(3m)$.*

[*You will find it useful to note that for $u, v \geq 0$,* $\text{sech}\, u \,\text{sech}\, v \geq \text{sech}\,(u+v)$.]

To test for an exact eigenfunction we need to consider the relevant differential equation (here the Schrödinger equation). This is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - \mu \frac{\hbar^2 \alpha^2}{m} \text{sech}^2(\alpha x)\, \psi = E\psi.$$

With $y = A \,\text{sech}\, \beta x$ as a trial function,

$$\begin{aligned}
\psi(x) &= A \,\text{sech}\, \beta x, \\
\psi'(x) &= -A\beta \,\text{sech}\, \beta x \tanh \beta x, \\
\psi''(x) &= A\beta^2 \text{sech}\, \beta x \tanh^2 \beta x - A\beta^2 \text{sech}^3 \beta x.
\end{aligned}$$

So ψ will be a solution provided (cancelling $\text{sech}\, \beta x$ throughout)

$$\begin{aligned}
\beta^2 \tanh^2 \beta x - \beta^2 \text{sech}^2 \beta x + 2\mu\alpha^2 \text{sech}^2 \alpha x &= -\frac{2mE}{\hbar^2}, \\
\beta^2 - 2\beta^2 \text{sech}^2 \beta x + 2\mu\alpha^2 \text{sech}^2 \alpha x &= -\frac{2mE}{\hbar^2}.
\end{aligned}$$

(a) For $\mu = 1$ this equation is satisfied if $\beta = \alpha$ and $E = -\dfrac{\hbar^2\alpha^2}{2m}$. The binding energy, which is the negative of the total energy, is therefore $\dfrac{\hbar^2\alpha^2}{2m}$, i.e. half the maximum depth of the well, $\dfrac{\hbar^2\alpha^2}{m} \text{sech}^2(0)$.

(b) For $\mu = 6$ an exact solution of the given form is not possible, but an upper limit can be placed on the ground state energy.

First,

$$\int \psi^* \psi \, dx = \int_{-\infty}^{\infty} A^2 \text{sech}^2 \beta x \, dx = A^2 \left[\frac{\tanh \beta x}{\beta} \right]_{-\infty}^{\infty} = \frac{2A^2}{\beta}.$$

Next, writing $H = T + V$, we have from the previous expression for ψ'' that

$$\begin{aligned}\langle T\rangle &= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} A\,\text{sech}\,\beta x(A\beta^2\text{sech}\,\beta x\tanh^2\beta x - A\beta^2\text{sech}\,^3\beta x)\,dx\\ &= -\frac{\hbar^2\beta^2}{2m}A^2\int_{-\infty}^{\infty}\text{sech}\,^2\beta x(\tanh^2\beta x - \text{sech}\,^2\beta x)\,dx\\ &= -\frac{\hbar^2\beta^2}{2m}A^2\int_{-\infty}^{\infty}\text{sech}\,^2\beta x(2\tanh^2\beta x - 1)\,dx\\ &= -\frac{\hbar^2\beta^2}{2m}A^2\left[\frac{2}{3\beta}\tanh^3\beta x - \frac{1}{\beta}\tanh\beta x\right]_{-\infty}^{\infty}\\ &= -\frac{\hbar^2\beta}{2m}A^2\left(\frac{4}{3}-2\right) = \frac{\hbar^2\beta}{3m}A^2.\end{aligned}$$

For the expectation of the contribution to H from the potential term we have

$$\begin{aligned}\langle V\rangle &= -\frac{6\hbar^2\alpha^2}{m}\int_{-\infty}^{\infty}A^2\text{sech}\,^2\alpha x\,\text{sech}\,^2\beta x\,dx\\ &= -\frac{12\hbar^2\alpha^2}{m}\int_0^{\infty}A^2(\text{sech}\,\alpha x\,\text{sech}\,\beta x)^2\,dx\\ &\le -\frac{12\hbar^2\alpha^2}{m}\int_0^{\infty}A^2\text{sech}\,^2[\,(\alpha+\beta)x\,]\,dx, \text{ using hint and } \langle V\rangle < 0,\\ &= -\frac{12\hbar^2\alpha^2}{m}A^2\left[\frac{\tanh[\,(\alpha+\beta)x\,]}{\alpha+\beta}\right]_0^{\infty}\\ &= -\frac{12\hbar^2\alpha^2}{m(\alpha+\beta)}A^2.\end{aligned}$$

Thus, recalling that $\displaystyle\int \psi^*\psi\,dx = \frac{2A^2}{\beta}$,

$$E = \frac{\langle T\rangle + \langle V\rangle}{\int\psi^*\psi\,dx} \le \frac{\hbar^2\beta^2}{6m} - \frac{6\hbar^2\alpha^2\beta}{m(\alpha+\beta)}.$$

The upper limit is minimised with respect to β when β satisfies

$$\begin{aligned}\frac{\beta}{3} - \frac{6\alpha^2}{\alpha+\beta} + \frac{6\alpha^2\beta}{(\alpha+\beta)^2} &= 0,\\ \beta(\alpha+\beta)^2 - 18\alpha^2(\alpha+\beta) + 18\alpha^2\beta &= 0,\\ \beta^3 + 2\alpha\beta^2 + \alpha^2\beta - 18\alpha^3 &= 0,\\ (\beta - 2\alpha)(\beta^2 + 4\alpha\beta + 9\alpha^2) &= 0.\end{aligned}$$

Thus $\beta = 2\alpha$ or $\beta = -2\alpha \pm i\sqrt{5}\alpha$; only the first is a real turning point. With this choice

$$E \le \frac{4\hbar^2\alpha^2}{6m} - \frac{12\hbar^2\alpha^3}{3m\alpha} = -\frac{10}{3}\frac{\hbar^2\alpha^2}{m}.$$

Since this gives an upper limit on the ground state energy, and $V(x) \to 0$ as $x \to \pm\infty$, the binding energy of the ground state must be at least $\frac{10}{3}\frac{\hbar^2\alpha^2}{m}$.

23

Integral equations

23.2 *Solve*

$$\int_0^\infty f(t)\exp(-st)\,dt = \frac{a}{a^2+s^2}.$$

Although this is an integral equation, we note that its LHS is also the definition of the Laplace transform of $f(t)$.

The solution to the equation is therefore the inverse Laplace transform of $\dfrac{a}{a^2+s^2}$. This is given in standard tables, which show that $f(t) = \sin at$.

23.4 *Use the fact that its kernel is separable to solve for $y(x)$ the integral equation*

$$y(x) = A\cos(x+a) + \lambda\int_0^\pi \sin(x+z)y(z)\,dz.$$

[*This equation is an inhomogeneous extension of the homogeneous Fredholm equation (23.13), and is similar to equation (23.57).*]

The kernel is separable because the equation can be written

$$\begin{aligned} y(x) &= A\cos(x+a) + \lambda\int_0^\pi \sin(x+z)y(z)\,dz \\ &= A\cos(x+a) + \lambda\int_0^\pi [\,\sin x\cos z\,y(z) + \cos x\sin z\,y(z)\,]\,dz, \end{aligned}$$

i.e. the kernel consists of a sum of terms each of which is the direct product of a function of x and a function of z.

So, we take $y(x)$ as a linear sum of the functions of x that appear in the integrand, explicitly $y(x) = c_1 \sin x + c_2 \cos x$. When this form is substituted into both sides of the integral equation (with z as its argument on the RHS), we obtain

$$\begin{aligned} c_1 \sin x + c_2 \cos x = A \cos x \cos a - A \sin x \sin a \\ + \lambda \sin x \int_0^\pi (c_1 \cos z \sin z + c_2 \cos^2 z)\, dz \\ + \lambda \cos x \int_0^\pi (c_1 \sin^2 z + c_2 \sin z \cos z)\, dz. \end{aligned}$$

Equating the coefficients of $\sin x$ and $\cos x$, and recalling that the average value of the square of a sinusoid over a whole number of half periods is $\frac{1}{2}$, gives

$$\begin{aligned} c_1 &= -A \sin a + \tfrac{1}{2}\lambda\pi c_2, \\ c_2 &= A \cos a + \tfrac{1}{2}\lambda\pi c_1. \end{aligned}$$

Solving this pair of simultaneous equations then yields

$$\begin{aligned} c_1 \left(1 - \frac{\lambda^2\pi^2}{4}\right) &= A\left(\frac{\lambda\pi}{2}\cos a - \sin a\right), \\ \text{and} \quad c_2 \left(1 - \frac{\lambda^2\pi^2}{4}\right) &= A\left(\cos a - \frac{\lambda\pi}{2}\sin a\right). \end{aligned}$$

Thus, the final solution is

$$\begin{aligned} y(x) &= \frac{A\left(\frac{\lambda\pi}{2}\cos a - \sin a\right)\sin x + A\left(\cos a - \frac{\lambda\pi}{2}\sin a\right)\cos x}{1 - (\lambda\pi/2)^2} \\ &= A\frac{(\lambda\pi/2)\sin(x-a) + \cos(x+a)}{1 - (\lambda\pi/2)^2}. \end{aligned}$$

We note that setting $\alpha = a + \frac{1}{2}\pi$ in equation (23.57) converts that integral equation into the current one with $A = 1$. Doing the same in its solution gives

$$\begin{aligned} y(x) &= \frac{(\lambda\pi/2)\cos(x - a - \frac{1}{2}\pi) + \cos(x+a)}{1 - (\lambda\pi/2)^2} \\ &= \frac{(\lambda\pi/2)\sin(x-a) + \cos(x+a)}{1 - (\lambda\pi/2)^2}, \end{aligned}$$

in agreement with the current solution.

> ***23.6*** *Consider the inhomogeneous integral equation*
>
> $$f(x) = g(x) + \lambda \int_a^b K(x,y) f(y)\,dy.$$
>
> *for which the kernel $K(x,y)$ is real, symmetric and continuous in $a \le x \le b$, $a \le y \le b$.*
>
> (a) *If λ is one of the eigenvalues λ_i of the homogeneous equation*
>
> $$f_i(x) = \lambda_i \int_a^b K(x,y) f_i(y)\,dy,$$
>
> *prove that the inhomogeneous equation can only a have non-trivial solution if $g(x)$ is orthogonal to the corresponding eigenfunction $f_i(x)$.*
>
> (b) *Show that the only values of λ for which*
>
> $$f(x) = \lambda \int_0^1 xy(x+y) f(y)\,dy$$
>
> *has a non-trivial solution are the roots of the equation*
>
> $$\lambda^2 + 120\lambda - 240 = 0.$$
>
> (c) *Solve*
>
> $$f(x) = \mu x^2 + \int_0^1 2xy(x+y) f(y)\,dy.$$

(a) Suppose $f(x)$ is a solution of

$$f(x) = g(x) + \lambda \int_a^b K(x,y) f(y)\,dy$$

with $\lambda = \lambda_i$, then

$$\begin{aligned}\int_a^b f_i(x) f(x)\,dx &= \int_a^b f_i(x) g(x)\,dx + \lambda_i \int_a^b f_i(x)\,dx \int_a^b K(x,y) f(y)\,dy \\ &= \int_a^b f_i(x) g(x)\,dx + \int_a^b \left[\lambda_i \int_a^b K(y,x) f_i(x)\,dx \right] f(y)\,dy,\end{aligned}$$

since $K(x,y) = K(y,x)$. Thus

$$\int_a^b f_i(x) f(x)\,dx = \int_a^b f_i(x) g(x)\,dx + \int_a^b f_i(y) f(y)\,dy,$$

$$\Rightarrow \quad \int_a^b f_i(x) g(x)\,dx = 0.$$

i.e. $g(x)$ being orthogonal to the eigenfunction $f_i(x)$ is a necessary condition for the inhomogeneous equation to have a solution whenever λ is equal to one of the eigenvalues λ_i.

(b) The kernel, $K(x,y) = yx^2 + y^2x$, is both symmetric and degenerate. To solve the equation we set $f(x) = a_1x^2 + a_2x$ giving

$$\begin{aligned} a_1x^2 + a_2x &= \lambda \int_0^1 (x^2y + y^2x)(a_1y^2 + a_2y)\,dy \\ &= \lambda \left(a_1x^2\frac{1}{4} + a_1x\frac{1}{5} + a_2x^2\frac{1}{3} + a_2x\frac{1}{4} \right). \end{aligned}$$

Equating coefficients gives

$$a_1 = \frac{\lambda a_1}{4} + \frac{\lambda a_2}{3} \quad \text{and} \quad a_2 = \frac{\lambda a_1}{5} + \frac{\lambda a_2}{4}.$$

For a non-trivial solution for a_1 and a_2 we need

$$\begin{aligned} \begin{vmatrix} 1 - \dfrac{\lambda}{4} & -\dfrac{\lambda}{3} \\ -\dfrac{\lambda}{5} & 1 - \dfrac{\lambda}{4} \end{vmatrix} &= 0, \\ 1 - \frac{\lambda}{2} + \frac{\lambda^2}{16} - \frac{\lambda^2}{15} &= 0, \\ 240 - 120\lambda - \lambda^2 &= 0, \qquad \text{as stated.} \end{aligned}$$

(c) As in part (b), the kernel is both symmetric and degenerate. Further, in the notation of part (a), $\lambda = 2$; but this is not a root of the equation derived in part (b). We therefore set $f(x) = \mu x^2 + a_1x^2 + a_2x$ and obtain [in the same way as in (b)]

$$a_1x^2 + a_2x = 2\left[x^2(a_1 + \mu)\frac{1}{4} + x(a_1 + \mu)\frac{1}{5} + x^2a_2\frac{1}{3} + xa_2\frac{1}{4} \right].$$

Equating the coefficients of x^2 and x and rationalising, we obtain

$$\begin{aligned} 6a_1 - 3a_1 - 4a_2 &= 3\mu, \\ -4a_1 + 10a_2 - 5a_2 &= 4\mu, \end{aligned}$$

yielding $a_1 = -31\mu$ and $a_2 = -24\mu$ and the solution as

$$f(x) = -30\mu x^2 - 24\mu x = -6\mu x(5x + 4).$$

This can be checked by substitution.

23.8 *By taking its Laplace transform, and that of $x^n e^{-ax}$, obtain the explicit solution of*

$$f(x) = e^{-x}\left[x + \int_0^x (x-u)e^u f(u)\,du\right].$$

Verify your answer by substitution.

Integrating by parts, we find the Laplace transform

$$\mathcal{L}\left[x^n e^{-ax}\right] = \int_0^\infty x^n e^{-ax} e^{-sx}\,dx = \frac{n(n-1)\cdots 2\,1}{(a+s)^{n+1}} = \frac{n!}{(a+s)^{(n+1)}}.$$

Setting $e^x f(x) = p(x)$ and $x = q(x)$, we can write the equation (after multiplying through by e^x) as

$$p(x) = q(x) + \int_0^x q(x-u)p(u)\,du,$$

in which the integral is a convolution. Thus, when the equation is Laplace transformed, the convolution theorem can be invoked and the transformed equation written in the form

$$\bar{p}(s) = \bar{q}(s) + \bar{p}(s)\bar{q}(s),$$

$$\Rightarrow \quad \bar{p}(s) = \frac{\bar{q}(s)}{1-\bar{q}(s)}.$$

Now $\bar{q}(s) = \mathcal{L}[x] = s^{-2}$ and so

$$\bar{p}(s) = \frac{1}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right),$$

$$\Rightarrow \quad p(x) = \frac{1}{2}\left(\frac{x^0e^x}{0!} - \frac{x^0e^{-x}}{0!}\right) = \sinh x,$$

$$\Rightarrow \quad f(x) = \tfrac{1}{2}(1-e^{-2x}).$$

This is the solution to the integral equation.

Verification:

$$\tfrac{1}{2}(1-e^{-2x}) = e^{-x}\left[x + \int_0^x (x-u)e^u\,\tfrac{1}{2}(1-e^{-2u})\,du\right],$$

$$\begin{aligned}
\sinh x &= x + \int_0^x (x-u)\sinh u\,du \\
&= x + [\,x\cosh u\,]_0^x - [\,u\cosh u\,]_0^x + \int_0^x \cosh u\,du \\
&= x + x\cosh x - x - x\cosh x + 0 + \sinh x - 0 \\
&= \sinh x, \text{ as expected.}
\end{aligned}$$

23.10 *Show that the equation*

$$f(x) = x^{-1/3} + \lambda \int_0^\infty f(y) \exp(-xy)\, dy$$

has a solution of the form $Ax^\alpha + Bx^\beta$. *Determine the values of* α *and* β *and show that those of* A *and* B *are*

$$\frac{1}{1 - \lambda^2 \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \qquad \text{and} \qquad \frac{\lambda\Gamma(\frac{2}{3})}{1 - \lambda^2 \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})},$$

where $\Gamma(z)$ *is the gamma function.*

We first find the Laplace transform of x^{-n} when $n < 1$ but is not necessarily integral. With $s > 0$,

$$\begin{aligned}
\mathscr{L}\left[x^{-n}\right] &= \int_0^\infty x^{-n} e^{-sx}\, dx, \text{ set } y = sx, \\
&= \int_0^\infty \frac{y^{-n}}{s^{-n}} e^{-y} \frac{dy}{s} \\
&= s^{n-1} \int_0^\infty y^{-n} e^{-y}\, dy \\
&= s^{n-1}\Gamma(1-n), \text{ provided } n < 1.
\end{aligned}$$

Next, we substitute the trial solution $f(x) = Ax^\alpha + Bx^\beta$ into the given equation:

$$\begin{aligned}
Ax^\alpha + Bx^\beta &= x^{-1/3} + \lambda \int_0^\infty (Ay^\alpha + By^\beta) e^{-xy}\, dy \qquad \text{(change variable to } u = xy) \\
&= x^{-1/3} + \lambda \left[A\Gamma(1+\alpha)x^{-\alpha-1} + B\Gamma(1+\beta)x^{-\beta-1} \right],
\end{aligned}$$

assuming that $\alpha, \beta > -1$. For this equation to be valid, one of α and β must be $-\frac{1}{3}$ and

$$\begin{aligned}
&\textit{either} \quad \alpha = -\alpha - 1, \quad \beta = -\beta - 1 \\
&\textit{or} \quad \alpha = -\beta - 1, \quad \beta = -\alpha - 1.
\end{aligned}$$

The first of these, which requires both α and β to have the value $-\frac{1}{2}$, is inconsistent with the other condition, but both it and the second are satisfied if $\alpha = -\frac{1}{3}$ and $\beta = -\frac{2}{3}$ (or vice versa). The assumption that $\alpha, \beta > -1$ is then also justified.

Thus, with the choice $\alpha = -\frac{1}{3}$,

$$A = 1 + \lambda B\Gamma(\tfrac{1}{3}), \quad \text{and} \quad B = \lambda A\Gamma(\tfrac{2}{3}),$$

yielding

$$A = \frac{1}{1 - \lambda^2 \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \qquad \text{and} \qquad B = \frac{\lambda\Gamma(\frac{2}{3})}{1 - \lambda^2 \Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}.$$

23.12 *By considering functions of the form* $h(x) = \int_0^x (x-y)f(y)\,dy$*, show that the solution* $f(x)$ *of the integral equation*

$$f(x) = x + \tfrac{1}{2}\int_0^1 |x-y|f(y)\,dy$$

satisfies the equation $f''(x) = f(x)$*.*

By examining the special cases $x = 0$ *and* $x = 1$*, show that*

$$f(x) = \frac{2}{(e+3)(e+1)}[(e+2)e^x - ee^{-x}].$$

To deal with the modulus sign we divide the integral into two parts:

$$\begin{aligned} f(x) &= x + \tfrac{1}{2}\int_0^1 |x-y|f(y)\,dy \\ &= x + \tfrac{1}{2}\int_0^x (x-y)f(y)\,dy + \tfrac{1}{2}\int_x^1 (y-x)f(y)\,dy. \end{aligned}$$

Thus the first and second derivatives of $f(x)$ are given by

$$\begin{aligned} f'(x) &= 1 + \tfrac{1}{2}\left[(x-x)f(x) + \int_0^x f(y)\,dy\right] \\ &\qquad + \tfrac{1}{2}\left[-(x-x)f(x) - \int_x^1 f(y)\,dy\right] \\ &= 1 + \tfrac{1}{2}\int_0^x f(y)\,dy - \tfrac{1}{2}\int_x^1 f(y)\,dy, \\ f''(x) &= \tfrac{1}{2}f(x) - [-\tfrac{1}{2}f(x)] = f(x). \end{aligned}$$

It follows from this differential equation that $f(x)$ must have the form $f(x) = Ae^x + Be^{-x}$.

Considering the integral equation in the special case $x = 0$:

$$\begin{aligned} A + B &= 0 + \frac{1}{2}\int_0^1 y(Ae^y + Be^{-y})\,dy \\ &= \frac{A}{2}\left\{[\,ye^y\,]_0^1 - \int_0^1 e^y\,dy\right\} + \frac{B}{2}\left\{\left[-ye^{-y}\right]_0^1 + \int_0^1 e^{-y}\,dy\right\} \\ &= \frac{A}{2}(e - 0 - e + 1) + \frac{B}{2}(-e^{-1} + 0 - e^{-1} + 1) \\ &= \frac{A}{2} + \frac{B}{2}(1 - 2e^{-1}), \end{aligned}$$

which gives the first relationship between A and B as

$$A = -B(1 + 2e^{-1}).$$

Now considering the case $x = 1$:

$$\begin{aligned} Ae + Be^{-1} &= 1 + \frac{1}{2}\int_0^1 (1-y)(Ae^y + Be^{-y})\,dy, \\ 2Ae + 2Be^{-1} - 2 &= A\left\{e - 1 - [\,ye^y\,]_0^1 + \int_0^1 e^y\,dy\right\} \\ &\quad + B\left\{1 - e^{-1} - \left[-ye^{-y}\right]_0^1 - \int_0^1 e^{-y}\,dy\right\} \\ &= A(e - 1 - e + 0 + e - 1) + B(1 - e^{-1} + e^{-1} - 0 + e^{-1} - 1) \\ &= A(e-2) + Be^{-1}, \end{aligned}$$

which gives the second relationship between A and B as

$$-2 = -(e+2)A - Be^{-1}.$$

Solving the two derived relationships as a pair of simultaneous equations, we obtain

$$A = \frac{2(1+2e^{-1})e}{(e+1)(e+3)} = \frac{2(e+2)}{(e+1)(e+3)},$$

$$\text{and} \quad B = \frac{-2}{e+4+3e^{-1}} = \frac{-2e}{(e+1)(e+3)}.$$

Thus, finally,

$$f(x) = \frac{2}{(e+3)(e+1)}[(e+2)e^x - ee^{-x}].$$

23.14 *For the integral equation*

$$y(x) = x^{-3} + \lambda \int_a^b x^2 z^2 y(z)\,dz,$$

show that the resolvent kernel is $5x^2z^2/[5-\lambda(b^5-a^5)]$ *and hence solve the equation. For what range of* λ *is the solution valid*?

We use the recurrence relation

$$K_n(x,z) = \int_a^b K(x,z_1)K_{n-1}(z_1,z)\,dz_1$$

to build up the terms of the infinite series representing the resolvent kernel

$$R(x,z:\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,z).$$

For this problem $y_0(x) = x^{-3}$ and $K(x,z) = x^2z^2$.

$$K_1(x,z) = x^2z^2,$$
$$K_2(x,z) = \int_a^b x^2u^2\,u^2z^2\,du = \frac{b^5-a^5}{5}\,x^2z^2,$$
$$K_3(x,z) = \int_a^b x^2u^2\left(\frac{b^5-a^5}{5}\right)u^2z^2\,du = \left(\frac{b^5-a^5}{5}\right)^2 x^2z^2.$$

Clearly,

$$K_n(x,z) = \left(\frac{b^5-a^5}{5}\right)^{n-1} x^2z^2,$$

$$\text{and}\quad R(x,z;\lambda) = \sum_{n=0}^{\infty}\lambda^n K_{n+1}(x,z) = \sum_{n=0}^{\infty}\lambda^n\left(\frac{b^5-a^5}{5}\right)^n x^2z^2 = \frac{5x^2z^2}{5-\lambda(b^5-a^5)},$$

provided $|\lambda|\,|b^5-a^5| < 5$ (so that the series is convergent).

The solution to the integral equation, has the general form

$$y(x) = y_0(x) + \lambda\int_a^b R(x,z;\lambda)y_0(z)\,dz,$$

and, in this particular case,

$$y(x) = x^{-3} + \lambda\int_a^b \frac{5x^2z^2z^{-3}}{5-\lambda(b^5-a^5)}\,dz = x^{-3} + \frac{5\lambda\ln(b/a)x^2}{5-\lambda(b^5-a^5)}.$$

23.16 *This exercise shows that following formal theory is not necessarily the best way to get practical results!*

(a) *Determine the eigenvalues $\lambda_\pm$ of the kernel $K(x,z) = (xz)^{1/2}(x^{1/2}+z^{1/2})$ and show that the corresponding eigenfunctions have the forms*

$$y_\pm(x) = A_\pm(\sqrt{2}x^{1/2} \pm \sqrt{3}x),$$

where $A_\pm^2 = 5/(10 \pm 4\sqrt{6})$.

(b) *Use Schmidt–Hilbert theory to solve*

$$y(x) = 1 + \tfrac{5}{2}\int_0^1 K(x,z)y(z)\,dz.$$

(c) *As will have been apparent, the algebra involved in the formal method used in* (b) *is long and error-prone, and it is in fact much more straightforward to use a trial function $1 + \alpha x^{1/2} + \beta x$. Check your answer by doing so.*

(a) To find the eigenvalues, given the (supposed) forms of the eigenfunctions, we may substitute $y(x) = a_1 x + a_2 x^{1/2}$ and require consistency.

$$\begin{aligned} a_1 x + a_2 x^{1/2} &= \lambda \int_0^1 (x z^{1/2} + z x^{1/2})(a_1 z + a_2 z^{1/2})\, dz \\ \Rightarrow \quad a_1 &= \tfrac{2}{5}\lambda a_1 + \tfrac{1}{2}\lambda a_2, \\ a_2 &= \tfrac{1}{3}\lambda a_1 + \tfrac{2}{5}\lambda a_2. \end{aligned}$$

These two equations have a non-trivial solution if

$$\left(1 - \frac{2\lambda}{5}\right)^2 = \frac{\lambda^2}{6} \quad \Rightarrow \quad \lambda = \frac{5\sqrt{6}}{2\sqrt{6} \pm 5} = \pm 25\sqrt{6} - 60.$$

To find the corresponding eigenfunctions, set $a_1 = 1$, say. Then

$$\begin{aligned} a_2 &= \frac{2}{\lambda}\left(1 - \frac{2\lambda}{5}\right) \\ &= \frac{2}{\pm 25\sqrt{6} - 60} - \frac{4}{5} \\ &= \frac{2 \mp 20\sqrt{6} + 48}{\pm 25\sqrt{6} - 60} \\ &= \frac{(50 \mp 20\sqrt{6})(\pm 25\sqrt{6} + 60)}{3750 - 3600} \\ &= \frac{\pm 50\sqrt{6}}{150} = \pm\sqrt{\frac{2}{3}}. \end{aligned}$$

Thus the normalised eigenfunctions are

$$y_\pm(x) = A_\pm(\sqrt{3}x \pm \sqrt{2}x^{1/2}),$$

with

$$\begin{aligned} 1 &= \int_0^1 y_\pm^2\, dx \\ &= \int_0^1 A_\pm^2 (\sqrt{3}x \pm \sqrt{2}x^{1/2})^2 dx \\ &= A_\pm^2 \left(3\,\frac{1}{3} \pm 2\sqrt{6}\,\frac{2}{5} + 2\,\frac{1}{2}\right) \\ &= A_\pm^2 \left(\frac{10 \pm 4\sqrt{6}}{5}\right) = A_\pm^2 \left(\frac{2 \pm \sqrt{6}}{\sqrt{5}}\right)^2, \end{aligned}$$

giving the stated values for $A_\pm^2$.

(b) We first need to calculate

$$\begin{aligned}\langle y_\pm|f\rangle &= \int_0^1 A_\pm(\sqrt{3}x \pm \sqrt{2}x^{1/2})\,1\,dx\\ &= \frac{\sqrt{5}}{2\pm\sqrt{6}}\,\frac{3\sqrt{3}\pm 4\sqrt{2}}{6}\\ &= \frac{\sqrt{5}(3\sqrt{3}\pm 4\sqrt{2})(2\mp\sqrt{6})}{6(4-6)} = \frac{\sqrt{5}}{12}(2\sqrt{3}\pm\sqrt{2}).\end{aligned}$$

The solution is given by

$$\begin{aligned}y(x) &= f + \lambda\sum_i \frac{\langle y_\pm|f\rangle}{\lambda_i-\lambda}\,y_i\\ &= 1 + \frac{5}{2}\,\frac{\sqrt{5}}{12}\,\frac{2\sqrt{3}+\sqrt{2}}{25\sqrt{6}-60-\frac{5}{2}}\,\frac{\sqrt{5}}{2+\sqrt{6}}\,(\sqrt{3}x+\sqrt{2}x^{1/2})\\ &\quad + \frac{5}{2}\,\frac{\sqrt{5}}{12}\,\frac{2\sqrt{3}-\sqrt{2}}{-25\sqrt{6}-60-\frac{5}{2}}\,\frac{\sqrt{5}}{2-\sqrt{6}}\,(\sqrt{3}x-\sqrt{2}x^{1/2}).\end{aligned}$$

The coefficient of x in this expression is

$$\begin{aligned}&\frac{25}{24}\sqrt{3}\left[\frac{2(2\sqrt{3}+\sqrt{2})}{(50\sqrt{6}-125)(2+\sqrt{6})} + \frac{2(2\sqrt{3}-\sqrt{2})}{(-50\sqrt{6}-125)(2-\sqrt{6})}\right]\\ &= \frac{\sqrt{3}}{12}\left[\frac{2\sqrt{3}+\sqrt{2}}{2-\sqrt{6}} + \frac{2\sqrt{3}-\sqrt{2}}{2+\sqrt{6}}\right]\\ &= \frac{\sqrt{3}}{12}\,\frac{4\sqrt{3}+2\sqrt{2}+6\sqrt{2}+2\sqrt{3}+4\sqrt{3}-2\sqrt{2}-6\sqrt{2}+2\sqrt{3}}{4-6} = -\frac{3}{2}.\end{aligned}$$

A similar (tedious) calculation shows that the coefficient of $x^{1/2}$ is $-4/3$, making the final solution

$$y(x) = 1 - \tfrac{3}{2}x - \tfrac{4}{3}x^{1/2}.$$

(c) Substituting the trial solution $1+\alpha x^{1/2}+\beta x$ directly into the equation gives

$$1+\alpha x^{1/2}+\beta x = 1 + \tfrac{5}{2}\int_0^1 (xz^{1/2}+zx^{1/2})\,(1+\alpha z^{1/2}+\beta z)\,dz.$$

Carrying out the integrations and equating the coefficients of x and $x^{1/2}$ then leads to

$$\begin{aligned}\alpha &= \tfrac{5}{2}\left(\tfrac{1}{2}+\tfrac{2}{5}\alpha+\tfrac{1}{3}\beta\right) \quad\Rightarrow\quad \beta = -\tfrac{3}{2},\\ \beta &= \tfrac{5}{2}\left(\tfrac{2}{3}+\tfrac{1}{2}\alpha+\tfrac{2}{5}\beta\right) \quad\Rightarrow\quad \alpha = -\tfrac{4}{3},\\ \Rightarrow\quad & y(x) = 1 - \tfrac{4}{3}x^{1/2} - \tfrac{3}{2}x.\end{aligned}$$

This is as in part (b) — but with much less effort!

24

Complex variables

> ***24.2*** *Find a function $f(z)$, analytic in a suitable part of the Argand diagram, for which*
> $$Re\ f = \frac{\sin 2x}{\cosh 2y - \cos 2x}.$$
> *Where are the singularities of $f(z)$?*

Let the required function be $f(z) = u + iv$, with $u = \sin 2x/(\cosh 2y - \cos 2x)$. Since y appears less often than x in the given expression, it will probably be easier to consider $\partial u/\partial y$ rather than $\partial u/\partial x$. This indicates that the relevant Cauchy–Riemann equation is

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = -\frac{\partial v}{\partial x}.$$

Having differentiated w.r.t y, we now integrate w.r.t x:

$$v = \int \frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}\,dx = -\frac{\sinh 2y}{\cosh 2y - \cos 2x} + f(y).$$

By inspection, or by substitution in the other C–R equation, $f(y)$ can be seen to be an ignorable constant. The required function is therefore

$$f(z) = \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}.$$

To determine what function of z this is, consider its form on the real axis where $y = 0$,

$$f(x) = \frac{\sin 2x}{1 - \cos 2x} = \frac{2\sin x \cos x}{2\sin^2 x} = \cot x \quad \Rightarrow \quad f(z) = \cot z.$$

This can be checked as follows.

$$\begin{aligned}
f(z) &= \frac{\cos(x+iy)}{\sin(x+iy)} \\
&= \frac{\cos x \cosh y - i \sin x \sinh y}{\sin x \cosh y + i \cos x \sinh y} \\
&= \frac{(\cos x \cosh y - i \sin x \sinh y)(\sin x \cosh y - i \cos x \sinh y)}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} \\
&= \frac{\sin x \cos x(\cosh^2 y - \sinh^2 y) - i \cosh y \sinh y(\cos^2 x + \sin^2 x)}{\cos^2 x(\sinh^2 y - \cosh^2 y) + \cosh^2 y} \\
&= \frac{\sin x \cos x - i \cosh y \sinh y}{\cosh^2 y - \frac{1}{2} - \cos^2 x + \frac{1}{2}} \\
&= \frac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}.
\end{aligned}$$

Since $f(z) = \cot z$ the poles can only occur at the zeros of $\sin z$, i.e at $z = n\pi$ where n is an integer; $\cos n\pi \neq 0$ and so there will be a (simple) pole at each such point. The same conclusion is reached by studying $\cosh 2y - \cos 2x$. Since $\cosh 2y \geq 1$ and $\cos 2x \leq 1$, this denomiator can only vanish if both terms equal 1; this requires $y = 0$ and $x = n\pi$.

24.4 *Find the Taylor series expansion about the origin of the function $f(z)$ defined by*

$$f(z) = \sum_{r=1}^{\infty} (-1)^{r+1} \sin\left(\frac{p\,z}{r}\right)$$

where p is a constant. Hence verify that $f(z)$ is a convergent series for all z.

Because every term in the series is a sine function, all of its even derivatives will also be sine functions and therefore vanish at $z = 0$. The odd derivatives will consist entirely of cosine functions and the $(2n+1)$th derivative of a typical term will be

$$f_r^{(2n+1)}(z) = (-1)^{r+1}(-1)^n \left(\frac{p}{r}\right)^{2n+1} \cos\left(\frac{p\,z}{r}\right),$$

with

$$f_r^{(2n+1)}(0) = (-1)^{n+r+1} \left(\frac{p}{r}\right)^{2n+1}.$$

The Taylor series expansion is therefore

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \sum_{r=1}^{\infty} (-1)^{n+r+1} \left(\frac{p}{r}\right)^{2n+1}.$$

giving the expansion coefficients as

$$a_{2n+1} = \frac{(-1)^{n+1} p^{2n+1}}{(2n+1)!} \sum_{r=1}^{\infty} \frac{(-1)^r}{r^{2n+1}},$$

with $a_{2n} = 0$.

As $n \to \infty$ the sum appearing in a_{2n+1} tends (rapidly) to -1 (only the $r = 1$ term contributing) and so the radius of convergence R is given by

$$\frac{1}{R^2} = \lim_{n\to\infty} \left| \frac{-p^{2n+3}(2n+1)!}{p^{2n+1}(2n+3)!} \right| = 0.$$

Thus $R = \infty$ and the series is convergent for all z.

24.6 *Identify the zeroes, poles and essential singularities of the following functions:*

(a) $\tan z$, (b) $[(z-2)/z^2]\sin[1/(1-z)]$, (c) $\exp(1/z)$,
(d) $\tan(1/z)$, (e) $z^{2/3}$.

(a) This function $\tan z = \dfrac{\sin z}{\cos z}$ has zeroes when

$$\sin z = \frac{1}{2i}(e^{ix-y} - e^{-ix+y}) = 0 \quad \Rightarrow \quad e^{ix}e^{-y} = e^{-ix}e^{y}.$$

The two terms can only be equal if they have equal magnitudes, i.e. $|\,e^{-y}\,| = |\,e^{y}\,| \quad \Rightarrow \quad y = 0$. We also need $e^{ix} = e^{-ix} \quad \Rightarrow \quad x = n\pi$, where n is an integer. Thus the zeroes of $\tan x$ occur at $z = n\pi$.

The poles of $\tan z$ will occur at the zeroes of $\cos z$. By a similar argument to that above, this needs $y = 0$ and $e^{ix} = -e^{-ix} = e^{i(-x+\pi+2n\pi)} \quad \Rightarrow \quad 2x = (2n+1)\pi$. Thus, the (simple) poles of $\tan z$ occur at $z = (n + \frac{1}{2})\pi$.

We note that both $\sin u$ and $\cos u$ have Maclaurin series that contain arbitrarily large powers of u and that they are not multiples of each other; we can conclude that their ratio will also have a Maclaurin series containing arbitrarily large powers of u. The same conclusion is reached by differentiating $\tan u$ and so constructing its Maclaurin series directly. Thus, when $z \to \infty$ is replaced by $z = 1/\xi$ with $\xi \to 0$, there will be arbitrarily large *inverse* powers of ξ in the series expansion; this establishes that $\xi = 0$ (i.e. $z = \infty$) is an essential singularity of $\tan z$.

For the remaining exercises we will not give such a detailed justification of our conclusions; most features are obvious and only the less obvious ones will be treated in any detail.

(b) This function, $\dfrac{z-2}{z^2}\sin\dfrac{1}{1-z}$, has obvious zeroes at $z=2$ and $z=\infty$. Equally clearly, at $z=0$ it has a 2nd-order pole. Further zeroes will occur when the sine term factor is zero; from the analysis in part (a), this will be when $(1-z)^{-1}=n\pi$, i.e. at $z=1-(n\pi)^{-1}$. The remaining singularity to classify is that at $z=1$. By a similar argument to that given in part (a), the Laurent expansion of the function about the point will have no largest negative power of $1-z$; the point is therefore an essential singularity.

(c) Since $\exp(0)=1$, the function is well behaved and analytic at ∞. The only non-analytic point is the origin, $z=0$, where the defining series for the exponential function generates a Laurent expansion with no largest negative power of z; the point is therefore an essential singularity.

(d) The singularities of $\tan(z^{-1})$ follow from those of $\tan z$ in part (a). They are therefore zeroes at $z=\infty$ and $(n\pi)^{-1}$, simple poles at $z=(n\pi+\frac{1}{2}\pi)^{-1}$ and an essential singularity at $z=0$.

(e) The origin, $z=0$ is both a zero and a branch point of the function $z^{2/3}$. To determine its behavior at ∞ we have to consider $1/\xi^{2/3}$ near $\xi=0$. There is clearly a singularity there, and, since the function cannot even be expressed as a Laurent series, the singularity is an essential singularity.

24.8 *Show that the transformation*

$$w=\int_0^z \frac{1}{(\zeta^3-\zeta)^{1/2}}\,d\zeta$$

transforms the upper half-plane into the interior of a square that has one corner at the origin of the w-plane and sides of length L, where

$$L=\int_0^{\pi/2}\operatorname{cosec}^{1/2}\theta\,d\theta.$$

This transformation is a Schwarz-Christoffel transformation of the upper half of the z-plane into a closed polygon. It can be written

$$w=\int_0^z \zeta^{-1/2}(\zeta-1)^{-1/2}(\zeta+1)^{-1/2}\,d\zeta.$$

Since each factor is raised to the same power, the interior angles at the corners of the polygon are all the same and given by $(\phi/\pi)-1=-\frac{1}{2}$, i.e. each $\phi=\pi/2$. To close the polygon a fourth vertex is needed (also with $\phi=\pi/2$); this must arise from transforming the point $x=\pm\infty$, $y=0$. Thus the four points on the x-axis that transform into the vertices of what is (for now) a rectangle are $x_1=-1$, $x_2=0$, $x_3=1$ and $x_4=\pm\infty$.

From the definition of the transform, the image of x_2 ($z = 0$) is clearly $w_2 = 0$. Thus one corner of the rectangle is at the origin in the w-plane. Further,

$$w_3 - w_2 = \int_0^1 \frac{1}{\zeta^{1/2}(\zeta^2 - 1)^{1/2}}\, d\zeta.$$

Setting $\zeta = \dfrac{1}{u}$ with $d\zeta = -\dfrac{1}{u^2}\, du$,

$$\begin{aligned} w_3 - w_2 &= \int_\infty^1 \frac{u^{1/2}}{(u^{-2} - 1)^{1/2}} \left(\frac{-1}{u^2}\right) du \\ &= \int_1^\infty \frac{u^{3/2}}{(1 - u^2)^{1/2} u^2}\, du \\ &= \int_1^\infty \frac{1}{(1 - u^2)^{1/2} u^{1/2}}\, du \\ &= \pm i(w_4 - w_3). \end{aligned}$$

Thus we have a rectangle with adjacent sides of equal length, i.e. a square.

The length of a side is given in magnitude by

$$L = \int_1^\infty \frac{1}{u^{1/2}(u^2 - 1)^{1/2}}\, du.$$

Setting $u = \text{cosec}\,\theta$ with $du = -\text{cosec}\,\theta \cot\theta\, d\theta$ and $u^2 - 1 = \cot^2\theta$, gives

$$\begin{aligned} L &= \int_{\pi/2}^0 \frac{-\text{cosec}\,\theta \cot\theta}{\text{cosec}^{\,1/2}\theta \cot\theta}\, d\theta \\ &= \int_0^{\pi/2} \text{cosec}^{\,1/2}\theta\, d\theta, \end{aligned}$$

as stated in the question.

Many of the remaining exercises in this chapter involve contour integration and the choice of a suitable contour. In order to save the space taken by drawing several broadly similar figures that differ only in notation, the positions of poles, the values of lengths or angles, or other minor details, we show in figure 24.1 a number of typical contour types to which reference can be made.

24.10 *Show that, if a is a positive real constant, the function* $\exp(iaz^2)$ *is analytic and* $\to 0$ *as* $|z| \to \infty$ *for* $0 < \arg z \le \pi/4$. *By applying Cauchy's theorem to a suitable contour prove that*

$$\int_0^\infty \cos(ax^2)\, dx = \sqrt{\frac{\pi}{8a}}.$$

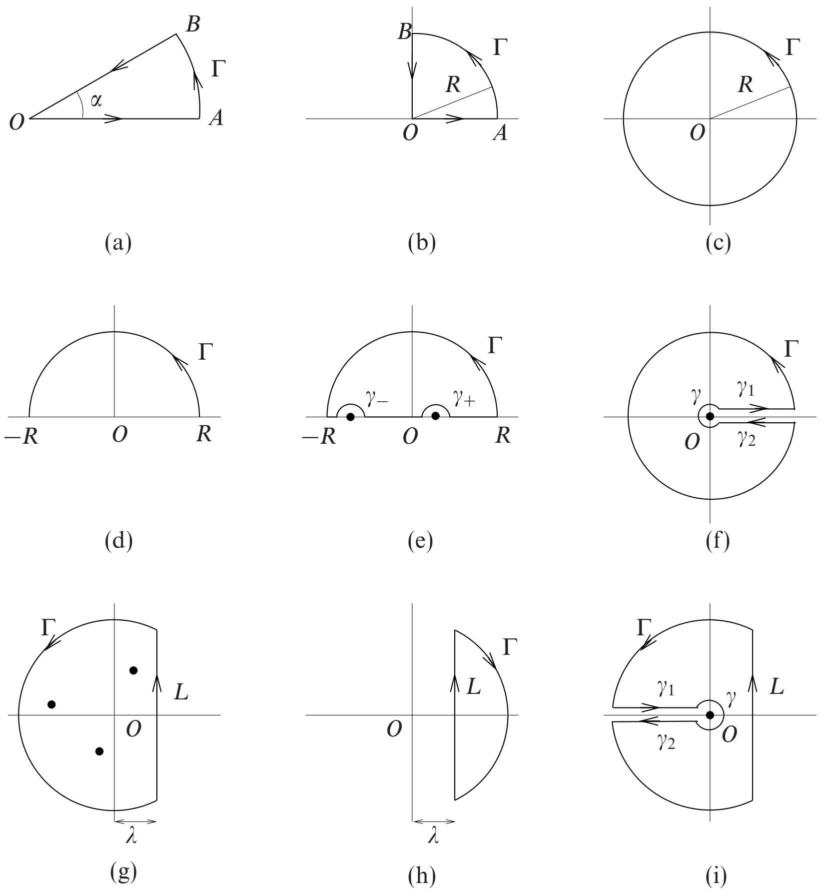

Figure 24.1 Typical contours for use in contour integration.

The function is explicitly a function of z and has no poles in the finite plane. By Cauchy's theorem, its integral around a closed loop will be zero. Writing z as $r\,e^{i\theta}$, we have

$$e^{iaz^2} = e^{iar^2(\cos 2\theta + i\sin 2\theta)} = e^{iar^2\cos 2\theta}e^{-ar^2\sin 2\theta}.$$

The real part of this, when $\theta = 0$, is the required integrand. Further, the function $\to 0$ as $r \to \infty$ provided $a\sin 2\theta$ is positive. Since a is positive, this requires $\sin 2\theta$ to be positive, i.e. $0 < \theta < \pi/2$.

To apply Cauchy's theorem we therefore need a closed contour which includes the positive real axis, $0 \leq x < \infty$, and some part of the semi-circle at infinity in the first quadrant; from the above result, this part of the contour will contribute nothing. The contour needs to be completed by a path along which the line integral is known or can be evaluated by means other than contour integration.

In view of the i and the square of z appearing in the exponent in the integrand we consider a contour such as that shown in 24.1(a) with $\alpha = \pi/4$. On the closing part of the contour, $z = ue^{i\pi/4}$. Cauchy's theorem now reads:

$$\int_0^\infty e^{iax^2}\,dx + 0 + \int_\infty^0 \exp[\,ia(ue^{i\pi/4})^2\,]\,e^{i\pi/4}\,du = 0,$$

$$\int_0^\infty e^{iax^2}\,dx - \int_0^\infty e^{ia(u^2 i)}\,\frac{1+i}{\sqrt{2}}\,du = 0.$$

Taking the real parts of both sides gives the equality

$$\int_0^\infty \cos(ax^2)\,dx = \frac{1}{\sqrt{2}}\int_0^\infty e^{-au^2}\,du$$

$$= \frac{1}{\sqrt{2}}\,\frac{1}{2}\sqrt{\frac{\pi}{a}} = \sqrt{\frac{\pi}{8a}}.$$

In the last line we have used the standard result for the infinite integral of $\exp(-ax^2)$, which can be found in any textbook if not already known. Apart from a change of scale, the overall result is a special case of a Fresnel integral $C(x)$ in which the argument $x = \infty$

24.12 *By considering the real part of*

$$\int \frac{-iz^{n-1}\;dz}{1 - a(z + z^{-1}) + a^2},$$

where $z = \exp i\theta$ *and* n *is a non-negative integer, evaluate*

$$\int_0^\pi \frac{\cos n\theta}{1 - 2a\cos\theta + a^2}\;d\theta,$$

for a *real and* > 1.

The integrand can be rewritten in a form that establishes the positions of any poles it possesses:

$$\frac{-iz^{n-1}}{1 - a(z + z^{-1}) + a^2} = \frac{i}{a}\,\frac{z^n}{z^2 - (a + a^{-1})z + 1} = \frac{i}{a}\,\frac{z^n}{(z-a)(z-a^{-1})}.$$

We use a contour C of type (c) in figure 24.1 with $R = 1$ (i.e. the unit circle) and integrate $f(z)$ around it. As the above form shows, the integrand has poles at $z = a$ (outside the contour) and $z = a^{-1}$ (inside it) and so we need the residue only at the latter (simple) pole. It is

$$\frac{i}{a}\lim_{z \to a^{-1}}\left(\frac{z^n}{z-a}\right) = \frac{i}{a}\,\frac{1}{a^n(a^{-1} - a)} = \frac{i}{a}\,\frac{1}{a^{n-1}(1 - a^2)}.$$

From the residue theorem it now follows that

$$\int_C \frac{i}{a}\,\frac{z^n\,dz}{(z-a)(z-a^{-1})}. = \frac{i}{a}\,\frac{2\pi i}{a^{n-1}(1-a^2)}.$$

On the unit circle, $z = e^{i\theta}$ and $dz = i\,e^{i\theta}\,d\theta$. Making this change of variable gives

$$\begin{aligned}\frac{-2\pi}{a^n(1-a^2)} &= \frac{i}{a}\int_0^{2\pi} \frac{e^{in\theta}\,i\,e^{i\theta}\,d\theta}{(\,e^{i\theta}-a)(\,e^{i\theta}-a^{-1})}\\ &= \int_0^{2\pi} \frac{e^{in\theta}\,d\theta}{(\,e^{i\theta}-a)(\,e^{-i\theta}-a)}\\ &= \int_0^{2\pi} \frac{e^{in\theta}\,d\theta}{a^2-2a\cos\theta+1}.\end{aligned}$$

On equating real parts,

$$\frac{2\pi}{a^n(a^2-1)} = \int_0^{2\pi} \frac{\cos n\theta\,d\theta}{a^2-2a\cos\theta+1}.$$

Finally, we note that the integrand is an even function of θ and so for the given limits of 0 and π the value of the integral is one-half of that calculated above, i.e. $\pi/(a^{n+2}-a^n)$.

> ***24.14*** *Prove that, for* $\alpha > 0$, *the integral*
>
> $$\int_0^{\infty} \frac{t\sin\alpha t}{1+t^2}\;dt$$
>
> *has the value* $(\pi/2)\exp(-\alpha)$.

We wish to evaluate

$$I = \int_0^{\infty} \frac{t\sin\alpha t}{1+t^2}\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{t\sin\alpha t}{1+t^2}\,dt = \frac{1}{2}\mathrm{Im}\,\int_{-\infty}^{\infty} \frac{t\,e^{i\alpha t}}{1+t^2}\,dt.$$

The complex function $f(z) = \dfrac{z}{1+z^2}$ has the properties:

(i) it is analytic in the upper half-plane except for a pole at $z = i$, and

(ii) $|f(z)| \sim |z^{-1}| \to 0$ as $|z| \to \infty$ in the upper half-plane.

Since $\alpha > 0$, all the conditions for Jordan's lemma are satisfied and we can usefully consider the integral

$$J = \int_C \frac{z\,e^{i\alpha z}}{1+z^2}\,dz,$$

where C is contour (d) in figure 24.1 with $R \to \infty$.

Jordan's lemma ensures that the integral along the semi-circle Γ goes to 0 as $R \to \infty$. The residue theorem then reads

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x\, e^{i\alpha x}}{1+x^2}\, dx + 0 &= 2\pi i(\text{residue at } z = i) \\ &= 2\pi i \lim_{z\to i} \frac{(z-i)z\, e^{i\alpha z}}{(z-i)(z+i)} \\ &= 2\pi i \frac{ie^{-\alpha}}{2i}.\end{aligned}$$

Equating the imaginary parts of both side of the equation shows that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\, dx = \pi e^{-\alpha}$$

and $I = \frac{1}{2}\pi e^{-\alpha}$, as stated in the question.

24.16 *Show that the principal value of the integral*

$$\int_{-\infty}^{\infty} \frac{\cos(x/a)}{x^2 - a^2}\, dx$$

is $-(\pi/a)\sin 1$.

The complex form of the integrand is

$$f(z) = \frac{e^{iz/a}}{z^2 - a^2};$$

this has two poles on the real axis, at $z = \pm a$. Consequently we need to work with a contour that has semicircular indentations into the upper half-plane at these points (see contour (e) in figure 24.1).

With this choice of contour, no poles are enclosed and the integral around the complete contour will be zero. However, the contributions from the separate parts of the contour are not individually zero. Since the conditions for Jordan's lemma are satisfied, the contribution from Γ *is* zero, but that still leaves the contributions from the indented semicircles as well as the principal value that we wish to evaluate.

Each semicircle contributes $-\pi i \times$ the residue at the corresponding (simple) pole, the minus sign arising from the fact that the semicircle is traversed in the negative sense. The residues are

$$\lim_{z\to -a} \frac{(z+a)e^{iz/a}}{(z+a)(z-a)} = \frac{e^{-i}}{-2a} \quad \text{and} \quad \lim_{z\to a} \frac{(z-a)e^{iz/a}}{(z+a)(z-a)} = \frac{e^{i}}{2a}.$$

The residue theorem (Cauchy's theorem in this case) then reads

$$-\frac{\pi i}{2a}(e^{i}-e^{-i})+P\int_{-\infty}^{\infty}\frac{\cos(x/a)}{x^2-a^2}\,dx+iP\int_{-\infty}^{\infty}\frac{\sin(x/a)}{x^2-a^2}\,dx=0.$$

Equating the real parts of the two sides of this equation yields the stated result,

$$P\int_{-\infty}^{\infty}\frac{\cos(x/a)}{x^2-a^2}\,dx=-\frac{\pi}{a}\sin 1.$$

24.18 *By applying the residue theorem around a wedge-shaped contour of angle $2\pi/n$, with one side along the real axis, prove that the integral*

$$\int_0^{\infty}\frac{dx}{1+x^n},$$

where n is real and ≥ 2, has the value $(\pi/n)\text{cosec}\,(\pi/n)$.

The contour needed is that shown in figure 24.1 (a) with $\alpha=2\pi/n$. The denominator of the complex integrand has zeroes when

$$z=\exp\left[\frac{(2m+1)i\pi}{n}\right],\ \text{for } m=0,1,\ldots,n-1.$$

Only one of these zeroes, the one at $z=e^{i\pi/n}$ with $m=0$, lies within the sector contour, and none lie on it.

On OA, $z=x$ and $dz=dx$.

On AB, $z=R\,e^{i\theta}$ for $0\leq\theta\leq 2\pi/n$ and $dz=iR\,e^{i\theta}\,d\theta$.

On BO, $z=ye^{2\pi i/n}$ and $dz=e^{2\pi i/n}\,dy$.

Applying the residue theorem to the contour integral gives

$$\int_0^R\frac{dx}{1+x^n}+\int_0^{2\pi/n}\frac{iR\,e^{i\theta}\,d\theta}{1+R^ne^{in\theta}}\,d\theta$$
$$+\int_R^0\frac{e^{2\pi i/n}\,dy}{1+y^ne^{2\pi i}}=2\pi i(\text{residue at } z=e^{i\pi/n}).$$

Letting $R\to\infty$ shows that the required integral I satisfies

$$I(1-e^{2\pi i/n})=2\pi i(\text{residue at } z=e^{i\pi/n}).$$

Now, since it is a simple pole at $z=e^{i\pi/n}$, the residue there is given by the inverse of the derivative of $1+z^n$, i.e.

$$\text{residue}=\frac{1}{nz^{n-1}}=\frac{1}{ne^{[i\pi(n-1)]/n}}=-\frac{e^{i\pi/n}}{n}.$$

Thus,

$$I(1-e^{2\pi i/n}) = \frac{-2\pi i}{n}\, e^{i\pi/n},$$

$$I\,\frac{1}{2i}(e^{-i\pi/n}-e^{i\pi/n}) = -\frac{\pi}{n}, \quad \text{after dividing through by } 2ie^{i\pi/n},$$

$$\Rightarrow \quad I\sin\frac{\pi}{n} = \frac{\pi}{n},$$

$$\Rightarrow \quad I = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n},$$

as stated in the question.

24.20 *Show that*

$$\int_0^\infty \frac{\ln x}{x^{3/4}(1+x)}\,dx = -\sqrt{2}\pi^2.$$

Denote the required integral by I. The complex form of the integrand $f(z)$ is not single-valued and by choosing contour (f) of figure 24.1 we can capitalise on that fact. We first consider the behaviour of the integrand on the various parts of the contour.

(i) Around γ, $|zf(z)| \sim \dfrac{z^{1/4}\ \ln z}{1} \to 0$ as $|z|\to 0$.

(ii) Around Γ, $|zf(z)| \sim \dfrac{\ln z}{z^{3/4}} \to 0$ as $|z|\to\infty$.

(iii) On γ_1, $z=x$ and $f(z) = \dfrac{\ln x}{x^{3/4}(1+x)}$.

(iv) On γ_2, $z = xe^{2\pi i}$ and $f(z) = \dfrac{\ln x + i2\pi}{x^{3/4}e^{3\pi i/2}(1+xe^{2\pi i})}$.

The only pole *inside* the contour is a simple one at $z = e^{i\pi}$; that at $z=0$ is excluded by the contour and its (non-) contribution is that calculated for γ. The residue at $z=e^{i\pi}$ is $(0+i\pi)e^{-3i\pi/4}$.

The residue theorem therefore reads [Note that $e^{-3\pi i/2} = i$]

$$0 + I + 0 - Ie^{-3\pi i/2} - \int_0^\infty \frac{2\pi i\, e^{-3\pi i/2}}{x^{3/4}(1+x)}\,dx = 2\pi i(i\pi e^{-3\pi i/4}),$$

$$I(1-i) + \int_0^\infty \frac{2\pi}{x^{3/4}(1+x)}\,dx = -2\pi^2\left(\frac{-1-i}{\sqrt{2}}\right).$$

Equating imaginary parts, $\qquad -I = \sqrt{2}\pi^2.$

As a bonus, we also deduce that

$$\int_0^\infty \frac{1}{x^{3/4}(1+x)}\,dx = \frac{1}{2\pi}(\sqrt{2}\pi^2 - I) = \sqrt{2}\pi.$$

24.22 *The equation of an ellipse in plane polar coordinates r, θ, with one of its foci at the origin, is*

$$\frac{\ell}{r} = 1 - \epsilon \cos\theta,$$

where ℓ is a length (that of the latus rectum) and ϵ $(0 < \epsilon < 1)$ is the eccentricity of the ellipse. Express the area of the ellipse as an integral around the unit circle in the complex plane, and show that the only singularity of the integrand inside the circle is a double pole at $z_0 = \epsilon^{-1} - (\epsilon^{-2} - 1)^{1/2}$.

By setting $z = z_0 + \xi$ and expanding the integrand in powers of ξ, find the residue at z_0 and hence show that the area is equal to $\pi\ell^2(1-\epsilon^2)^{-3/2}$.

[Note*: In terms of the semi-axes a and b of the ellipse, $\ell = b^2/a$ and $\epsilon^2 = (a^2 - b^2)/a^2$.*]

The area A is given by

$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \frac{\ell^2}{2} \int_0^{2\pi} \frac{d\theta}{(1 - \epsilon\cos\theta)^2}.$$

Now, if we set $z = e^{i\theta}$, with $dz = i\, e^{i\theta}\, d\theta = iz\, d\theta$, the integral becomes a contour integral around the unit circle C [contour (c) in figure 24.1 with $R = 1$]. The area integral is then given by

$$\begin{aligned}
\frac{2A}{\ell^2} &= \int_C \frac{-iz^{-1}\, dz}{[\,1 - \frac{1}{2}\epsilon(z + z^{-1})\,]^2} \\
&= \frac{4}{\epsilon^2} \int_C \frac{-iz\, dz}{(z^2 - 2\epsilon^{-1}z + 1)^2} \\
\frac{A\epsilon^2}{2\ell^2} &= \int_C \frac{-iz\, dz}{(z - z_0)^2 (z - z_1)^2}, \text{ where } z_{1,0} = \frac{1}{\epsilon} \pm \sqrt{\frac{1}{\epsilon^2} - 1}.
\end{aligned}$$

Since $0 < \epsilon < 1$, $\epsilon^{-1} > 1$ and only the pole at $z_0 = \epsilon^{-1} - (\epsilon^{-2} - 1)^{1/2}$ lies inside the circle $|z| = 1$. Clearly it is a double pole of the integrand.

To determine the residue at z_0 we set $z = z_0 + \xi$:

$$\begin{aligned}
\frac{-iz}{(z - z_0)^2 (z - z_1)^2} &= \frac{-i(z_0 + \xi)}{\xi^2 (z_0 - z_1 + \xi)^2} \\
&= \frac{-i(z_0 + \xi)}{\xi^2 (z_0 - z_1)^2} \left[1 - \frac{2\xi}{z_0 - z_1} + \frac{6\xi^2}{2!\,(z_0 - z_1)^2} - \cdots \right].
\end{aligned}$$

The residue, equal to the coefficient of ξ^{-1}, is

$$\frac{-i}{(z_0 - z_1)^2} \left(1 - \frac{2z_0}{z_0 - z_1} \right) = \frac{i(z_0 + z_1)}{(z_0 - z_1)^3} = \frac{2i\epsilon^{-1}}{-8(\epsilon^{-2} - 1)^{3/2}}.$$

Thus, by the residue theorem,

$$\frac{A\epsilon^2}{2\ell^2} = 2\pi i \frac{i\epsilon^{-1}}{-4(\epsilon^{-2} - 1)^{3/2}},$$

giving the area A as

$$A = \frac{\pi\ell^2\epsilon^{-1}}{\epsilon^2(\epsilon^{-2} - 1)^{3/2}} = \frac{\pi\ell^2}{(1 - \epsilon^2)^{3/2}}.$$

25

Applications of complex variables

Many of the exercises in this chapter involve contour integration and the choice of a suitable contour. In order to save the space taken by drawing several broadly similar figures that differ only in notation, the positions of poles, the values of lengths or angles, or other minor details, we showed in figure 24.1 of the previous chapter a number of typical contour types to which reference can be made.

> **25.2** *A long straight fence made of conducting wire mesh separates two fields and stands 1 metre high. Sometimes, on fine days, there is a vertical electric field over flat open countryside. Well away from the fence the strength of the field is E_0. By considering the effect of the transformation $w = (1-z^2)^{1/2}$ on the real and imaginary z-axes, find the strengths of the field* (a) *at a point one metre directly above the fence,* (b) *at ground level one metre to the side of the fence, and* (c) *at a point that is level with the top of the fence but one metre to the side of it. What is the direction of the field in case* (c)?

We first consider the situation of a uniform vertical field (conventionally in the positive y-direction) of strength E_0. The corresponding potential is $\phi = -E_0 y$ and, as this is the real part of $iE_0 z$, the appropriate complex potential is $f(z) = iE_0 z$.

Now consider what happens to the real z-axis ($y = 0$) under the transformation $w = (1 - z^2)^{1/2}$. For $-1 < x < 1$, $w = u + iv$ is real and covers $0 < u < 1$ twice, once in each direction. For $-\infty < x < -1$ and $1 < x < \infty$, w is imaginary and covers the v-axis in its entirety. Under the same transformation, the imaginary z-axis maps (twice) onto the part of the positive real w-axis given by $1 < u < \infty$.

Thus, apart from a rotation of the whole coordinate system through $\pi/2$, the transformation maps the equipotential (ground) surface $y = 0$ into another equipotential (ground) surface $u = 0$, but this time with a unit height 'fence' situated at $v = 0$, the fence being at the same potential as the surface.

Under the transformation, $w = (1 - z^2)^{1/2}$, or equivalently $z = (1 - w^2)^{1/2}$, the potential

$$f(z) = iE_0 z \to F(w) = iE_0(1 - w^2)^{1/2} = -E_0(w^2 - 1)^{1/2}.$$

The magnitude of the derivative $F' = dF/dw$ gives the strength of the field in the tranformed situation; the field's direction makes an angle $\pi - \arg F'$ with the u-axis, which corresponds to the upward vertical directly above the fence. The derivative is

$$\frac{dF}{dw} = -\frac{w\,E_0}{(w^2 - 1)^{1/2}}.$$

Its magnitude, and hence the strength of the field, is

$$\text{(a) for } w = 2 + 0i, \qquad \left| -\frac{2E_0}{(4-1)^{1/2}} \right| = \frac{2\,E_0}{\sqrt{3}},$$

$$\text{(b) for } w = 0 + 1i, \qquad \left| -\frac{iE_0}{(-1-1)^{1/2}} \right| = \frac{E_0}{\sqrt{2}},$$

$$\text{(c) for } w = 1 + 1i, \qquad \left| -\frac{(1+i)E_0}{(2i-1)^{1/2}} \right| = \frac{\sqrt{2}\,E_0}{5^{1/4}}.$$

In case (c),

$$\arg\left[-\frac{(1+i)E_0}{(2i-1)^{1/2}} \right] = \pi + \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{2}{-1} = \frac{5\pi}{4} - 1.017.$$

Thus the direction of the field at $1 + i$ makes an angle $\pi - (\frac{5\pi}{4} - 1.017) = 0.232$ radians with the upward vertical.

Finally, we note that the equipotential surfaces are closely packed just above the top of the fence but separate as they spread out to become asymptotically parallel to the ground (without, of course, ever crossing each other).

25.4 *Find a complex potential in the z-plane appropriate to a physical situation in which the half-plane $x > 0$, $y = 0$ has zero potential and the half-plane $x < 0$, $y = 0$ has potential V.*

By making the transformation $w = a(z + z^{-1})/2$, with a real and positive, find the electrostatic potential associated with the half-plane $r > a$, $s = 0$ and the half-plane $r < -a$, $s = 0$ at potentials 0 and V respectively.

We require a function whose real or imaginary part takes the value 0 whenever $y = 0$ and $x > 0$, and takes a constant non-zero value whenever $y = 0$ but $x < 0$. The argument of z does this but, unfortunately, $\arg z$ is not an analytic function.

However $\arg z$ is (within a constant multiplier) the imaginary part of $\ln z$, which *is* an analytic function over (nearly all of) the complex plane. To get the scale correct and avoid problems with discontinuities across the negative x-axis, we need to take as the solution of Laplace's equation

$$\text{either } \phi(x,y) = \text{Im}\left(\frac{V}{\pi}\ln z\right) \text{ or } \phi(x,y) = \text{Re}\left(-i\frac{V}{\pi}\ln z\right)$$

with $z = |z|e^{i\theta}$ restricted by $0 \le \theta \le \pi$. The solution for the half-space $y < 0$ is to be given by symmetry, rather than by continuing the function into $-\pi < \theta < 0$.

Now consider the conformal transformation

$$r + is = w = \frac{a}{2}\left(z + \frac{1}{z}\right).$$

The half-plane $y = 0$, $0 < x < \infty$ becomes

$$r + is = \frac{a}{2}\left(x + \frac{1}{x}\right) \quad \Rightarrow \quad s = 0 \text{ and } a < r < \infty.$$

Similarly the half-plane $y = 0$, $-\infty < x < 0$ becomes

$$r + is = \frac{a}{2}\left(-|x| - \frac{1}{|x|}\right) \quad \Rightarrow \quad s = 0 \text{ and } -\infty < r < -a.$$

Thus the transformation maps the original half-planes (virtually touching at the origin) into two half-planes symmetrically separated about the origin by $2a$. This is as neeeded. To find the corresponding complex potential we must express $(V/\pi)\ln z$ in terms of w. We start by inverting the transformation,

$$z^2 - \frac{2w}{a}z + 1 = 0,$$

$$z = \frac{w}{a} \pm \sqrt{\frac{w^2}{a^2} - 1},$$

and then substitute for z,

$$F(w) = \frac{V}{\pi}\ln\left(\frac{w \pm \sqrt{w^2 - a^2}}{a}\right).$$

This, or $-i$ times it, (depending upon whether the imaginary or real part of the complex potential is taken as the physical potential) is the appropriate complex potential. Consideration of the particular case $w = is$ with $s > 0$, which must yield $+\frac{1}{2}V$ for the physical potential, shows that the $+$ sign for the square root is the correct choice. [Choosing the minus sign would make the expression in parentheses both imaginary and negative, and lead to a physical potential of $-\frac{1}{2}V$.]

25.6 *For the equation* $8z^3 + z + 1 = 0$*:*

(a) *show that all three roots lie between the circles* $|z| = 3/8$ *and* $|z| = 5/8$;
(b) *find the approximate location of the real root, and hence deduce that the complex ones lie in the first and fourth quadrants and have moduli greater than* 0.5.

(a) We start by considering $h(z) = 8z^3 + z + 1$ as $f(z) + g(z)$ where $f(z) = 8z^3$ and $g(z) = z + 1$. Now, on the circle $|z| = \frac{5}{8}$,

$$|8z^3| = \frac{1000}{512} = \frac{125}{64} \quad \text{and} \quad |z + 1| \leq \frac{5}{8} + 1 = \frac{104}{64}.$$

Thus $|f(z)| > |g(z)|$ at all points on the circle. It then follows from Rouché's theorem that $h(z)$ and $f(z)$ have the same number of zeroes inside the circle; $f(z)$ clearly has three (all at the origin), implying that all three zeroes of $h(z)$ lie within the same circle.

We next consider $h(z) = 8z^3+z+1$ as $f(z)+g(z)$ where $f(z) = 1$ and $g(z) = 8z^3+z$. On the circle $|z| = \frac{3}{8}$,

$$|1| = 1 \quad \text{and} \quad |8z^3 + z| \leq \frac{216}{512} + \frac{3}{8} = \frac{51}{64}.$$

Thus $|f(z)| > |g(z)|$ at all points on the circle. As before, it follows that $h(z)$ and $f(z)$ have the same number of zeroes inside the circle; $f(z)$ clearly has no zeroes inside the circle, and so, therefore, neither has $h(z)$.

Combining these results shows that all roots of the equation lie *between* the circles $|z| = 3/8$ and $|z| = 5/8$.

(b) Since the order of the cubic $h(z)$ is odd, the equation must have at least one real root; further, since the signs of all the coefficients in the cubic are the same, the real root must be negative. Let it be at x and write the equation in the form

$$8z^3 + z + 1 = 8(z - x)(z - \alpha - i\beta)(z - \alpha + i\beta) = 0.$$

Considering the coefficients of z^2 gives $-x - 2\alpha = 0$, showing that, since x is negative, α must be positive, i.e. the complex roots occur in the first and fourth quadrants.

At $x = -\frac{3}{8}$, $h(z) = \frac{13}{64}$, whilst at $z = -\frac{1}{2}$, $h(z) = -\frac{1}{2}$. Thus the real root lies between these two negative values of x. From the constant terms in the above expression of $h(z) = 0$ we deduce that $-8x(\alpha^2 + \beta^2) = 1$. Since $|x| < \frac{1}{2}$ it follows that $\alpha^2 + \beta^2 > \frac{1}{4}$ and that the modulus of either complex root $\sqrt{\alpha^2 + \beta^2} > 0.5$.

25.8 *The following is a method of determining the number of zeroes of an nth-degree polynomial $f(z)$ inside the contour C given by $|z| = R$:*

(a) *put $z = R(1+it)/(1-it)$, with $t = \tan(\theta/2)$ in the range $-\infty \le t \le \infty$;*
(b) *obtain $f(z)$ as*

$$\frac{A(t)+iB(t)}{(1-it)^n}\frac{(1+it)^n}{(1+it)^n};$$

(c) *it follows that $\arg f(z) = \tan^{-1}(B/A) + n\tan^{-1} t$;*
(d) *and that $\Delta_C[\arg f(z)] = \Delta_C[\tan^{-1}(B/A)] + n\pi$;*
(e) *determine $\Delta_C[\tan^{-1}(B/A)]$ by evaluating $\tan^{-1}(B/A)$ at $t = \pm\infty$ and finding the discontinuities in B/A by inspection or using a sketch graph.*

Then, by the principle of the argument, the number of zeroes inside C is given by the integer $(2\pi)^{-1}\Delta_C[\arg f(z)]$.

It can be shown that the zeroes of z^4+z+1 lie one in each quadrant. Use the above method to show that the zeroes in the second and third quadrants have $|z| < 1$.

(a) and (b). In this exercise we are concerned with the contour $|z| = 1$ and so we set

$$z = \frac{1+it}{1-it} = \frac{(1+it)^2}{(1-it)(1+it)} = \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}, \qquad t = \tan\frac{\theta}{2}, \qquad -\infty < t < \infty.$$

As θ increases from 0 to 2π, $z = \cos\theta + i\sin\theta$ and prescribes the unit circle, as can be verified from the half-angle identities. In terms of t increasing from $-\infty$ to $+\infty$, it is easier to think of θ increasing from $-\pi$ to $+\pi$, but the result is the same, as the circle is traversed once in the positive direction in either case.

The expression for $f(z)$ in terms of t is

$$\begin{aligned} g(t) &= \frac{1}{(1-it)^4}[\,(1+it)^4 + (1+it)(1-it)^3 + (1-it)^4\,] \\ &= \frac{1}{(1-it)^4}[\,t^4(1-1+1) + t^3(-4i+i-3i+4i) \\ &\qquad + t^2(-6-3+3-6) + t(4i-3i+i-4i) + (1+1+1)\,] \\ &= \frac{1}{(1-it)^4}(t^4 - 2it^3 - 12t^2 - 2it + 3) \\ &= \frac{A(t)+iB(t)}{(1-it)^4}\,\frac{(1+it)^4}{(1+it)^4}, \end{aligned}$$

where $A(t) = t^4 - 12t^2 + 3$ and $B(t) = -2t^3 - 2t$.

(c) Since its denominator is real and its numerator contains two factors, the argument of $g(t)$ is the sum of the arguments of these two factors.

$$\begin{aligned}\arg g(t) &= \arg[\,A(t) + iB(t)\,] + \arg[\,(1+it)^4\,] \\ &= \tan^{-1}\frac{B}{A} + 4\arg(1+it) \\ &= \tan^{-1}\frac{B}{A} + 4\tan^{-1} t \\ &= \tan^{-1}\frac{B}{A} + 4\frac{\theta}{2}.\end{aligned}$$

(d) The change in $\arg g$ around C is therefore

$$\begin{aligned}\Delta_C(\arg g) &= \Delta_C\left(\tan^{-1}\frac{B}{A}\right) + 4\Delta_C\left(\frac{\theta}{2}\right) \\ &= \Delta_C\left(\tan^{-1}\frac{B}{A}\right) + 4\pi \\ &= \Delta_C\left(\tan^{-1}\frac{-2t^3-2t}{t^4-12t^2+3}\right) + 4\pi \qquad -\infty < t < \infty, \\ &\equiv \Delta_C(\alpha) + 4\pi, \qquad \text{thus defining } \alpha.\end{aligned}$$

(e) Taking account of the magnitude of α and the signs of the numerator and denominator of $\tan\alpha$ separately, we deduce that as $t \to -\infty$, $\alpha \to 0_+$ and as $t \to \infty$, $\alpha \to 0_-$. We also note that, for real t, the numerator of $\tan\alpha$ is zero only when $t = 0$ and that the denominator is zero when $t^2 = 6 \pm \sqrt{33}$. Thus, as t increases from $-\infty$ through $-(6+\sqrt{33})^{1/2}$, a graph of α increases from 0 and passes through $\pi/2$. However, it decreases through $\pi/2$ at $-(6-\sqrt{33})^{1/2}$ without reaching π in between. It then passes through 0 at $t = 0$. The rest of the graph is the antisymmetric reflection in the line $t = 0$ of the first half.

Thus there are no discontinuities in α and, as it starts and ends at 0, $\Delta_C(\alpha) = 0$. It follows that $\Delta_C(\arg g) = 4\pi$ and that there are 2 zeroes inside $|z| = 1$.

To determine in which quadrants the two zeroes of $f(z) = z^4+z+1$ occur requires some numerical work. On the circle $|z| = 1$,

$$\arg f(z) = \tan^{-1}\frac{\sin 4\theta + \sin\theta}{\cos 4\theta + \cos\theta + 1}.$$

For a contour such as (b) in figure 24.1 with $R = 1$, $\Delta_{OA}(\arg f) = 0$ since z, and hence f, is purely real. On BO, where $z = iy$, with $1 \geq y \geq 0$, the argument of f is equal to $\tan^{-1}[\,y/(y^4+1)\,]$ and varies smoothly from $\tan^{-1}\frac{1}{2}$ to 0. On AB it is given by the above expression. Numerical investigation (using a spreadsheet, say) shows that the variation from 0 at $\theta = 0$ to $\tan^{-1}\frac{1}{2}$ at $\theta = \pi/2$, whilst having several turning points, has no discontinuities. [By contrast, there is a discontinuity at $\theta \approx 2.095$ in the second quadrant.]

Putting these observations together shows that $\Delta_{OABO}[\arg f(z)] = 0$ and that the contour of unit radius encloses no zeroes. Since the zeroes of polynomials with real coefficients occur in complex conjugate pairs, it follows that a similar contour in the fourth quadrant also encloses no zeroes. Hence the two zeroes inside $|z| = 1$ must lie in the second and third quadrants.

25.10 *This exercise illustrates a method of summing some infinite series.*

(a) *Determine the residues at all the poles of the function*

$$f(z) = \frac{\pi \cot \pi z}{a^2 + z^2},$$

where a is a positive real constant.

(b) *By evaluating, in two different ways, the integral I of f(z) along the straight line joining* $-\infty - ia/2$ *and* $+\infty - ia/2$, *show that*

$$\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi \coth \pi a}{2a} - \frac{1}{2a^2}.$$

(c) *Deduce the value of* $\sum_1^\infty n^{-2}$.

(a) This function has simple poles at $z = n$ whenever n is an integer. At each one the residue is

$$\frac{\pi \cos \pi z}{d(\sin \pi z)/dz} \frac{1}{a^2 + z^2} = \frac{\pi \cos \pi z}{\pi \cos \pi z} \frac{1}{a^2 + n^2} = \frac{1}{a^2 + n^2}.$$

There are two other poles, at $z = \pm ia$, and they have (equal) residues of

$$\frac{\pi \cot(\pm i\pi a)}{\pm 2ia} = -\frac{\pi \coth(\pi a)}{2a}.$$

(b) As $|z| \to \infty$, $\cot \pi z$ is bounded in both the upper and lower half-planes (i.e. away from the real axis) and so the integrals along the semicircular paths Γ_1 and Γ_2 shown in figure 25.1 tend to zero as the radius of the circle $\to \infty$. Now take as a closed contour the line L, joining $-\infty - ia/2$ and $+\infty - ia/2$, together with Γ_1 and apply the residue theorem

$$I + 0 = 2\pi i \left[2 \sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} + \frac{1}{a^2} - \frac{\pi \coth(\pi a)}{2a} \right].$$

Next take a closed contour consisting of L and Γ_2 and again apply the theorem (taking account of the sense of integration)

$$-I + 0 = 2\pi i \left[-\frac{\pi \coth(\pi a)}{2a} \right].$$

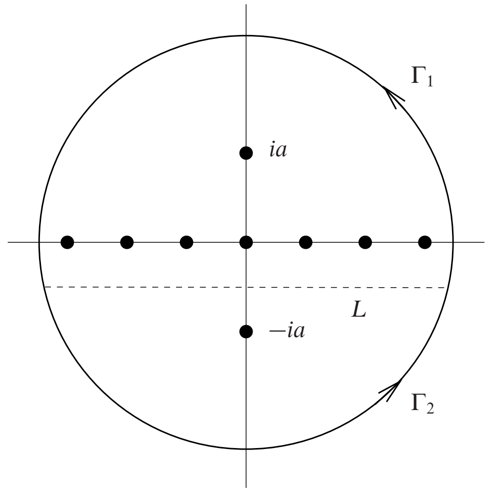

Figure 25.1 The contours used in exercise 25.10.

Adding these results gives

$$2\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}+\frac{1}{a^2}-\frac{2\pi\coth(\pi a)}{2a}=0,$$

i.e.

$$\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}=\frac{\pi\coth(\pi a)}{2a}-\frac{1}{2a^2}.$$

(c) To deduce the value of $S=\sum_{n=1}^{\infty}n^{-2}$ we need to evaluate

$$\lim_{a\to 0}\left(\frac{\pi\coth(\pi a)}{2a}-\frac{1}{2a^2}\right).$$

For this we employ l'Hôpital's rule.

$$\begin{aligned}
S&=\lim_{a\to 0}\frac{a\pi\cosh(a\pi)-\sinh(a\pi)}{2a^2\sinh(a\pi)}\\
&=\lim_{a\to 0}\frac{\pi\cosh(a\pi)+a\pi^2\sinh(a\pi)-\pi\cosh(a\pi)}{4a\sinh(a\pi)+2a^2\pi\cosh(a\pi)}\\
&=\lim_{a\to 0}\frac{\pi^2\sinh(a\pi)+a\pi^3\cosh(a\pi)}{4\sinh(a\pi)+8\pi a\cosh(a\pi)+2a^2\pi^2\sinh(a\pi)}\\
&=\lim_{a\to 0}\frac{2\pi^3\cosh(a\pi)+a\pi^4\sinh(a\pi)}{4\pi\cosh(a\pi)+8\pi\cosh(a\pi)+12\pi^2a\sinh(a\pi)+2a^2\pi^3\cosh(a\pi)}\\
&=\frac{2\pi^3}{12\pi}=\frac{\pi^2}{6}.
\end{aligned}$$

Not the most straightforward way to derive this particular result!

25.12 *Use the Bromwich inversion, and contours similar to that shown in figure 24.1(g) to find the functions of which the following are the Laplace transforms:*

(a) $s(s^2 + b^2)^{-1}$;
(b) $n!(s - a)^{-(n+1)}$, *with n a positive integer and $s > a$;*
(c) $a(s^2 - a^2)^{-1}$, *with $s > |a|$.*

Compare your answers with those given in a table of standard Laplace transforms.

(a) Inside the suggested contour $s(s^2 + b^2)^{-1}$ has simple poles at $s = \pm ib$. At $s = ib$, the residue of $se^{sx}(s^2 + b^2)^{-1}$ is $ib\, e^{ibx}/(ib + ib) = \frac{1}{2}\, e^{ibx}$. At $s = -ib$, the residue of $se^{sx}(s^2 + b^2)^{-1}$ is $-ib\, e^{-ibx}/(-ib - ib) = \frac{1}{2}\, e^{-ibx}$. Thus the function $f(x)$ whose Laplace transform is $s(s^2 + b^2)^{-1}$ is $\frac{1}{2}(\, e^{ibx} + e^{-ibx}) = \cos bx$.

(b) For the inverse Laplace transform of $\dfrac{n!}{(s-a)^{n+1}}$ the contour formed by closing the Bromwich line in the left half-plane contains a pole in the right half-plane, at $z = a$. We therefore make a change of variable to $t = s - a$ with $t > 0$ and again use a closed contour such as (g) in figure 24.1. The integrand becomes

$$\frac{n!e^{x(t+a)}}{t^{n+1}} = \frac{n!e^{ax}}{t^{n+1}} \sum_{r=0}^{\infty} \frac{(xt)^r}{r!}.$$

The only pole enclosed by the new contour is at $t = 0$ and the coefficient of t^{-1} in the Taylor expansion about that point is

$$a_{-1} = \frac{n!\, e^{ax}\, x^n}{n!} = e^{ax} x^n.$$

This is the residue at that point and also the required function $f(x) = e^{ax}x^n$; note that the given function of s is its Laplace transform only for $s > a$.

(c) As in part (b), the integrand, in this case $ae^{sx}(s^2 - a^2)^{-1}$, has a pole in the right half-plane. The same formal device as in part (b) could be used to convert the integral to one of type (g) with $\lambda = \epsilon > 0$ for any small ϵ. However, we can obtain the correct result by ignoring this 'nicety' and merely noting that the integrand has simple poles with residues

$$\frac{ae^{ax}}{a+a} \text{ at } s = a \qquad \text{and} \qquad \frac{ae^{-ax}}{-a-a} \text{ at } s = -a.$$

The function of which $a(s^2 - a^2)^{-1}$ is the Laplace transform is therefore $f(x) = \frac{1}{2}(e^{ax} - e^{-ax}) = \sinh ax$. This conclusion is, however, valid only for $s > |a|$.

25.14 *A function $f(t)$ has the Laplace transform*

$$F(s) = \frac{1}{2i} \ln\left(\frac{s+i}{s-i}\right),$$

the complex logarithm being defined by a finite branch cut running along the imaginary axis from $-i$ to i.

(a) *Convince yourself that, for $t > 0$, $f(t)$ can be expressed as a closed contour integral that encloses only the branch cut.*
(b) *Calculate $F(s)$ on either side of the branch cut, evaluate the integral and hence determine $f(t)$.*
(c) *Confirm that the derivative with respect to s of the Laplace transform integral of your answer is the same as that given by dF/ds.*

From the standard Bromwich integral representation, $f(t)$ can be written

$$f(t) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{1}{2i} \ln\left(\frac{s+i}{s-i}\right) e^{st}\, ds.$$

For $t < 0$ the contour has to be closed, as usual, in the right half-plane [contour (h) of figure 24.1]. There are no poles with Re $s > \lambda$ and so the integral, and hence $f(t)$, are both zero for $t < 0$.

(a) For $t > 0$ the contour has to be closed in the left half-plane [contour (g)]. The integrand is

$$\frac{1}{2i}\left\{ \ln\left|\frac{s+i}{s-i}\right| e^{st} + i[\arg(s+i) - \arg(s-i)]e^{st} \right\}.$$

This has no singularities in the left half-plane (excluding the imaginary axis) and so, by Cauchy's theorem, the contour can be deformed to be a line $z = -\lambda' + iy$ for any real $\lambda' > 0$. There are no poles on the imaginary axis beyond the cut and so the contour can be further deformed (shrunk) to one that just encloses the cut.

(b) We introduce plane-polar angles θ_1 and θ_2 centred respectively on the points $s = -i$ and $s = i$. Each is restricted to a range of 2π but where the zero of each is taken does not matter. Let a point P close to the cut but to the right of it, and given by $s = \epsilon + iy$, where $\epsilon > 0$ and $-1 < y < 1$, correspond to $\theta_1 = \phi_1$ and $\theta_2 = \phi_2$. Then

$$F(\epsilon + iy) = \frac{1}{2i}\left[\ln\left(\frac{1+y}{1-y}\right) + i\phi_1 - i\phi_2 \right].$$

As P moves (upward) along a path beside the cut, rounds the point $s = i$ and moves (downwards) beside, but to the left of, the cut to reach $s = -\epsilon + iy$, the value of θ_1 hardly changes and is finally the same as it started. However, θ_2 increases by 2π (as P rounds $s = i$). The new value of $F(s)$ is

$$F(-\epsilon + iy) = \frac{1}{2i}\left[\ln\left(\frac{1+y}{1-y}\right) + i\phi_1 - i(\phi_2 + 2\pi)\right].$$

In the evaluation of the integral y runs from -1 to 1 on the right of the cut and from 1 to -1 on the left. The two contributions from the terms containing logarithms clearly cancel and we are left with

$$\int_{-1}^{1} \frac{1}{2}(\phi_1 - \phi_2)e^{iyt} i\,dy + \int_{1}^{-1} \frac{1}{2}(\phi_1 - \phi_2 - 2\pi)\,e^{iyt} i\,dy.$$

The integrals involving ϕ_1 and ϕ_2 cancel and leave as the only contribution to the Bromwich integral

$$J = \int_{-1}^{1} \pi\, e^{iyt} i\,dy = \pi i \frac{e^{it} - e^{-it}}{it} = 2\pi i \frac{\sin t}{t}.$$

Hence $f(t)$, which is equal to $J/(2\pi i)$, has the form $\sin t/t$.

(c) Firstly, the derivative with respect to s of the Laplace transform of the solution just obtained is

$$\begin{aligned}
\frac{d}{ds}\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt &= -\int_0^\infty e^{-st}\sin t\,dt \\
&= -\text{Im}\int_0^\infty e^{-st}\,e^{it}\,dt \\
&= -\text{Im}\left[\frac{e^{-st+it}}{-s+i}\right]_0^\infty \\
&= -\text{Im}\left(\frac{1}{s-i}\right) = -\frac{1}{1+s^2}.
\end{aligned}$$

Secondly, the corresponding derivative of the given form for $F(s)$ is

$$\begin{aligned}
\frac{dF(s)}{ds} &= \frac{1}{2i}\frac{d}{ds}[\ln(s+i) - \ln(s-i)] \\
&= \frac{1}{2i}\left(\frac{1}{s+i} - \frac{1}{s-i}\right) \\
&= \frac{1}{2i}\frac{-2i}{s^2+1} = -\frac{1}{s^2+1}.
\end{aligned}$$

Thus, $\displaystyle\frac{d}{ds}\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \frac{dF(s)}{ds}$, confirming the result stated in the question.

25.16 *Transverse vibrations of angular frequency ω on a string stretched with constant tension T are described by $u(x,t) = y(x)\,e^{-i\omega t}$ where*

$$\frac{d^2y}{dx^2} + \frac{\omega^2\, m(x)}{T}\, y(x) = 0.$$

Here, $m(x) = m_0 f(x)$ is the mass per unit length of the string and, in the general case, is a function of x. Find the first-order W.K.B. solution for $y(x)$.

Due to imperfections in its manufacturing process, a particular string has a small periodic variation in its linear density of the form $m(x) = m_0[\,1 + \epsilon \sin(2\pi x/L)\,]$, where $\epsilon \ll 1$. A progressive wave (i.e. one in which no energy is lost) travels in the positive x-direction along the string. Show that its amplitude fluctuates by $\pm\frac{1}{4}\epsilon$ of its value A_0 at $x = 0$ and that, to first order in ϵ, the phase of the wave is

$$\frac{\epsilon\,\omega\, L}{2\pi}\sqrt{\frac{m_0}{T}}\,\sin^2\frac{\pi x}{L}$$

ahead of what it would be if the string were uniform with $m(x) = m_0$.

We first write $\alpha^2 = m_0\omega^2/T$ and assume that $\alpha \gg 1$, so that the W.K.B. method is appropriate. We now try as a solution to

$$\frac{d^2y}{dx^2} + \alpha^2 f(x)y = 0 \qquad (*)$$

the form $y(x) = A(x)e^{i\alpha\phi(x)}$. The necessary derivatives are

$$y' = A'e^{i\alpha\phi} + i\alpha A\phi' e^{i\alpha\phi},$$
$$y'' = A''e^{i\alpha\phi} + 2i\alpha A'\phi' e^{i\alpha\phi} + i\alpha A\phi'' e^{i\alpha\phi} - \alpha^2 A(\phi')^2 e^{i\alpha\phi}.$$

Substituting these into $(*)$ and cancelling a factor of $e^{i\alpha\phi}$ throughout, yields

$$A'' + i\alpha(2A'\phi' + A\phi'') + \alpha^2[\,Af - A(\phi')^2\,] = 0.$$

The first W.K.B. approximations is obtained by setting the coefficients of α^2 and α (both $\gg 1$, but of different orders of magnitude) separately equal to zero (and assuming that A'' can be ignored).

$$(\phi')^2 = f \quad \Rightarrow \quad \phi(x) = \int_0^x \sqrt{f(u)}\, du,$$
$$2A'\phi' + A\phi'' = 0 \quad \Rightarrow \quad \frac{2A'}{A} + \frac{\phi''}{\phi'} = 0 \quad \Rightarrow \quad \ln(A^2\phi') = k_1$$
$$\Rightarrow \quad A^2 = \frac{k_2}{f^{1/2}} \quad \Rightarrow \quad A(x) = \frac{c}{[\,m(x)\,]^{1/4}}.$$

Collecting these results together gives the first-order W.K.B. solution as

$$y(x) = \frac{c}{[\,m(x)\,]^{1/4}} \exp\left[\, i\alpha \int_0^x \sqrt{f(u)}\, du \,\right],$$

with α as defined above.

We now substitute $m(x) = m_0[\,1 + \epsilon \sin(2\pi x/L)]$ in the above general result and expand the expressions containing $m(x)$ up to first order in ϵ:

$$\begin{aligned} y(x) &= \frac{c}{[\,m(x)\,]^{1/4}} \exp\left(i\alpha \int_0^x \sqrt{1 + \epsilon \sin\frac{2\pi u}{L}}\, du \right), \\ &= \frac{c}{m_0^{1/4}} (1 + \epsilon \sin\frac{2\pi x}{L})^{-1/4} \exp\left[i\alpha \int_0^x \left(1 + \frac{\epsilon}{2} \sin\frac{2\pi u}{L}\right) du \right] \\ &= \frac{c}{m_0^{1/4}} (1 - \frac{\epsilon}{4} \sin\frac{2\pi x}{L}) \exp\left\{ i\alpha \left[x + \frac{\epsilon L}{4\pi} \left(1 - \cos\frac{2\pi x}{L}\right) \right] \right\}. \end{aligned}$$

Thus, the amplitude at $x = 0$ is $A_0 = c/m_0^{1/4}$ and its variation at other values of x is up to $\pm\frac{1}{4}\epsilon A_0$.

With $m(x) = m_0$, the phase would be $\phi = \alpha x$ but, with the variation in linear density, it is ahead of this by

$$\alpha\frac{\epsilon L}{4\pi} \left(1 - \cos\frac{2\pi x}{L}\right) = \frac{\epsilon\, \omega\, L}{2\pi} \sqrt{\frac{m_0}{T}} \sin^2\frac{\pi x}{L}.$$

25.18 *A W.K.B. solution of Bessel's equation of order zero,*

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + y = 0, \qquad (*)$$

valid for large $|z|$ *and* $-\pi/2 < \arg z < 3\pi/2$ *is* $y(z) = Az^{-1/2}e^{iz}$. *Obtain an improvement on this by finding a multiplier of* $y(z)$ *in the form of an asymptotic expansion in inverse powers of* z *as follows.*

(a) *Substitute for* $y(z)$ *in* $(*)$ *and show that the equation is satisfied to* $O(z^{-5/2})$.

(b) *Now replace the constant* A *by* $A(z)$ *and find the equation that must be satisfied by* $A(z)$. *Look for a solution of the form* $A(z) = z^{\sigma} \sum_{n=0}^{\infty} a_n z^{-n}$ *where* $a_0 = 1$. *Show that* $\sigma = 0$ *is the only acceptable solution to the indicial equation and obtain a recurrence relation for the* a_n.

(c) *To within a (complex) constant, the expression* $y(z) = A(z)z^{-1/2}e^{iz}$ *is the asymptotic expansion of the Hankel function* $H_0^{(1)}(z)$. *Show that it is a divergent expansion for all values of* z *and estimate, in terms of* z, *the value of* N *such that* $\sum_{n=0}^{N} a_n z^{-n-1/2} e^{iz}$ *gives the best estimate of* $H_0^{(1)}(z)$.

(a) We first calculate the required derivatives appearing in Bessel's equation, which are given by

$$\frac{dy}{dz} = -\frac{A}{2z^{3/2}}\,e^{iz} + \frac{iA}{z^{1/2}}\,e^{iz},$$
$$\frac{d^2y}{dz^2} = \frac{(3/2)A}{2z^{5/2}}\,e^{iz} - 2\frac{iA}{2z^{3/2}}\,e^{iz} + \frac{i^2A}{z^{1/2}}\,e^{iz}.$$

Substituting these into (∗), and cancelling common factors A and e^{iz}, gives

$$\left(\frac{3}{4z^{5/2}} - \frac{i}{z^{3/2}} - \frac{1}{z^{1/2}}\right) + \left(-\frac{1}{2z^{5/2}} + \frac{i}{z^{3/2}}\right) + \frac{1}{z^{1/2}} = 0.$$

The equation is not satisfied exactly, the LHS having a value $z^{-5/2}/4$. Thus the error is $\mathrm{O}(z^{-5/2})$.

(b) The first task is to find the equation satisfied by $A(z)$. We can make use of the result of part (a) by denoting the new $y(z)$ by $y(z) = A(z)y_1(z) \equiv A(z)z^{-1/2}e^{iz}$, where

$$\frac{d^2y_1}{dz^2} + \frac{1}{z}\frac{dy_1}{dz} + y_1 = \frac{e^{iz}}{4z^{5/2}}.$$

Using Leibnitz' theorem to calculate the LHS of (∗) gives

$$A''y_1 + 2A'y_1' + Ay_1'' + z^{-1}(A'y_1 + Ay_1') + Ay_1 = 0,$$
$$\Rightarrow \qquad A''y_1 + 2A'y_1' + \frac{1}{z}\,A'y_1 + \frac{Ae^{iz}}{4z^{5/2}} = 0,$$
$$\Rightarrow \qquad \frac{A''}{z^{1/2}} + A'\left(-\frac{2}{2z^{3/2}} + \frac{2i}{z^{1/2}} + \frac{1}{z^{3/2}}\right) + \frac{A}{4z^{5/2}} = 0,$$
$$\Rightarrow \qquad 4z^2A'' + 8iz^2A' + 1 = 0. \qquad (**)$$

This is the equation to be satisfied by $A(z)$ and so, setting $A(z) = z^{\sigma}\sum_{n=0}^{\infty} a_n z^{-n}$, we have

$$4\sum_{n=0}^{\infty}(\sigma - n)(\sigma - n - 1)a_n z^{\sigma-n} + 8i\sum_{n=0}^{\infty}(\sigma - n)a_n z^{\sigma-n+1} + \sum_{n=0}^{\infty} a_n z^{\sigma-n} = 0.$$

The highest power of z present on the LHS is that of $z^{\sigma-n+1}$ when $n = 0$. Its coefficient is $8i(\sigma - 0)a_0$ and, since $a_0 \neq 0$, we must have $\sigma = 0$.

When this value is used the equation reduces to

$$4\sum_{n=0}^{\infty} n(n+1)a_n z^{-n} - 8i\sum_{n=0}^{\infty} na_n z^{-n+1} + \sum_{n=0}^{\infty} a_n z^{-n} = 0.$$

Equating the coefficients of z^{-n} gives

$$4n(n+1)a_n - 8i(n+1)a_{n+1} + a_n = 0 \quad \Rightarrow \quad a_{n+1} = -i\frac{(2n+1)^2}{8(n+1)}\,a_n.$$

(c) Successive terms t_n in the series (which has alternate real and imaginary terms when z is real) are related by

$$t_{n+1} = a_{n+1} z^{-(n+1)} = -i\frac{(2n+1)^2}{8(n+1)}\, a_n z^{-(n+1)} = -i\frac{(2n+1)^2}{8(n+1)}\, z^{-1} t_n.$$

The numerator of the fraction on the RHS varies as n^2 whilst the denominator is linear in n. Consequently, no matter how large $|z|$ is, $|t_{n+1}| > |t_n|$ for sufficiently large n, i.e. the expansion is divergent. The closest estimate to $H_0^{(1)}(z)$ given by a sum of the form $\sum_{n=0}^{N} a_n z^{-n-1/2} e^{iz}$ is obtained when N is chosen so that

$$\frac{(2N+1)^2}{8(N+1)}\, \frac{1}{|z|} \approx 1,$$

i.e. $4N^2/8N \approx |z|$, or $N \approx 2|z|$. For N larger than this, the additional terms, and hence the uncertainty in the value of the function, begin to get bigger, rather than smaller.

25.20 *Use the method of steepest descents to show that an approximate value for the integral*

$$F(z) = \int_{-\infty}^{\infty} \exp[\, iz(\tfrac{1}{5}t^5 + t)\,]\, dt,$$

where z is real and positive, is

$$\left(\frac{2\pi}{z}\right)^{1/2} \exp(-\beta z)\, \cos(\beta z - \tfrac{1}{8}\pi),$$

where $\beta = 4/(5\sqrt{2})$.

Although this is an integral with respect to the real variable t, we will consider it as one along the real axis in the complex t-plane and then distort its path so that it passes from $t = -\infty$ to $t = +\infty$ via one or more of the saddle points t_i of the complex function.

The saddle points are situated where the gradient of the integrand is zero. The values of t at which this happens are given by

$$0 = \frac{d}{dt}[\, iz(\tfrac{1}{5}t^5 + t)\,] = iz(t^4 + 1) \quad \Rightarrow \quad t_i = \pm e^{i\pi/4} \text{ or } \pm e^{3i\pi/4}.$$

We will use the two in the upper half plane, $t_1 = e^{3i\pi/4}$ and $t_2 = e^{i\pi/4}$.

The values of the exponents and their second derivatives at the saddles are

$$\begin{aligned}
f_1 = f(t_1) &= iz\left(\tfrac{1}{5}e^{15i\pi/4} + e^{3i\pi/4}\right) = \tfrac{4}{5}iz\, e^{3i\pi/4},\\
f_2 = f(t_2) &= iz\left(\tfrac{1}{5}e^{5i\pi/4} + e^{i\pi/4}\right) = \tfrac{4}{5}iz\, e^{i\pi/4},\\
f_1'' = f''(t_1) &= 4izt_1^3 = 4iz\, e^{9i\pi/4}\\
&\Rightarrow\ A_1 = 4z \text{ and } \alpha_1 = \tfrac{9}{4}\pi + \tfrac{1}{2}\pi = \tfrac{3}{4}\pi,\\
f_2'' = f''(t_2) &= 4izt_2^3 = 4iz\, e^{3i\pi/4}\\
&\Rightarrow\ A_2 = 4z \text{ and } \alpha_2 = \tfrac{3}{4}\pi + \tfrac{1}{2}\pi = \tfrac{5}{4}\pi.
\end{aligned}$$

We now need to determine the directions of the l.s.d. at the saddles and the senses in which they are traversed. At $t = t_1$ the directions θ of the l.s.d. are given in the usual way by $\sin(2\theta + \alpha_1) = 0$ and the appropriate pair of choices amongst these by the requirement that $\cos(2\theta + \alpha_1)$ is negative. With $\alpha_1 = 3\pi/4$, the two acceptable values of θ are $\pi/8$ and $9\pi/8$, with the contour, which starts at $-\infty$, clearly passing through the saddle and leaving it in the direction $\theta = \pi/8$. Since this lies in the range $-\frac{1}{2}\pi < \theta \le \frac{1}{2}\pi$, the contribution to the approximate value of the integral from this saddle point will be positive.

A similar analysis at $t = t_2$ shows that the contour following the l.s.d. there approaches the saddle from the direction $\theta = 7\pi/8$ and leaves it in the direction $\theta = -\pi/8$, again making a positive contribution to the value of the integral.

Finally, we substitute these calculated data into the standard formula for the steepest descents estimation,

$$\pm\left(\frac{2\pi}{A_i}\right)^{1/2} \exp(f_i)\exp[\,\tfrac{1}{2}i(\pi - \alpha_i)\,],$$

and obtain, with $4/(5\sqrt{2})$ written as β,

$$\begin{aligned}
F_1(z) &= +\left(\frac{2\pi}{4z}\right)^{1/2} \exp\left[\frac{4iz}{5}\,\frac{(-1+i)}{\sqrt{2}}\right] \exp[\,\tfrac{1}{2}i(\pi - \tfrac{3}{4}\pi)\,]\\
&= \left(\frac{\pi}{2z}\right)^{1/2} \exp(-\beta z)\, \exp\left(-i\beta z + \frac{i\pi}{8}\right),\\
F_2(z) &= +\left(\frac{2\pi}{4z}\right)^{1/2} \exp\left[\frac{4iz}{5}\,\frac{(1+i)}{\sqrt{2}}\right] \exp[\,\tfrac{1}{2}i(\pi - \tfrac{5}{4}\pi)\,]\\
&= \left(\frac{\pi}{2z}\right)^{1/2} \exp(-\beta z)\, \exp\left(i\beta z - \frac{i\pi}{8}\right).
\end{aligned}$$

Adding these two contributions together gives the stated result

$$F(z) = \left(\frac{2\pi}{z}\right)^{1/2} \exp(-\beta z)\, \cos(\beta z - \tfrac{1}{8}\pi).$$

25.22 *The Bessel function* $J_\nu(z)$ *is given for* $|\arg z| < \frac{1}{2}\pi$ *by the integral around a contour C of the function*

$$g(z) = \frac{1}{2\pi i}\, t^{-(\nu+1)} \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right].$$

The contour starts and ends along the negative real t-axis and encircles the origin in the positive sense. It can be considered as made up of two contours. One of them, C_2*, starts at* $t = -\infty$*, runs through the third quadrant to the point* $t = -i$ *and then approaches the origin in the fourth quadrant in a curve that is ultimately anti-parallel to the positive real axis. The other contour,* C_1*, is the mirror image of this in the real axis; it is confined to the upper half plane, passes through* $t = i$ *and is anti-parallel to the real t-axis at both of its extremities. The contribution to* $J_\nu(z)$ *from the curve* C_k *is* $\frac{1}{2}H_\nu^{(k)}$*, the function* $H_\nu^{(k)}$ *being known as a Hankel function.*

Using the method of steepest descents, establish the leading term in an asymptotic expansion for $H_\nu^{(1)}$ *for z real, large and positive. Deduce, without detailed calculation, the corresponding result for* $H_\nu^{(2)}$*. Hence establish the asymptotic form of* $J_\nu(z)$ *for the same range of z.*

We first note that, in the neighbourhood of any saddle point, we will treat that part of the integrand that is not exponentiated, $(2\pi i)^{-1}t^{-(\nu+1)}$, by assigning it the value $(2\pi i)^{-1}t_0^{-(\nu+1)}$, where t_0 is the location of that saddle point.

The ends of contour C_1 are at $t = -\infty$ and $t = 0$ and we next check the values of the integrand at these two points, remembering that z is real and positive,

$$t = -\infty,\ \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \exp\left[\frac{z}{2}(-\infty + 0)\right] = 0,$$
$$t = 0,\ \exp\left[\frac{z}{2}(0 - \infty)\right] = 0.$$

These are both satisfactory in so far as not invalidating the method is concerned.

The saddle points of the integrand $f(t)$ are given by

$$\frac{d}{dt}\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = 0 \quad \Rightarrow \quad 1 + \frac{1}{t^2} = 0, \text{ i.e. } t = \pm i.$$

Clearly we need $t = +i$ for the C_1 contour; $t = -i$ will be appropriate to the C_2 contour giving $H_\nu^{(2)}$.

The other derivatives and values needed at $t_0 = i$ are

$$\frac{d^2}{dt^2}\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] = \frac{z}{2}\,\frac{-2}{t^3} = -\frac{z}{t^3} = -iz, \text{ at } t = i,$$

$$f_0 \equiv f(t_0) = \frac{z}{2}\left(i - \frac{1}{i}\right) = iz,$$

$$t_0^{-(\nu+1)} = i^{-(\nu+1)} = e^{-i\pi(\nu+1)/2}.$$

In the standard notation, $f''(t_0) \equiv Ae^{i\alpha}$, we have $A = z$ and $\alpha = 3\pi/2$.

We now use the standard approach that if $t - t_0 = se^{i\theta}$ then

$$\left|\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right]\right| = |\exp(f_0)|\,\exp[\,\tfrac{1}{2}As^2(\cos 2\theta + \alpha) + O(s^3)\,].$$

and the line of steepest descents (l.s.d.) is given by the condition that the argument of $\exp\left[\,\frac{1}{2}z\left(t - t^{-1}\right)\,\right]$ is independent of s. This is $\sin(2\theta + \alpha) = 0$, leading to $\theta = \pm\frac{1}{4}\pi$ or $\theta = \pm\frac{3}{4}\pi$.

It is not immediately apparent which pair of directions from these four is the correct choice — for the contour to pass through the saddle point at $t = i$ from '(i) top right to bottom left' or '(ii) from bottom right to top left'. To determine which is correct, we evaluate $h(s) = \frac{1}{2}(t - t_0)^2 f''(t_0)$, where $t - t_0 = se^{i\theta}$, in each case

$$\begin{aligned}
\text{(i) } t - t_o \quad &= \quad se^{i\pi/4} \text{ (approaching) or } se^{-3i\pi/4} \text{ (leaving)}\\
&\Rightarrow \quad h(s) = \tfrac{1}{2}s^2 e^{i\pi/2}(-iz) = \tfrac{1}{2}s^2 z, \text{ i.e. real and } > 0,\\
\text{(ii) } t - t_o \quad &= \quad se^{-i\pi/4} \text{ (approaching) or } se^{3i\pi/4} \text{ (leaving)}\\
&\Rightarrow \quad h(s) = \tfrac{1}{2}s^2 e^{-i\pi/2}(-iz) = -\tfrac{1}{2}s^2 z, \text{ i.e. real and } < 0.
\end{aligned}$$

For approximating the integral by a Gaussian with its maximum at the saddle point, clearly we must have case (ii).

Finally, we use the standard form of the integral

$$I \approx \pm g(t_0)\left(\frac{2\pi}{A}\right)^{1/2}\exp(f_0)\exp[\,\tfrac{1}{2}i(\pi - \alpha)\,],$$

with the $\pm$ choice being resolved by the direction in which the l.s.d. passes through the saddle-point; it is positive if $|\theta| < \pi/2$ and negative otherwise. In this particular case, as we have just shown, the l.s.d. traverses the saddle-point in the direction $3\pi/4$ and the minus sign is appropriate. Putting in the specific values gives

$$\frac{1}{2}H_\nu^{(1)} = -\frac{1}{2\pi i}\left(\frac{2\pi}{z}\right)^{1/2}\exp(iz)\exp[\,\tfrac{1}{2}i(\pi - \tfrac{3}{2}\pi)\,]\,e^{-i\pi(\nu+1)/2},$$

$$H_\nu^{(1)} = \left(\frac{2}{\pi z}\right)^{1/2} e^{iz}e^{-i\pi/4}e^{-i\pi\nu/2}.$$

For $H_\nu^{(2)}$ we can deduce from symmetry/antisymmetry that $t_0 = -i$, $A = z$, $\alpha = \pi/2$, that the contour C_2 traverses the saddle point in the direction $+i\pi/4$ and, consequently, that the contribution is a positive one.

$$\begin{aligned} H_\nu^{(2)} &= +\frac{1}{\pi i}\left(\frac{2\pi}{z}\right)^{1/2} \exp(-iz)\exp[\,\tfrac{1}{2}i(\pi - \tfrac{1}{2}\pi)\,]\, e^{i\pi(\nu+1)/2} \\ &= \left(\frac{2}{\pi z}\right)^{1/2} e^{-iz} e^{i\pi/4} e^{i\pi\nu/2}. \end{aligned}$$

Now adding together the asymptotic forms of $\frac{1}{2}H_\nu^{(1)}$ and $\frac{1}{2}H_\nu^{(2)}$ to form that for $J_\nu(z)$ gives

$$\begin{aligned} J_\nu(z) &\sim \frac{1}{2}\left(\frac{2}{\pi z}\right)^{1/2} \left(e^{iz} e^{-i\pi/4} e^{-i\pi\nu/2} + e^{-iz} e^{i\pi/4} e^{i\pi\nu/2}\right) \\ &= \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi}{4} - \frac{\nu\pi}{2}\right). \end{aligned}$$

26

Tensors

26.2 *The components of two vectors* **A** *and* **B** *and a second-order tensor* **T** *are given in one coordinate system by*

$$\mathsf{A} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In a second coordinate system, obtained from the first by rotation, the components of **A** *and* **B** *are*

$$\mathsf{A}' = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix}, \quad \mathsf{B}' = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}.$$

Find the components of **T** *in this new coordinate system and hence evaluate, with a minimum of calculation,*

$$T_{ij}T_{ji}, \quad T_{ki}T_{jk}T_{ij}, \quad T_{ik}T_{mn}T_{ni}T_{km}.$$

Since we must have $x_i' = L_{ij}x_j$ and **A** and **B** have their components tranformed into the given values, L must have the form

$$\mathsf{L} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 & a \\ 0 & 0 & b \\ 1 & \sqrt{3} & c \end{pmatrix}.$$

We determine a, b and c by requiring that L is orthogonal and has $|\mathsf{L}| = +1$.

$$\mathsf{L}\mathsf{L}^{\mathrm{T}} = \frac{1}{4}\begin{pmatrix} \sqrt{3} & -1 & a \\ 0 & 0 & b \\ 1 & \sqrt{3} & c \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 1 \\ -1 & 0 & \sqrt{3} \\ a & b & c \end{pmatrix} = \mathsf{I},$$

giving $a = 0$, $b = \pm 2$ and $c = 0$. The determinant of L is $\frac{1}{8}(-3b - b + 0)$, thus requiring that $b = -2$. Hence the required orthoganal matrix L is

$$\mathsf{L} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix}.$$

The third column of L could have been obtained by considering $\mathsf{A}' \times \mathsf{B}'$. The matrix product $\mathsf{T}' = \mathsf{L}\mathsf{T}\mathsf{L}^{\mathrm{T}}$ is given by

$$\begin{aligned} \mathsf{T}' &= \frac{1}{4}\begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 1 \\ -1 & 0 & \sqrt{3} \\ 0 & -2 & 0 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 5 \\ -1 & 0 & 5\sqrt{3} \\ 0 & -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}. \end{aligned}$$

As a check, we note that $\mathrm{Tr}\,\mathsf{T}' = 1 + 2 + 5 = 2 + 4 + 2 = \mathrm{Tr}\,\mathsf{T}$.

In this new coordinate system T is diagonal – and therefore very convenient for calculating the following invariants (tensors of order 0); their values are independent of the frame in which they are calculated.
$T_{ij}T_{ji} = T'_{ij}T'_{ji} = 1 + 4 + 25 = 30.$
$T_{ki}T_{jk}T_{ij} = T_{ij}T_{jk}T_{ki} = T'_{ij}T'_{jk}T'_{ki} = 1 + 8 + 125 = 134.$
$T_{ik}T_{mn}T_{ni}T_{km} = T_{ik}T_{km}T_{mn}T_{ni} = T'_{ik}T'_{km}T'_{mn}T'_{ni} = 1 + 16 + 625 = 642.$

26.4 *Show how to decompose the Cartesian tensor T_{ij} into three tensors,*

$$T_{ij} = U_{ij} + V_{ij} + S_{ij},$$

where U_{ij} is symmetric and has zero trace, V_{ij} is isotropic and S_{ij} has only three independent components.

We start by writing T_{ij} as the sum of its even and odd parts:

$$T_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}) + \tfrac{1}{2}(T_{ij} - T_{ji}) \equiv \tfrac{1}{2}(T_{ij} + T_{ji}) + S_{ij}.$$

Clearly S_{ij} has zeroes on its leading diagonal and $S_{ji} = -S_{ij}$; it therefore has only 3 independent components, S_{12}, S_{13} and S_{23}.

Now, if the trace T_{ii} is written as T_0, then $\frac{1}{3}T_0\delta_{ij}$ is an isotropic tensor V_{ij}. Subtracting this from the symmetric part of T_{ij} leaves

$$U_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}T_0\delta_{ij}.$$

Since $U_{ij} = U_{ji}$, U_{ij} is symmetric. Further,

$$\text{Tr } U_{ij} = \frac{1}{2}(T_0 + T_0) - \frac{1}{3}T_0 3 = 0,$$

i.e. U_{ij} is traceless. This completes the decomposition.

If T_{ij} were a second-order tensor in n dimensions, S_{ij} would have $\frac{1}{2}n(n-1)$ independent components and the factor in V_{ij} would be $1/n$.

26.6 *Use tensor methods to establish the following vector identities:*

(a) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$;

(b) $\text{curl}\,(\phi\mathbf{u}) = \phi\,\text{curl}\,\mathbf{u} + (\text{grad}\,\phi) \times \mathbf{u}$;

(c) $\text{div}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}$;

(d) $\text{curl}\,(\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \text{grad})\mathbf{u} - (\mathbf{u} \cdot \text{grad})\mathbf{v} + \mathbf{u}\,\text{div}\,\mathbf{v} - \mathbf{v}\,\text{div}\,\mathbf{u}$;

(e) $\text{grad}\,\frac{1}{2}(\mathbf{u} \cdot \mathbf{u}) = \mathbf{u} \times \text{curl}\,\mathbf{u} + (\mathbf{u} \cdot \text{grad})\mathbf{u}$.

All of the expressions for vector operators in tensor notation are given in the text and should be known but, for convenience, the principal ones and the double epsilon identity are repeated here:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk}a_j b_k,$$
$$\nabla \cdot \mathbf{a} = \frac{\partial a_i}{\partial x_i},$$
$$(\nabla \times \mathbf{a})_i = \epsilon_{ijk}\frac{\partial a_k}{\partial x_j},$$
$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

In addition, it should be remembered that ϵ_{ijk} is merely a number and can be moved from under a differentiation sign.

Where the identity is a vector equation, our proofs will consider only its ith component; but, as i is general, this will establish the full vector identity. Case (c) is a scalar identity.

(a) Consider the ith component:

$$\begin{aligned}[(\mathbf{u}\times\mathbf{v})\times\mathbf{w}]_i &= \epsilon_{ijk}(\mathbf{u}\times\mathbf{v})_j w_k\\ &= \epsilon_{ijk}\epsilon_{jlm}u_l v_m w_k\\ &= (\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})u_l v_m w_k\\ &= v_i u_l w_l - u_i v_m w_m\\ &= [(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}]_i\end{aligned}$$

(b) Since ∇ is involved in this identity [and in those in parts (c)-(e)], we must take particular care with the order in which we write the differential operator and the functions (or products of functions) on which it might act. Again consider the ith component:

$$\begin{aligned}[\nabla\times(\phi\mathbf{u})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\phi u_k)\\ &= \epsilon_{ijk}\phi\frac{\partial u_k}{\partial x_j}+\epsilon_{ijk}u_k\frac{\partial\phi}{\partial x_j}\\ &= \phi(\nabla\times\mathbf{u})_i+(\nabla\phi\times\mathbf{u})_i\end{aligned}$$

(c) As noted above, this is a scalar quantity and any tensor expression for it must have all of its indices contracted.

$$\begin{aligned}\nabla\cdot(\mathbf{u}\times\mathbf{v}) &= \frac{\partial}{\partial x_i}(\epsilon_{ijk}u_j v_k)\\ &= \epsilon_{ijk}\frac{\partial u_j}{\partial x_i}v_k+\epsilon_{ijk}u_j\frac{\partial v_k}{\partial x_i}\\ &= v_k(\nabla\times\mathbf{u})_k-u_j(\nabla\times\mathbf{v})_j\\ &= \mathbf{v}\cdot(\nabla\times\mathbf{u})-\mathbf{u}\cdot(\nabla\times\mathbf{v}).\end{aligned}$$

(d) For the ith component of this vector identity,

$$\begin{aligned}[\nabla\times(\mathbf{u}\times\mathbf{v})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\mathbf{u}\times\mathbf{v})_k\\ &= \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}u_l v_m\\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\left(u_l\frac{\partial v_m}{\partial x_j}+v_m\frac{\partial u_l}{\partial x_j}\right)\\ &= u_i\frac{\partial v_j}{\partial x_j}+v_j\frac{\partial u_i}{\partial x_j}-u_j\frac{\partial v_i}{\partial x_j}-v_i\frac{\partial u_j}{\partial x_j}\\ &= u_i\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla u_i-\mathbf{u}\cdot\nabla v_i-v_i\nabla\cdot\mathbf{u}\\ &= [\mathbf{u}(\nabla\cdot\mathbf{v})+(\mathbf{v}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{v}-\mathbf{v}(\nabla\cdot\mathbf{u})]_i\end{aligned}$$

(e) Since simplification and reduction are usually easier to recognise than the best way to make an expression more complicated but still valid, we start with the most complicated of the terms in the identity:

$$\begin{aligned}[\mathbf{u}\times(\nabla\times\mathbf{u})]_i &= \epsilon_{ijk}u_j(\nabla\times\mathbf{u})_k \\ &= \epsilon_{ijk}u_j\,\epsilon_{klm}\frac{\partial u_m}{\partial x_l}.\end{aligned}$$

We can now employ the double ϵ formula to convert this expression into one containing Kronecker deltas.

$$\begin{aligned}[\mathbf{u}\times(\nabla\times\mathbf{u})]_i &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})u_j\frac{\partial u_m}{\partial x_l} \\ &= u_j\frac{\partial u_j}{\partial x_i}-u_j\frac{\partial u_i}{\partial x_j} \\ &= \frac{1}{2}\frac{\partial}{\partial x_i}(u_ju_j)-(\mathbf{u}\cdot\nabla)u_i \\ &= [\tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\nabla)\mathbf{u}]_i.\end{aligned}$$

This completes the proof.

26.8 *A column matrix* a *has components* a_x, a_y, a_z *and* A *is the matrix with elements* $A_{ij} = -\epsilon_{ijk}a_k$.

(a) *What is the relationship between column matrices* b *and* c *if* $\mathsf{Ab} = \mathsf{c}$?
(b) *Find the eigenvalues of* A *and show that* a *is one of its eigenvectors. Explain why this must be so.*

(a) The matrix equation $\mathsf{Ab} = \mathsf{c}$ will have the explicit form

$$\mathsf{c} = \mathsf{Ab} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} -a_zb_y + a_yb_z \\ a_zb_x - a_xb_z \\ -a_yb_x + a_xb_y \end{pmatrix} = \mathsf{a}\times\mathsf{b},$$

i.e. $\mathsf{c} = \mathsf{a}\times\mathsf{b}$.

(b) The characteristic equation for A is

$$\begin{aligned}0 = |\mathsf{A}-\lambda\mathsf{I}| &= \begin{vmatrix} -\lambda & -a_z & a_y \\ a_z & -\lambda & -a_x \\ -a_y & a_x & -\lambda \end{vmatrix} \\ &= -\lambda^3 - a_x^2\lambda - a_z(a_xa_y+\lambda a_z) + a_y(a_za_x-\lambda a_y).\end{aligned}$$

Thus $\lambda = 0$ or $-\lambda^2 - a_x^2 - a_z^2 - a_y^2 = 0$. The second possibility gives $\lambda = \pm i(a_x^2+a_y^2+a_z^2)^{1/2} = \pm i|\mathbf{a}|$.

Now consider Aa. From part (a) this is

$$\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} -a_z a_y + a_y a_z \\ a_z a_x - a_x a_z \\ -a_y a_x + a_x a_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\mathsf{a}.$$

Thus a is an eigenvector of A corresponding to eigenvalue 0. That this must be so also follows from the general conclusion of part (a) that if $\mathsf{c} = \mathsf{Ab}$ then $\mathsf{c} = \mathsf{a} \times \mathsf{b}$; if $\mathsf{b} = \mathsf{a}$, then $\mathsf{c} = \mathsf{a} \times \mathsf{a} = \mathsf{0} = 0\mathsf{a}$.

26.10 *A symmetric second-order Cartesian tensor is defined by*

$$T_{ij} = \delta_{ij} - 3x_i x_j.$$

Evaluate the following surface integrals, each taken over the surface of the unit sphere:

$$\text{(a)} \int T_{ij}\, dS; \quad \text{(b)} \int T_{ik} T_{kj}\, dS; \quad \text{(c)} \int x_i T_{jk}\, dS.$$

We will need the following integrals over the unit sphere U:

$$\int_U 1\, dS = 4\pi,$$

$$\int_U x_i\, dS = 0, \text{ on antisymmetry grounds},$$

$$\int_U x_i x_j\, dS = 0, \text{ on antisymmetry grounds for } i \neq j,$$

$$\int_U x_i^2\, dS = \frac{1}{3} \int_U (x_1^2 + x_2^2 + x_3^2)\, dS = \frac{1}{3} \int_U 1\ dS = \frac{4\pi}{3},$$

$$\int_U x_i^3\, dS = 0, \text{ on antisymmetry grounds}.$$

Thus, combining the third and fourth of these,

$$\int_U x_i x_j\, dS = \frac{4\pi}{3}\, \delta_{ij}.$$

We note that the integrands in (a) and (b) each have two uncontracted subscripts and that that in (c) has three. As the integrations are with respect to the scalar quantity S, our answers must have corresponding properties.

(a) $$\int_U T_{ij}\,dS = \int_U \delta_{ij}\,dS - \int 3x_i x_j\,dS$$
$$= 4\pi\delta_{ij} - 3\,\frac{4\pi}{3}\delta_{ij} = 0 \quad \text{for all } i \text{ and } j.$$

(b) $$\int_U T_{ik}T_{kj}\,dS = \int_U (\delta_{ik} - 3x_i x_k)(\delta_{kj} - 3x_k x_j)\,dS$$
$$= \int_U (\delta_{ij} - 3x_i x_j - 3x_i x_j + 9x_i x_k x_k x_j)\,dS$$
$$= \int_U (\delta_{ij} + 3x_i x_j)\,dS, \text{ since } x_k x_k = 1 \text{ on } U,$$
$$= 4\pi\delta_{ij} + 3\,\frac{4\pi}{3}\delta_{ij} = 8\pi\delta_{ij}.$$

(c) $$\int_U x_i T_{jk}\,dS = \int_U (x_i\delta_{jk} - 3x_i x_j x_k)\,dS$$
$$= \begin{cases} 0 - 0 \text{ if } i \neq j \neq k \neq i, \text{ on antisymmetry grounds,} \\ 0 - \int_U x_k\,dS = 0 \text{ if } i = j \neq k \text{ or } j = k \neq i \text{ or } i = k \neq j, \\ 0 - \int_U x_k^3\,dS = 0 \text{ if } i = j = k. \end{cases}$$

26.12 *In four dimensions define second-order antisymmetric tensors F_{ij} and Q_{ij} and a first-order tensor S_i as follows:*

(a) *$F_{23} = H_1$, $Q_{23} = B_1$ and their cyclic permutations;*
(b) *$F_{i4} = -D_i$, $Q_{i4} = E_i$ for $i = 1, 2, 3$;*
(c) *$S_4 = \rho$, $S_i = J_i$ for $i = 1, 2, 3$.*

Then, taking x_4 as t and the other symbols to have their usual meanings in electromagnetic theory, show that the equations $\sum_j \partial F_{ij}/\partial x_j = S_i$ and $\partial Q_{jk}/\partial x_i + \partial Q_{ki}/\partial x_j + \partial Q_{ij}/\partial x_k = 0$ reproduce Maxwell's equations. In the latter i, j, k is any set of three subscripts selected from 1, 2, 3, 4, but chosen in such a way that they are all different.

We can write the defining equations for F and Q as

$$F_{ij} = \epsilon_{ijk}H_k \quad \text{and} \quad Q_{ij} = \epsilon_{ijk}B_k,$$

where none of i, j, k is equal to 4.

First, for $i = 1, 2, 3$,

$$J_i = S_i = \sum_j \frac{\partial F_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(\epsilon_{ijk} H_k) + \frac{\partial F_{i4}}{\partial t} = (\nabla \times \mathbf{H})_i - \frac{\partial D_i}{\partial t}.$$

Thus the given equation is the ith component of the Maxwell equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}.$$

Second, for $i = 4$, and noting that, since F is antisymmetric, $F_{44} = 0$,

$$\rho = S_4 = \sum_j \frac{\partial F_{4j}}{\partial x_j} = -\sum_{j=1}^{3} \frac{\partial F_{j4}}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial D_j}{\partial x_j}.$$

This is the Maxwell equation $\nabla \cdot \mathbf{D} = \rho$.

For the equation involving Q, with i, j, k some non-repeating selection from $1, 2, 3$, we have, say,

$$\begin{aligned} \frac{\partial Q_{23}}{\partial x_1} + \frac{\partial Q_{31}}{\partial x_2} + \frac{\partial Q_{12}}{\partial x_3} &= 0, \\ \Rightarrow \quad \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

This is one of Maxwell's equations. Reassigning $1, 2, 3$ amongst i, j, k produces the same equation.

Finally, for $k = 4$ and $i \neq j$ with neither equal to 4,

$$\frac{\partial Q_{j4}}{\partial x_i} + \frac{\partial Q_{4i}}{\partial x_j} + \frac{\partial Q_{ij}}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial E_j}{\partial x_i} + \frac{\partial(-E_i)}{\partial x_j} + \frac{\partial}{\partial t}(\epsilon_{ijm} B_m) = 0.$$

We now contract this second equation with ϵ_{ijn} and use the double epsilon identity, remembering also that $\epsilon_{ijn}\partial E_j/\partial x_i$ is the nth component of $\nabla \times \mathbf{E}$:

$$\begin{aligned} (\nabla \times \mathbf{E})_n - (-\nabla \times \mathbf{E})_n + (\delta_{jj}\delta_{nm} - \delta_{jm}\delta_{nj})\frac{\partial B_m}{\partial t} &= 0, \\ 2(\nabla \times \mathbf{E})_n + (3\delta_{nm} - \delta_{nm})\frac{\partial B_m}{\partial t} &= 0, \\ \Rightarrow \quad (\nabla \times \mathbf{E})_n + (\dot{\mathbf{B}})_n &= 0, \quad \text{for } n = 1, 2, 3. \end{aligned}$$

This is the nth component of the Maxwell equation $(\nabla \times \mathbf{E}) + (\dot{\mathbf{B}}) = \mathbf{0}$.

26.14 *Assuming that the current density* **j** *and the electric field* **E** *appearing in equation (26.43),* $j_i = \sigma_{ij}E_j$*, are first-order Cartesian tensors, show explicitly that the electrical conductivity tensor* σ_{ij} *transforms according to the law appropriate to a second-order tensor.*

The rate W *at which energy is dissipated per unit volume, as a result of the current flow, is given by* $\mathbf{E}\cdot\mathbf{j}$*. Determine the limits between which* W *must lie for a given value of* $|\mathbf{E}|$ *as the direction of* **E** *is varied.*

This result follows immediately from the quotient law but to show it directly we proceed as follows.

$$\begin{aligned}
j_i &= \sigma_{ij}E_j, &&\text{given,}\\
j'_i &= \sigma'_{ij}E'_j, &&\text{in the transformed frame,}\\
L_{ip}j_p &= \sigma'_{ij}L_{jq}E_q, &&j \text{ and } E \text{ are first-order tensors,}\\
L_{ip}\sigma_{pk}E_k &= \sigma'_{ij}L_{jq}E_q, &&\text{substituting for } j_p,\\
L_{ir}L_{ip}\sigma_{pk}E_k &= L_{ir}L_{jq}\sigma'_{ij}E_q, &&\text{multiply by } L_{ir} \text{ and sum,}\\
(\delta_{rp}\sigma_{pk} - L_{ir}L_{jk}\sigma'_{ij})E_k &= 0, &&\mathsf{L} \text{ is orthogonal; relabel dummy suffix,}\\
\sigma_{rk} &= L_{ir}L_{jk}\sigma'_{ij}, &&\text{since true for all } E_k.
\end{aligned}$$

This is a sufficient proof, but to put the final result in the usual form we continue with

$$L_{mr}L_{nk}\sigma_{rk} = L_{mr}L_{nk}L_{ir}L_{jk}\sigma'_{ij} = \delta_{mi}\delta_{nj}\sigma'_{ij} = \sigma'_{mn},$$

showing that σ is a second-order tensor.

The rate of dissipation is $W = E_ij_i = E_i\sigma_{ij}E_j$. The problem is to extremise this, subject to the constraint $E_kE_k = |\mathbf{E}|^2$, a constant. Introducing a Lagrange multiplier, we consider the extremes of

$$W' = E_i\sigma_{ij}E_j - \lambda E_kE_k.$$

They are given by

$$\frac{\partial W'}{\partial E_j} = 0 \quad \Rightarrow \quad 2(\sigma_{ij}E_i - \lambda E_j) = 0.$$

This shows that the extremising directions of **E** are the eigenvectors $\hat{\mathbf{E}}$ of σ. For these directions

$$\sigma_{ij}\hat{E}_i\hat{E}_j - \lambda\hat{E}_j\hat{E}_j = 0 \quad \Rightarrow \quad \hat{W} = \lambda|\mathbf{E}|^2.$$

Thus the maximum and minimum values of W are determined by the maximum and minimum eigenvalues of σ_{ij}.

> ***26.16*** *A rigid body consists of four particles of masses m, $2m$, $3m$, $4m$, respectively situated at the points (a,a,a), $(a,-a,-a)$, $(-a,a,-a)$, $(-a,-a,a)$ and connected together by a light framework.*
>
> (a) *Find the inertia tensor at the origin and show that the principal moments of inertia are $20ma^2$ and $(20 \pm 2\sqrt{5})ma^2$.*
> (b) *Find the principal axes and verify that they are orthogonal.*

The masses are

$$m \text{ at } a(1,1,1), \qquad 2m \text{ at } a(1,-1,-1),$$
$$3m \text{ at } a(-1,1,-1), \quad 4m \text{ at } a(-1,-1,1).$$

(a) The inertia tensor components (recall that the off-diagonal elements have an intrinsic minus sign) are calculated as

$$\begin{aligned} I_{11} &= ma^2[\,1(2)+2(2)+3(2)+4(2)\,] = 20ma^2, \\ I_{12} = I_{21} &= ma^2[\,1(-1)+2(1)+3(1)+4(-1)\,] = 0, \\ I_{13} = I_{31} &= ma^2[\,1(-1)+2(1)+3(-1)+4(1)\,] = 2ma^2, \\ I_{22} &= ma^2[\,1(2)+2(2)+3(2)+4(2)\,] = 20ma^2, \\ I_{23} = I_{32} &= ma^2[\,1(-1)+2(-1)+3(1)+4(1)\,] = 4ma^2, \\ I_{33} &= ma^2[\,1(2)+2(2)+3(2)+4(2)\,] = 20ma^2. \end{aligned}$$

Thus,

$$I = 2ma^2 \begin{pmatrix} 10 & 0 & 1 \\ 0 & 10 & 2 \\ 1 & 2 & 10 \end{pmatrix},$$

and the principal moments are given by $2ma^2\lambda$ where

$$\begin{aligned} 0 &= \begin{vmatrix} 10-\lambda & 0 & 1 \\ 0 & 10-\lambda & 2 \\ 1 & 2 & 10-\lambda \end{vmatrix} \\ &= (10-\lambda)(\lambda^2 - 20\lambda + 96) - 10 + \lambda \\ &= (10-\lambda)(\lambda^2 - 20\lambda + 95). \end{aligned}$$

Thus $\lambda = 10$ or $\lambda = 10 \pm \sqrt{100-95}$ and the principal moments are $20ma^2$ and $(20 \pm 2\sqrt{5})ma^2$. It is clear that these add up to the trace of I (as a check).

(b) For $\lambda = 10$ the (unnormalised) axis vector, $\mathbf{v}_1$, satisfies

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

For $\lambda = 10 \pm \sqrt{5}$ the (unnormalised) axis vectors, $\mathbf{v}_2$, and $\mathbf{v}_3$ satisfy

$$\begin{pmatrix} \mp\sqrt{5} & 0 & 1 \\ 0 & \mp\sqrt{5} & 2 \\ 1 & 2 & \mp\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_{2,3} = \begin{pmatrix} \pm 1 \\ \pm 2 \\ \sqrt{5} \end{pmatrix}.$$

Further,

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2 - 2 + 0) = 0, \ \mathbf{v}_1 \cdot \mathbf{v}_3 = (-2 + 2 + 0) = 0, \ \mathbf{v}_2 \cdot \mathbf{v}_3 = (-1 - 4 + 5) = 0,$$

showing that the axes vectors are mutually orthogonal.

26.18 *The paramagnetic tensor χ_{ij} of a body placed in a magnetic field, in which its energy density is $-\frac{1}{2}\mu_0 \mathbf{M} \cdot \mathbf{H}$ with $M_i = \sum_j \chi_{ij} H_j$, is*

$$\begin{pmatrix} 2k & 0 & 0 \\ 0 & 3k & k \\ 0 & k & 3k \end{pmatrix}.$$

Assuming depolarizing effects are negligible, find how the body will orientate itself if the field is horizontal, in the following circumstances:

(a) *the body can rotate freely;*
(b) *the body is suspended with the (1, 0, 0) axis vertical;*
(c) *the body is suspended with the (0, 1, 0) axis vertical.*

The equilibrium orientation of the body will be such as to minimise the total energy (per unit volume)

$$E = -\tfrac{1}{2}\mu_0 \mathbf{M} \cdot \mathbf{H} = -\tfrac{1}{2}\mu_0 \chi_{ij} H_i H_j,$$

subject to any constraints imposed by the method of suspension. We therefore need to maximise (assuming that $k > 0$) the quadratic form $\hat{\mathbf{n}}^{\mathrm{T}} \chi \hat{\mathbf{n}}$. This could be done by finding the eigenvalues and eigenvectors of χ or directly from the quadratic form. We will adopt the latter approach (and omit the factor k which is merely a scaling factor).

(a) If the body can rotate freely there are no constraints on the components n_i of the unit vector $\hat{\mathbf{n}}$ fixed in the body that aligns itself with the external field. We therefore maximise

$$Q = (n_1, n_2, n_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 2n_1^2 + 3n_2^2 + 3n_3^2 + 2n_2 n_3,$$

subject to $n_1^2 + n_2^2 + n_3^2 = 1$. Now, $Q = 2 + (n_2 + n_3)^2$ which is clearly maximal, given the constraint, when $n_1 = 0$ and $n_2 = n_3 = 1/\sqrt{2}$. Thus the body aligns itself with the $(0, 1, 1)$ direction parallel to the field.

(b) With the $(1, 0, 0)$ axis vertical (and the field horizontal), $\hat{\mathbf{n}}$ must have the form $(0, n_2, n_3)$ with $n_2^2 + n_3^2 = 1$. We consider

$$Q = (0, n_2, n_3)\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ n_2 \\ n_3 \end{pmatrix} = 3n_2^2 + 3n_3^2 + 2n_2n_3.$$

This is $Q = 3 + 2n_2n_3$ and, again, is clearly maximal, given the constraint, when $n_2 = n_3 = 1/\sqrt{2}$. Thus the equilibrium orientation is as in part (a).

(c) With the $(0, 1, 0)$ axis vertical (and the field horizontal), $\hat{\mathbf{n}}$ must have the form $(n_1, 0, n_3)$ with $n_1^2 + n_3^2 = 1$. We consider

$$Q = (n_1, 0, n_3)\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} n_1 \\ 0 \\ n_3 \end{pmatrix} = 2n_1^2 + 3n_3^2,$$

This is $Q = 2 + n_3^2$ and to obtain a maximum we must take $n_1 = 0$ and $n_3 = 1$. Thus the body aligns itself with the $(0, 0, 1)$ direction parallel to the field.

26.20 *For tin, the conductivity tensor is diagonal, with entries a, a, and b when referred to its crystal axes. A single crystal is grown in the shape of a long wire of length L and radius r, the axis of the wire making polar angle θ with respect to the crystal's 3-axis. Show that the resistance of the wire is*

$$\frac{L}{\pi r^2 ab}\left(a\cos^2\theta + b\sin^2\theta\right).$$

Since the conductivity tensor σ_{ij} is diagonal, the usual equation $J_i = \sigma_{ij}E_j$ can easily be inverted to read $E_i = \rho_{ij}J_j$ where ρ_{ij}, the 'resistance tensor', has the form

$$\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}.$$

The potential difference between the ends of the wire is given by

$$V = \mathbf{L}\cdot\mathbf{E} = L_i\rho_{ij}J_j,$$

where $\mathbf{L} = L(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Now, although $\mathbf{E}$ is not necessarily parallel to the wire, the current density $\mathbf{J}$ must be and can be written

$$\mathbf{J} = \frac{I}{\pi r^2}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

It follows that $V = \mathbf{L}^{\mathrm{T}}\rho\mathbf{J}$ can be expressed as

$$\frac{IL}{\pi r^2}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix},$$

and that the resistance of the wire is

$$R = \frac{V}{I} = \frac{L}{\pi r^2}\left(\frac{\sin^2\theta}{a} + \frac{\cos^2\theta}{b}\right) = \frac{L}{\pi r^2 ab}(b\sin^2\theta + a\cos^2\theta).$$

26.22 *For an isotropic elastic medium under dynamic stress, at time t the displacement u_i and the stress tensor p_{ij} satisfy*

$$p_{ij} = c_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \qquad \textit{and} \qquad \frac{\partial p_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2},$$

where c_{ijkl} is the isotropic tensor given in equation (26.47) and ρ is a constant. Show that both $\nabla\cdot\mathbf{u}$ and $\nabla\times\mathbf{u}$ satisfy wave equations and find the corresponding wave speeds.

Using the given equations and the form of the most general isotropic fourth-order tensor, we have

$$\begin{aligned} p_{ij} &= (\lambda\delta_{ij}\delta_{kl} + \eta\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk})\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \\ &= \lambda\delta_{ij}2\nabla\cdot\mathbf{u} + \eta\frac{\partial u_i}{\partial x_j} + \eta\frac{\partial u_j}{\partial x_i} + \nu\frac{\partial u_j}{\partial x_i} + \nu\frac{\partial u_i}{\partial x_j} \\ &= 2\lambda\delta_{ij}\nabla\cdot\mathbf{u} + (\eta + \nu)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \end{aligned}$$

We now differentiate this equation with respect to x_j and sum over j. We also abbreviate $\eta + \nu$ to μ.

$$\begin{aligned} \frac{\partial p_{ij}}{\partial x_j} &= 2\lambda\frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_i}\right), \\ \Rightarrow \quad \rho\frac{\partial^2 u_i}{\partial t^2} &= 2\lambda\frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_i}\right). \end{aligned}$$

Next we differentiate this equation with respect to x_i and sum over i to obtain

$$\begin{aligned} \rho\frac{\partial^2(\nabla\cdot\mathbf{u})}{\partial t^2} &= 2\lambda\nabla^2(\nabla\cdot\mathbf{u}) + \mu[\nabla^2(\nabla\cdot\mathbf{u}) + \nabla^2(\nabla\cdot\mathbf{u})] \\ &= 2(\lambda + \mu)\nabla^2(\nabla\cdot\mathbf{u}). \end{aligned}$$

This is a wave equation for $(\nabla \cdot \mathbf{u})$ with wave speed $[\,2(\lambda + \eta + \nu)/\rho\,]^{1/2}$.

To find a similar equation for $\nabla \times \mathbf{u}$, we start from an expression for its ith component:

$$(\nabla \times \mathbf{u})_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j},$$

$$\frac{\partial^2}{\partial t^2}(\nabla \times \mathbf{u})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial^2 u_k}{\partial t^2} \right) = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{1}{\rho} \frac{\partial p_{kl}}{\partial x_l} \right),$$

where the given time-dependent equation has been used to make the final step.

Now substituting the alternative expression for p_{ij} derived earlier, we have

$$\begin{aligned}
\rho \frac{\partial^2}{\partial t^2}(\nabla \times \mathbf{u})_i &= \epsilon_{ijk} \frac{\partial^2}{\partial x_j \partial x_l} \left[2\lambda \delta_{kl} \nabla \cdot \mathbf{u} + \mu \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \right] \\
&= \epsilon_{ijk} \frac{\partial^2}{\partial x_j \partial x_k} [\,2\lambda \nabla \cdot \mathbf{u}\,] + \mu \epsilon_{ijk} \frac{\partial^2}{\partial x_j \partial x_l} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \\
&= 0 + \mu \frac{\partial^2}{\partial x_l \partial x_l} \left(\epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \right) + \mu \epsilon_{ijk} \frac{\partial^2 (\nabla \cdot \mathbf{u})}{\partial x_j \partial x_k} \\
&= \mu \nabla^2 (\nabla \times \mathbf{u})_i + 0.
\end{aligned}$$

This is a wave equation for the ith component of $(\nabla \times \mathbf{u})$ with a wave speed $[\,(\eta + \nu)/\rho\,]^{1/2}$ that is independent of i. To obtain the third and fourth lines of the final derivation we twice used the fact that, because ϵ_{ijk} is antisymmetric in j and k whilst $\partial^2/\partial x_j \partial x_k$ is symmetric in them, the contracted expression containing both is identically zero.

> ***26.24*** *Working in cylindrical polar coordinates ρ, ϕ, z, parameterise the straight line (geodesic) joining $(1, 0, 0)$ to $(1, \pi/2, 1)$ in terms of s, the distance along the line. Show by substitution that the geodesic equations derived at the end of section 26.22 are satisfied.*

Clearly, the length of the line joining $(1, 0, 0)$ to $(1, \pi/2, 1)$ in cylindrical polars, i.e. $(1, 0, 0)$ to $(0, 1, 1)$ in Caretesians, is $\sqrt{3}$. Points along the line are given in terms of s, the distance along the line, by

$$x = 1 - \frac{s}{\sqrt{3}}, \quad y = \frac{s}{\sqrt{3}}, \quad z = \frac{s}{\sqrt{3}}.$$

In cylindrical polars, z is as given and (defining the shorthand $(**)$ as in the first line below)

$$\rho = (x^2 + y^2)^{1/2} = \frac{1}{\sqrt{3}} (3 - 2\sqrt{3}s + 2s^2)^{1/2} \equiv \frac{1}{\sqrt{3}} (**)^{1/2},$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{s}{\sqrt{3} - s}.$$

The required derivatives are

$$\frac{d\rho}{ds} = \frac{1}{2\sqrt{3}}\,\frac{4s - 2\sqrt{3}}{(**)^{1/2}},$$

$$\frac{d^2\rho}{ds^2} = \frac{1}{2\sqrt{3}}\,\frac{(**)^{1/2}(4) - 2(2s - \sqrt{3})\frac{1}{2}(**)^{-1/2}(4s - 2\sqrt{3})}{(**)}$$

$$= \frac{1}{2\sqrt{3}}\,\frac{12 - 8\sqrt{3}s + 8s^2 - 8s^2 + 8\sqrt{3}s - 6}{(**)^{3/2}} = \frac{\sqrt{3}}{(**)^{3/2}},$$

$$\frac{d\phi}{ds} = \frac{1}{1 + \frac{s^2}{(\sqrt{3}-s)^2}}\,\frac{(\sqrt{3} - s)(1) - s(-1)}{(\sqrt{3} - s)^2} = \frac{\sqrt{3}}{(**)},$$

$$\frac{d^2\phi}{ds^2} = \frac{-\sqrt{3}(4s - 2\sqrt{3})}{(**)^2}.$$

Using these results, the first equation,

$$\frac{d^2\rho}{ds^2} - \rho\left(\frac{d\phi}{ds}\right)^2 = 0,$$

reads

$$\frac{\sqrt{3}}{(**)^{3/2}} - \frac{(**)^{1/2}}{\sqrt{3}}\,\frac{3}{(**)^2} = 0, \text{ which is satisfied.}$$

The second equation,

$$\frac{d^2\phi}{ds^2} + \frac{2}{\rho}\,\frac{d\rho}{ds}\,\frac{d\phi}{ds} = 0,$$

reads

$$\frac{-\sqrt{3}(4s - 2\sqrt{3})}{(**)^2} + \frac{2\sqrt{3}}{(**)^{1/2}}\,\frac{(4s - 2\sqrt{3})}{2\sqrt{3}(**)^{1/2}}\,\frac{\sqrt{3}}{(**)} = 0, \text{ which is also satisfied.}$$

The third equation $\dfrac{d^2z}{ds^2} = \dfrac{d^2}{ds^2}\left(\dfrac{s}{\sqrt{3}}\right) = 0$ is trivially satisfied, thus completing the verification.

26.26 *By writing down the expression for the square of the infinitesimal arc length* $(ds)^2$ *in spherical polar coordinates, find the components* g_{ij} *of the metric tensor in this coordinate system. Hence, using (26.97), find the expression for the divergence of a vector field* $\mathbf{v}$ *in spherical polars. Calculate the Christoffel symbols (of the second kind)* $\Gamma^i{}_{jk}$ *in this coordinate system.*

Since $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$, we have that

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and $g = |g_{ij}| = r^4\sin^2\theta$. Further, for an orthogonal system, $g_{ij} = h_i\delta_{ij}$ with $g^{ij} = h_i^{-1}\delta_{ij}$.

For the divergence, as given in equation (26.97), we have

$$\begin{aligned}
\nabla\cdot\mathbf{v} = v^i_{\;;i} &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^i}(\sqrt{g}\,v^i)\\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial u^i}(r^2\sin\theta\,v^i)\\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial u^i}(r^2\sin\theta\,g^{ik}v_k)\\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial u^i}[r^2\sin\theta\,(v_i/h_i)], \text{ still summed over } i,\\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\,v_r) + \frac{\partial}{\partial\theta}(r\sin\theta\,v_\theta) + \frac{\partial}{\partial\phi}(rv_\phi)\right]\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi},
\end{aligned}$$

which recovers the familiar form.

The Christoffel symbols of the second kind are calculated from

$$\Gamma^m_{ij} = \frac{1}{2}g^{mk}\left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right).$$

Because g^{mk} is diagonal, only terms with $m = k$ can contribute; $g_{11} = 1$, $g_{22} = r^2$ and $g_{33} = r^2\sin^2\theta$. For each value of k we have $g^{kk} = (g_{kk})^{-1}$. Further, in the present case the only non-zero derivatives are

$$\frac{\partial g_{22}}{\partial u^1} = 2r, \quad \frac{\partial g_{33}}{\partial u^1} = 2r\sin^2\theta, \quad \frac{\partial g_{33}}{\partial u^2} = 2r^2\sin\theta\cos\theta.$$

Using these expressions in the general formula gives, for those cases in which $i = j$,

$$\Gamma^m_{11} = \frac{1}{2} g^{mm} \left(\frac{\partial g_{1m}}{\partial u^1} + \frac{\partial g_{m1}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^m} \right) = 0, \text{ for all } m,$$

$$\begin{aligned}\Gamma^m_{22} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{2m}}{\partial u^2} + \frac{\partial g_{m2}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^m} \right) \\ &= 0, \text{ for } m = 2, 3, \\ &= \tfrac{1}{2} g^{11}(0 + 0 - 2r) = -r, \text{ for } m = 1.\end{aligned}$$

$$\begin{aligned}\Gamma^m_{33} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{3m}}{\partial u^3} + \frac{\partial g_{m3}}{\partial u^3} - \frac{\partial g_{33}}{\partial u^m} \right) \\ &= 0, \text{ for } m = 3, \\ &= \tfrac{1}{2} g^{11}(0 + 0 - 2r \sin^2\theta) = -r \sin^2\theta, \text{ for } m = 1, \\ &= \tfrac{1}{2} g^{22}(0 + 0 - 2r^2 \sin\theta \cos\theta) = -\sin\theta\cos\theta, \text{ for } m = 2.\end{aligned}$$

These account for 9 of the 27 possible Christoffel symbols. The other 18 are those in which i and j are different, but since the symbols are symmetric under i-j interchange, only 9 calculations are needed. They are as follows.

$$\begin{aligned}\Gamma^m_{21} = \Gamma^m_{12} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{2m}}{\partial u^1} + \frac{\partial g_{m1}}{\partial u^2} - \frac{\partial g_{12}}{\partial u^m} \right) \\ &= 0, \text{ for } m = 1, 3, \\ &= \tfrac{1}{2} g^{22}(2r + 0 - 0) = r^{-1}, \text{ for } m = 2,\end{aligned}$$

$$\begin{aligned}\Gamma^m_{31} = \Gamma^m_{13} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{3m}}{\partial u^1} + \frac{\partial g_{m1}}{\partial u^3} - \frac{\partial g_{13}}{\partial u^m} \right) \\ &= 0, \text{ for } m = 1, 2, \\ &= \tfrac{1}{2} g^{33}(2r \sin^2\theta + 0 - 0) = r^{-1}, \text{ for } m = 3,\end{aligned}$$

$$\begin{aligned}\Gamma^m_{23} = \Gamma^m_{32} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{2m}}{\partial u^3} + \frac{\partial g_{m3}}{\partial u^2} - \frac{\partial g_{23}}{\partial u^m} \right) \\ &= 0, \text{ for } m = 1, 2, \\ &= \tfrac{1}{2} g^{33}(0 + 2r^2 \sin\theta\cos\theta - 0) = \cot\theta, \text{ for } m = 3.\end{aligned}$$

This completes the calculation of all 27 Christoffel symbols for this coordinate system; only 9 of them are non-zero.

26.28 *A curve* $\mathbf{r}(t)$ *is parameterised by a scalar variable* t*. Show that the length of the curve between two points,* A *and* B*, is given by*

$$L = \int_A^B \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\, dt.$$

Using the calculus of variations (see chapter 22), show that the curve $\mathbf{r}(t)$ *that minimises* L *satisfies the equation*

$$\frac{d^2u^i}{dt^2} + \Gamma^i{}_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = \frac{\ddot{s}}{\dot{s}}\frac{du^i}{dt},$$

where s *is the arc length along the curve,* $\dot{s} = ds/dt$ *and* $\ddot{s} = d^2s/dt^2$*. Hence, show that if the parameter* t *is of the form* $t = as + b$*, where* a *and* b *are constants, then we recover the equation for a geodesic (26.101).*

[*A parameter which, like* t*, is the sum of a linear transformation of* s *and a translation is called an* affine *parameter.*]

Denoting derivatives with respect to t by a dot notation, the element of curve length is given by

$$(ds)^2 = g_{ij}\,du^i\,du^j = g_{ij}\,\dot{u}^i\,dt\,\dot{u}^j\,dt \quad \Rightarrow \quad \dot{s}^2 = g_{ij}\,\dot{u}^i\dot{u}^j.$$

Thus

$$L = \int_A^B ds = \int_A^B \dot{s}\,dt = \int_A^B (g_{ij}\,\dot{u}^i\dot{u}^j)^{1/2}\,dt \equiv \int_A^B F(u^i, \dot{u}^i, t)\,dt.$$

The Euler–Lagrange equation for minimising L is

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}^k}\right) - \frac{\partial F}{\partial u^k} \\
&= \frac{d}{dt}\left(\frac{1}{2}\,\frac{2g_{ik}\,\dot{u}^i}{\dot{s}}\right) - \frac{\dot{u}^i\,\dot{u}^j}{2\dot{s}}\,\frac{\partial g_{ij}}{\partial u^k} \\
&= -\frac{\ddot{s}}{\dot{s}^2}\,g_{ik}\dot{u}^i + \frac{g_{ik}\,\ddot{u}^i}{\dot{s}} + \frac{\dot{u}^i}{\dot{s}}\frac{dg_{ik}}{dt} - \frac{\dot{u}^i\,\dot{u}^j}{2\dot{s}}\,\frac{\partial g_{ij}}{\partial u^k} \\
&= -\frac{\ddot{s}}{\dot{s}^2}\,g_{ik}\dot{u}^i + \frac{g_{ik}\,\ddot{u}^i}{\dot{s}} + \frac{\dot{u}^i}{\dot{s}}\frac{\partial g_{ik}}{\partial u^m}\dot{u}^m - \frac{\dot{u}^i\,\dot{u}^j}{2\dot{s}}\,\frac{\partial g_{ij}}{\partial u^k}.
\end{aligned}$$

Now, in the double summation over dummy variables i and m in the third term on the RHS, we can set

$$\dot{u}^i\dot{u}^m\frac{\partial g_{ik}}{\partial u^m} = \dot{u}^i\dot{u}^m\frac{\partial g_{mk}}{\partial u^i}$$

and re-write the Euler–Lagrange equation as

$$0 = -\frac{\ddot{s}}{\dot{s}^2}\,g_{ik}\dot{u}^i + \frac{g_{ik}\,\ddot{u}^i}{\dot{s}} + \frac{\dot{u}^i\dot{u}^m}{\dot{s}}\,\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^m} + \frac{\partial g_{mk}}{\partial u^i}\right) - \frac{\dot{u}^i\,\dot{u}^j}{2\dot{s}}\,\frac{\partial g_{ij}}{\partial u^k}.$$

Next we multiply throughout by g^{lk} and contract over k to obtain

$$\begin{aligned} 0 &= -\frac{\ddot{s}}{\dot{s}}\,\delta^l_i \dot{u}^i + \delta^l_i\,\ddot{u}^i + \frac{1}{2}\dot{u}^i\dot{u}^j g^{lk}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \\ &= -\frac{\ddot{s}}{\dot{s}}\,\dot{u}^l + \ddot{u}^l + \Gamma^l{}_{ij}\dot{u}^i\dot{u}^j, \end{aligned}$$

thus establishing the stated result.

If $t = as + b$, then $\dot{s} = a^{-1}$ and $\ddot{s} = 0$; further $\dfrac{d}{dt} = \dfrac{1}{a}\dfrac{d}{ds}$. With this substitution the minimising equation becomes

$$\frac{1}{a^2}\frac{d^2u^l}{ds^2} + \Gamma^l{}_{ij}\frac{1}{a}\frac{du^i}{ds}\frac{1}{a}\frac{du^j}{ds} = 0.$$

This is the equation for a geodesic as given in (26.101).

27

Numerical methods

27.2 *Using the Newton–Raphson procedure find, correct to three decimal places, the root nearest to 7 of the equation* $4x^3 + 2x^2 - 200x - 50 = 0$.

The Newton–Raphson scheme has 2nd-order convergence and so we expect rapid convergence if a reasonable first guess is made. The iteration scheme is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{4x_n^3 + 2x_n^2 - 200x_n - 50}{12x_n^2 + 4x_n - 200} \\ &= \frac{8x_n^3 + 2x_n^2 + 50}{12x_n^2 + 4x_n - 200}. \end{aligned}$$

Starting with $x_1 = 7$, $x_2 = 6.951923077$ and $x_3 = x_4 = \cdots = 6.951436674$ to 10 s.f. To 3 decimal places $x = 6.951$, a result achieved (but not verified) after only one iteration; as can be seen, a result accurate to 9 decimal places is achieved after only two iterations.

27.4 *The square root of a number N is to be determined by means of the iteration scheme*

$$x_{n+1} = x_n \left[1 - \left(N - x_n^2\right) f(N)\right].$$

Determine how to choose $f(N)$ *so that the process has second-order convergence. Given that* $\sqrt{7} \approx 2.65$*, calculate* $\sqrt{7}$ *as accurately as a single application of the formula will allow.*

Writing the iteration scheme in the standard form $x_{n+1} = F(x_n)$, we see that $F(x) = x - x(N - x^2)f(N)$. For second-order convergence we require that $F'(\sqrt{N}) = 0$. Now

$$\begin{aligned} F'(x) \quad &= \quad 1 - (N - x^2)f(N) + 2x^2 f(N), \\ F'(\sqrt{N}) = 0 \quad &\Rightarrow \quad f(N) = -\frac{1}{2N}, \end{aligned}$$

i.e an iteration scheme that has second-order convergence is

$$x_{n+1} = x_n \left(1 + \frac{N - x_n^2}{2N}\right).$$

With $x_1 = 2.65$ as a first approximation to $\sqrt{7}$, the second approximation is

$$x_2 = 2.65 \left(1 + \frac{7 - (2.65)^2}{14}\right) = 2.645741071.$$

To the same 10-figure accuracy, the correct answer is 2.645751311.

27.6 *The following table of values of a polynomial $p(x)$ of low degree contains an error. Identify and correct the erroneous value and extend the table up to $x = 1.2$.*

x	*p(x)*	*x*	*p(x)*
0.0	*0.000*	*0.5*	*0.165*
0.1	*0.011*	*0.6*	*0.216*
0.2	*0.040*	*0.7*	*0.245*
0.3	*0.081*	*0.8*	*0.256*
0.4	*0.128*	*0.9*	*0.243*

Since the function is a polynomial of low degree, we expect the n-th difference of the entries to be constant, where n is a relatively low number. To test for this we set out (in two overlapping parts) a table of calculable differences (working in units of 0.001) as follows:

x	0.0		0.1		0.2		0.3		0.4		0.5
$p(x)$	0		11		40		81		128		165
1st diff		11		29		41		47		37	
2nd diff			18		12		6		−10		14
3rd diff				−6		−6		−16		24	

x	0.5		0.6		0.7		0.8		0.9
$p(x)$	165		216		245		256		243
1st diff		51		29		11		−13	
2nd diff	14		−22		−18		−24		
3rd diff		−36		4		−6			

Several of the 3rd differences are equal at −6, suggesting that the others should have been but for the error in the given table. To test this we construct an inverted table assuming that all third differences are in fact −6.

x	0.0		0.1		0.2		0.3		0.4		0.5
3rd diff				−6		−6		−6		−6	
2nd diff			18		12		6		0		−6
1st diff		11		29		41		47		47	
$p(x)$	0		11		40		81		128		175

x	0.5		0.6		0.7		0.8		0.9
3rd diff		−6		−6		−6			
2nd diff	−6		−12		−18		−24		
1st diff		41		29		11		−13	
$p(x)$	175		216		245		256		243

This has reconstructed the original table except for the value of $p(0.5)$ which is now 0.175, rather than the erroneous value of 0.165.

We continue the table up to $x = 1.2$ by starting from $x = 0.8$, where all values and differences are known.

x	0.8		0.9		1.0		1.1		1.2
3rd diff		−6		−6		−6		−6	
2nd diff	−24		−30		−36		−42		
1st diff		−13		−43		−79		−121	
$p(x)$	256		243		200		121		0

The corrected and extended table now reads

x	$p(x)$	x	$p(x)$
0.0	0.000	0.7	0.245
0.1	0.011	0.8	0.256
0.2	0.040	0.9	0.243
0.3	0.081	1.0	0.200
0.4	0.128	1.1	0.121
0.5	0.175	1.2	0.000
0.6	0.216		

27.8 *A possible rule for obtaining an approximation to an integral is the* mid-point rule, *given by*

$$\int_{x_0}^{x_0+\Delta x} f(x)\,dx = \Delta x\, f(x_0 + \tfrac{1}{2}\Delta x) + \mathrm{O}(\Delta x^3).$$

Writing h for Δx, and evaluating all derivates at the mid-point of the interval $(x, x+\Delta x)$, use a Taylor series expansion to find, up to $O(h^5)$, the coefficients of the higher-order errors in both the trapezium and mid-point rules. Hence find a linear combination of these two rules that gives $O(h^5)$ accuracy for each step Δx.

With all derivatives evaluated at $x + \frac{1}{2}h$ the Taylor series up to terms in h^5 for $f_0 = f(x)$ and $f_1 = f(x+h)$ are

$$f_{1,0} = f \pm \frac{h}{2}f' + \frac{1}{2!}\left(\frac{h}{2}\right)^2 f'' \pm \frac{1}{3!}\left(\frac{h}{2}\right)^3 f^{(3)} + \frac{1}{4!}\left(\frac{h}{2}\right)^4 f^{(4)} \pm \frac{1}{5!}\left(\frac{h}{2}\right)^5 f^{(5)},$$

with the upper sign corresponding to f_1 and the lower to f_0.

By definition, the midpoint rule gives

$$I_{\mathrm{mid}} = hf.$$

as the integral for the interval $(x, x+h)$. For the same interval the trapezium rule is evaluated as

$$I_{\mathrm{trap}} = \frac{h}{2}(f_0 + f_1) = h\left[f + \frac{1}{2!}\frac{h^2}{4}f'' + \frac{1}{4!}\frac{h^4}{16}f^{(4)} + \mathrm{O}(h^6)\right].$$

The exact integral over the same interval is

$$\begin{aligned} I_{\mathrm{ex}} &= \int_{-h/2}^{h/2}\left(f + yf' + \frac{y^2}{2!}f'' + \frac{y^3}{3!}f^{(3)} + \frac{y^4}{4!}f^{(4)} + \frac{y^5}{5!}f^{(5)} + \cdots\right) dy \\ &= hf + \frac{2}{2!}\frac{1}{3}\frac{h^3}{8}f'' + \frac{2}{4!}\frac{1}{5}\frac{h^5}{32}f^{(4)} + \cdots . \end{aligned}$$

Thus, to $\mathrm{O}(h^5)$,

$$\begin{aligned} I_{\mathrm{ex}} &= hf + \frac{1}{24}h^3 f'', \\ I_{\mathrm{mid}} &= hf, \\ I_{\mathrm{trap}} &= hf + \frac{1}{8}h^3 f'', \end{aligned}$$

and the best linear combination of the trapezium and mid-point rules that approximates the exact result to this order is $I = \frac{1}{3}I_{\mathrm{trap}} + \frac{2}{3}I_{\mathrm{mid}}$.

27.10 *Using the points and weights given in table 27.9, answer the following questions.*

(a) *A table of unnormalised Hermite polynomials* $H_n(x)$ *has been spattered with ink blots and gives* $H_5(x)$ *as* $32x^5 - ?x^3 + 120x$ *and* $H_4(x)$ *as* $?x^4 - ?x^2 + 12$, *where the coefficients marked ? cannot be read. What should they read?*

(b) *What is the value of the integral*

$$I = \int_{-\infty}^{\infty} \frac{e^{-2x^2}}{1 + 4x^2 + 3x}\,dx,$$

as given by a 7-point integration routine?

(a) Since the integration points for an n-point Gauss–Hermite integration are those values of x that make $H_n(x) = 0$, the given sampling points for a 5-point routine are those that give the expression $32x^5 - ax^3 + 120x$ zero value. Thus

$$a = \frac{32x^5 + 120x}{x^3}, \text{ where } x \text{ is either of } 0.95857\ldots \text{ and } 2.0218\ldots.$$

As they must, both cases give the same value; that value is 160.

Let $H_4(x) = bx^4 - cx^2 + 12$. Then, since all four sampling points $\pm x_i$ for a 4-point Gauss–Hermite scheme must satisfy $H_4(x_i) = 0$, we can write

$$\begin{aligned}
bx_1^4 - cx_1^2 + 12 &= 0,\\
bx_2^4 - cx_2^2 + 12 &= 0,\\
\Rightarrow \quad bx_1^4x_2^2 - cx_1^2x_2^2 + 12x_2^2 &= 0,\\
\text{and} \quad bx_2^4x_1^2 - cx_2^2x_1^2 + 12x_1^2 &= 0,\\
\Rightarrow \quad bx_1^2x_2^2(x_1^2 - x_2^2) + 12(x_2^2 - x_1^2) &= 0,\\
\Rightarrow \quad b = \frac{12}{x_1^2x_2^2}, \text{ and } c &= \frac{bx_i^4 + 12}{x_i^2}
\end{aligned}$$

for both $i = 1$ and $i = 2$. The first equation gives $b = 16$ and the second pair both yield $c = 48$.

(b) The denominator of the integrand is a quadratic form with '$4ac > b^2$'. It therefore has no real zeroes and we may use Gauss–Hermite integration. To cast the exponential in the appropriate form, we need to make the change of variable $y = \sqrt{2}\,x$; the exponential then has the form e^{-y^2}, as assumed in the quadrature formula. The integral becomes

$$I = \int_{-\infty}^{\infty} \frac{e^{-y^2}}{2y^2 + (3/\sqrt{2})y + 1}\,\frac{dy}{\sqrt{2}} = \int_{-\infty}^{\infty} \frac{e^{-y^2}}{2\sqrt{2}y^2 + 3y + \sqrt{2}}\,dy.$$

This is now in the form $\int_{-\infty}^{\infty} e^{-y^2} g(y)\,dy$ to which the Gauss-Hermite procedure can be applied directly. Using the points and weights for a 7-point calculation gives a value of 1.1642.

27.12 *In normal use only a single application of n-point Gaussian quadrature is made, using a value of n that is estimated from experience to be 'safe'. However, it is instructive to examine what happens when n is changed in a controlled way.*

(a) *Evaluate the integral*

$$I_n = \int_2^5 \sqrt{7x - x^2 - 10}\,dx$$

using n-point Gauss–Legendre formulae for $n = 2, 3, \ldots, 6$. *Estimate (to* 4 *s.f.) the value* I_∞ *you would obtain for very large n and compare it with the result I obtained by exact integration. Explain why the variation of* I_n *with n is monotonically decreasing.*

(b) *Try to repeat the processes described in* (a) *for the integrals*

$$J_n = \int_2^5 \frac{1}{\sqrt{7x - x^2 - 10}}\,dx.$$

Why is it very difficult to estimate J_∞?

(a) Since the integral is not over the (finite) range $-1 \le x \le 1$, we must first make the transformation

$$z = \frac{2x - 5 - 2}{5 - 2} \qquad \Rightarrow \qquad x = \frac{7}{2} + \frac{3z}{2}.$$

This results in the integral being

$$I = \tfrac{5-2}{2} \int_{-1}^{1} g(z)\,dz$$

with $g(z) = f(x)$.

Using the tables of points and weights given in the text, we obtain the following results as n is varied.

n	2	3	4	5	6
I_n	3.674	3.581	3.556	3.546	3.541

Clearly, a constant value good to 4 s.f. has not been achieved but, either by rough plotting (preferably *versus* an inverse power of n) or by extrapolating the rate of change of the last two significant figures, we can estimate I_∞ as lying in the range 3.533 to 3.535.

To calculate the exact result, we note that $7x - x^2 - 10$ can be written (by 'completing the square') as $\frac{9}{4} - (x - \frac{7}{2})^2$ and so the substitution $x = \frac{7}{2} + \frac{3}{2}\sin\theta$ gives

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \tfrac{3}{2}\sqrt{1 - \sin^2\theta}\, \tfrac{3}{2}\cos\theta\, d\theta \\ &= \tfrac{9}{4}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = \tfrac{9}{4}\,\tfrac{1}{2}\,\pi = 3.5343 \end{aligned}$$

This does come within our estimated range, though, obviously, we could not be more certain of the value at a level better than ± 0.001 without using much higher values of n. [$n = 20$ gives $I_{20} = 3.5345$.]

The monotonic behaviour of I_n with n comes about because the integrand starts from zero, rises to a single maximum, and falls to zero again as x nears the end of its range. This, coupled with the fact that as n increases the sampling points are continually pushed further outwards from the middle of the range, means that smaller actual values of the integrand gain increasing weight in the sum, thus lowering it. Consequently, the series I_n approaches I monotonically from above.

(b) The procedure is exactly the same as in part (a); only the form of the integrand is different. The results are:

n	2	3	4	5	6	20
J_n	2.449	2.646	2.755	2.825	2.874	3.057

This time the results are monotonically increasing. This is no surprise, as the integrand now increases at the extremes of its range; indeed, it has an (integrable) singularity at each end point. These do not make the Gauss–Legendre sum infinite for any finite n since the sample points never include the end points of the range. However, they do prevent the estimates from converging rapidly for small values of n, thus making it virtually impossible to extrapolate to J_∞.

The accurate value of J is obtained by making the same substitution as in part (a) and produces

$$J = \int_{-\pi/2}^{\pi/2} \frac{\frac{3}{2}\cos\theta}{\frac{3}{2}\sqrt{1 - \sin^2\theta}}\, d\theta = \int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

Thus we see that, even with n as high as 20, the second significant figure of the value of J is not yet established with any certainty.

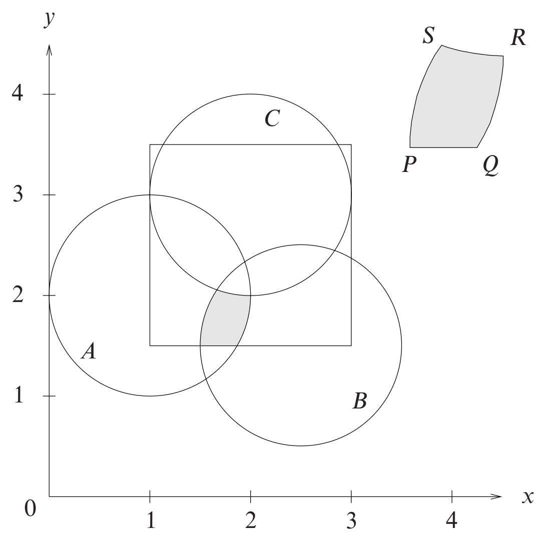

Figure 27.1 The area to be estimated using the hit or miss Monte Carlo method of exercise 27.14.

27.14 *A, B and C are three circles of unit radius with centres in the xy-plane at* $(1, 2), (2.5, 1.5)$ *and* $(2, 3)$*, respectively. Devise a hit or miss Monte Carlo calculation to determine the size of the area that lies outside C but inside A and B, as well as inside the square centred on* $(2, 2.5)$ *that has sides of length 2 parallel to the coordinate axes. You should choose your sampling region so as to make the estimation as efficient as possible. Take the random number distribution to be uniform on* $(0, 1)$ *and determine the inequalities that have to be tested using the random numbers chosen.*

Figure 27.1 shows the three circles and the square, with the area whose size has to be determined shown shaded. The same shaded area is repeated in the top right-hand corner of the figure, but on a larger scale and labelled $PQRS$.

In order to make the estimation as efficient as possible, we sample in the minimal rectangular region that encloses $PQRS$. We must therefore find the coordinates of these 'corners'. Those of P are determined by the circle B and the square as $P = (2.5 - 1, 2.5 - 1) = (1.5, 1.5)$. The x-coordinate of Q is determined by the circle A as $1 + \sqrt{1 - (2 - 1.5)^2} = 1 + \sqrt{3}/2$; its y-coordinate is 1.5. It is therefore $(1 + \sqrt{3}/2, 1.5)$. The circles A and C determine R as $(2, 3 - 1) = (2, 2)$.

The coordinates of S are a little more difficult to find; they are determined by the circles B and C as follows:

$$
\begin{aligned}
(x-2.5)^2+(y-1.5)^2 &= 1,\\
(x-2)^2+(y-3)^2 &= 1,\\
-5x+4x+6.25-4-3y+6y+2.25-9 &= 0, \text{ by subtraction,}\\
-x+3y-4.5 &= 0, \text{ substitute this into } C,\\
(3y-6.5)^2+(y-3)^2 &= 1,\\
10y^2-45y+50.25 &= 0,\\
y = \frac{45 \pm \sqrt{2025-2010}}{20} &= 2.056 \quad (2.444, \text{ rejected}),\\
x = (3\times 2.056)-4.5 &= 1.669.
\end{aligned}
$$

The minimal enclosing rectangle is thus $1.5 \le x \le 2$, $1.5 \le y \le 2.056$. We note that it is entirely contained within the square and so no subsequent test of individual sample points will be needed in this regard.

The sampling scheme is to select a pair of random numbers, (ξ_1, ξ_2), and then set $x = 1.5 + \alpha\xi_1$ and $y = 1.5 + \beta\xi_2$. The optimal values for α and β are 0.5 and 0.556 respectively. These values of x and y are then tested to see if they lie inside/outside the defining circles, using the inequalities:

$$
\begin{aligned}
(\alpha\xi_1+1.5-1)^2+(\beta\xi_2+1.5-2)^2 &\le 1,\\
(\alpha\xi_1+1.5-2.5)^2+(\beta\xi_2+1.5-1.5)^2 &\le 1,\\
(\alpha\xi_1+1.5-2)^2+(\beta\xi_2+1.5-3)^2 &\ge 1.
\end{aligned}
$$

If all three conditions are satisfied for n out of N pairs of randomly chosen numbers, (ξ_1, ξ_2), the area of $PQRS$ can be estimated as $n\alpha\beta/N$.

27.16 *Consider the application of the predictor–corrector method described near the end of subsection 27.6.3 to the equation*

$$\frac{dy}{dx} = x + y, \qquad y(0) = 0.$$

Show, by comparison with a Taylor series expansion, that the expression obtained for y_{i+1} in terms of x_i and y_i by applying the three steps indicated (without any repeat of the last two) is correct to $O(h^2)$. Using steps of $h = 0.1$, compute the value of $y(0.3)$ and compare it with the value obtained by solving the equation analytically.

Since $y' = x + y$, $y'' = 1 + y' = 1 + x + y$, the first few terms of the Taylor series expansion of $y(x)$ are

$$y(x+h) = y(x) + hy'(x) + \tfrac{1}{2}h^2 y''(x) + \cdots$$
$$= y(x) + h(x+y) + \tfrac{1}{2}h^2(1+x+y) + \cdots .$$

We now consider the predictor–corrector method and indicate intermediate (predicted) values by placing a bar over them. As in the main text, we denote the function of x and y (here $x+y$) that determines the derivative dy/dx by $f(x,y)$.

$$\overline{y_{i+1}} = y_i + hf_i,$$
$$\overline{f_{i+1}} = x_{i+1} + \overline{y_{i+1}} = x_{i+1} + y_i + h(x_i + y_i).$$

Now, using $y_{i+1} = y_i + \frac{1}{2}h(f_i + \overline{f_{i+1}})$,

$$y_{i+1} = y_i + \tfrac{1}{2}h[\,x_i + y_i + x_{i+1} + y_i + h(x_i+y_i)\,]$$
$$= y_i + \tfrac{1}{2}h(x_i + x_{i+1}) + hy_i + \tfrac{1}{2}h^2(x_i + y_i)$$

Now, since $x_{i+1} = x_i + h$,

$$y_{i+1} = y_i + hx_i + hy_i + \tfrac{1}{2}h^2(1 + x_i + y_i).$$

This coincides with the Taylor series expansion up to $O(h^2)$ and proves the stated result.

We calculate the required value of $y(0.3)$ in the following table (in which each column is completed before moving to the next and the last entry in each column becomes the third entry in the next):

i	0	1	2
x_i	0.0	0.1	0.2
y_i	0.00000	0.00500	0.02103
$h(x_i+y_i)$	0.00000	0.01050	0.02210
$\frac{1}{2}h^2(1+x_i+y_i)$	0.00500	0.00553	0.00611
y_{i+1}	0.00500	0.02103	0.04924

The calculated value of $y(0.3)$ is thus 0.04924.

For the marginally reorganised initial equation

$$\frac{dy}{dx} - y = x,$$

we see by inspection that the CF is $y(x) = Ae^x$ and that a PI is $y(x) = -x - 1$. The given boundary condition, $y(0) = 0$, implies that $A = 1$ and that the exact solution is $y(x) = e^x - x - 1$. The correct value of $y(0.3)$ to 4 s.f. is therefore 0.04986.

27.18 *If $dy/dx = f(x, y)$ then show that*

$$\frac{d^2 f}{dx^2} = \frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + f^2\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2.$$

Hence verify, by substitution and the subsequent expansion of arguments in Taylor series of their own, that the scheme given in (27.79) coincides with the Taylor expansion (27.68), i.e.

$$y_{i+1} = y_i + hy_i^{(1)} + \frac{h^2}{2!}y_i^{(2)} + \frac{h^3}{3!}y_i^{(3)} + \cdots .$$

up to terms in h^3.

The scheme (a third-order Runge–Kutta calculation) is

$$y_{i+1} = y_i + \tfrac{1}{6}(b_1 + 4b_2 + b_3),$$

where

$$\begin{aligned} b_1 &= hf(x_i, y_i), \\ b_2 &= hf(x_i + \tfrac{1}{2}h,\ y_i + \tfrac{1}{2}b_1), \\ b_3 &= hf(x_i + h,\ y_i + 2b_2 - b_1), \end{aligned}$$

To find the first and second total derivatives of $f(x, y)$ with respect to x we use the chain rule:

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}, \\ \frac{d^2 f}{dx^2} &= \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 f}{\partial y \partial x} + f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f^2\frac{\partial^2 f}{\partial y^2} + f\left(\frac{\partial f}{\partial y}\right)^2, \end{aligned}$$

i.e. as stated.

The (accurate) Taylor expansion is

$$\begin{aligned} y_{i+1} &= y_i + hy_i^{(1)} + \frac{h^2}{2!}y_i^{(2)} + \frac{h^3}{3!}y_i^{(3)} + \cdots \qquad (*) \\ &= y_i + hf_i + \frac{h^2}{2!}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\ &\quad + \frac{h^3}{3!}\left[\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f^2\frac{\partial^2 f}{\partial y^2} + f\left(\frac{\partial f}{\partial y}\right)^2\right] + \cdots \end{aligned}$$

We next need to find explicit expressions for the quantities b_1, b_2 and b_3 in terms of f and its various derivatives. The first is simple, $b_1 = hf_i$. For the other two, with f evaluated at points other than (x_i, y_i), we expand in local Taylor series, retaining only those terms that will be of order h^3 or less in the final expression. For b_2:

$$\begin{aligned} b_2 &= hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hf_i) \\ &= hf_i + h^2\left(\frac{1}{2}\frac{\partial f_i}{\partial x} + \frac{1}{2}f_i\frac{\partial f_i}{\partial y}\right) \\ &\quad + h^3\left(\frac{1}{2}\,\frac{1}{4}\,\frac{\partial^2 f_i}{\partial x^2} + \frac{1}{2}\,\frac{1}{4}\,\frac{\partial^2 f_i}{\partial y^2}f_i^2 + \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,2\frac{\partial^2 f_i}{\partial x \partial y}f_i\right). \end{aligned}$$

For b_3, still working to order h^3:

$$\begin{aligned} b_3 &= h\,f(\,x_i + h,\ y_i + 2hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hf_i) - hf_i\,) \\ &= h\,f\left(x_i + h,\ y_i + 2h\left[f_i + \frac{1}{2}h\frac{\partial f_i}{\partial x} + \frac{1}{2}hf_i\frac{\partial f_i}{\partial y}\right] - hf_i\right) \\ &= h\,f\left(x_i + h,\ y_i + hf_i + h^2\frac{\partial f_i}{\partial x} + h^2 f_i\frac{\partial f_i}{\partial y} + \mathrm{O}(h^3)\right). \end{aligned}$$

We now need a two-variable Taylor expansion of this last function about (x_i, y_i). The leading term is clearly hf_i and the contributions from partial derivatives with respect to x only are

$$h^2\frac{\partial f_i}{\partial x} + \frac{1}{2}\,h^3\frac{\partial^2 f_i}{\partial x^2} + \mathrm{O}(h^4).$$

The contributions from partial derivatives with respect to y only are

$$h\left[hf_i + h^2\left(\frac{\partial f_i}{\partial x} + f_i\,\frac{\partial f_i}{\partial y}\right)\right]\frac{\partial f_i}{\partial y} + \mathrm{O}(h^4) + \frac{1}{2}h(hf_i)^2\frac{\partial^2 f_i}{\partial y^2} + \mathrm{O}(h^4),$$

whilst, to order h^3, the only contribution from the mixed derivatives is

$$h\,h\,hf_i\frac{\partial^2 f_i}{\partial x \partial y}.$$

Collecting these together gives b_3 as

$$\begin{aligned} b_3 &= hf_i + h^2\left(\frac{\partial f_i}{\partial x} + f_i\frac{\partial f_i}{\partial y}\right) \\ &\qquad + h^3\left[\frac{1}{2}\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial f_i}{\partial x}\,\frac{\partial f_i}{\partial y} + f_i\left(\frac{\partial f_i}{\partial y}\right)^2 + \frac{1}{2}f_i^2\frac{\partial^2 f_i}{\partial y^2} + f_i\frac{\partial^2 f_i}{\partial x \partial y}\right]. \end{aligned}$$

Finally we must form the sum $\frac{1}{6}(b_1 + 4b_2 + b_3)$ and check it against the accurate Taylor expansion for $y_{i+1} - y_i$. The collected multipliers of the three powers of h

are:

$$\begin{aligned}
h &: \ \frac{1}{6}(f_i + 4f_i + f_i) = f_i, \\
h^2 &: \ \frac{1}{6}\left(\frac{4}{2}\frac{\partial f_i}{\partial x} + \frac{4}{2}f_i\frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial x} + f_i\frac{\partial f_i}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial f_i}{\partial x} + f_i\frac{\partial f_i}{\partial y}\right), \\
h^3 &: \ \frac{1}{6}\left[\frac{4}{8}\frac{\partial^2 f_i}{\partial x^2} + \frac{4}{8}\frac{\partial^2 f_i}{\partial y^2}f_i^2 + \frac{4}{4}\frac{\partial^2 f_i}{\partial x \partial y}f_i + \frac{1}{2}\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial f_i}{\partial x}\frac{\partial f_i}{\partial y}\right. \\
&\qquad\qquad \left. + f_i\left(\frac{\partial f_i}{\partial y}\right)^2 + \frac{1}{2}f_i^2\frac{\partial^2 f_i}{\partial y^2} + f_i\frac{\partial^2 f_i}{\partial x \partial y}\right].
\end{aligned}$$

These can now be compared with the expression given in the last line of (∗). The multipliers of h and h^2 have explicitly been shown to be equal and those of h^3 are also shown to be so when the final expression above is simplified; this establishes the validity of this third-order Runge–Kutta scheme.

27.20 *Set up a finite difference scheme to solve the ordinary differential equation*

$$x\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} = 0$$

in the range $1 \le x \le 4$ *and subject to the boundary conditions* $\phi(1) = 2$ *and* $d\phi/dx = 2$ *at* $x = 4$*. Using* N *equal increments,* Δx*, in* x*, obtain the general difference equation and state how the boundary conditions are incorporated into the scheme. Setting* Δx *equal to the (crude) value* 1*, obtain the relevant simultaneous equations and so obtain rough estimates for* $\phi(2), \phi(3)$ *and* $\phi(4)$*.*

Finally, solve the original equation analytically and compare your numerical estimates with the accurate values.

We will use central differences for both derivatives except at $x = 4$ where the backward difference is used to fit the boundary condition on $d\phi/dx$.

Defining Δx by $N\Delta x = 3$ and denoting $\phi(1 + j\Delta x)$ by ϕ_j, we have as boundary conditions, $\phi_0 = 2$ and $\phi_N = \phi_{N-1} + 2\Delta x$. We thus have to calculate ϕ_j for $j = 1, 2, \ldots, N-2$ using a difference representation of the differential equation taking the form

$$(1 + j\Delta x)\frac{\phi_{j+1} + \phi_{j-1} - 2\phi_j}{(\Delta x)^2} + \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} = 0,$$

which can be re-arranged as

$$[\,2 + (2j+1)\Delta x\,]\phi_{j+1} - (4 + 4j\Delta x)\phi_j + [\,2 + (2j-1)\Delta x\,]\phi_{j-1} = 0.$$

Now setting $\Delta x = 1$ (and consequently $N = 3$) we have, for $j = 1, 2$, in turn,

$$5\phi_2 - 8\phi_1 + 3 \times 2 = 5\phi_2 - 8\phi_1 + 6 = 0,$$

$$7(\phi_2 + 2) - 12\phi_2 + 5\phi_1 = -5\phi_2 + 5\phi_1 + 14 = 0.$$

Solving this pair of simultaneous equations gives

$$\phi_1 = \frac{20}{3} = 6.67 \quad \text{and} \quad \phi_2 = \frac{142}{15} = 9.47.$$

Re-substitution of these values in the recurrence relation then gives the value of ϕ at the upper boundary, where only its slope has been specified, as

$$\phi_3 = 11.47.$$

For an analytical solution we write the equation as

$$\frac{\phi''}{\phi'} + \frac{1}{x} = 0 \quad \Rightarrow \quad \phi' x = k = 2 \times 4 = 8, \text{ using } \phi'(4) = 2,$$

$$\frac{d\phi}{dx} = \frac{8}{x} \quad \Rightarrow \quad \phi = c + 8 \ln x = 2 + 8 \ln x.$$

The boundary condition at $x = 4$ has already been incorporated into the first line above and that at $x = 1$ is used in the second. The accurately calculated values are therefore

$$\phi(2) = \phi_1 = 7.55, \quad \phi(3) = \phi_2 = 10.79, \quad \phi(4) = \phi_3 = 13.09.$$

The estimated values follow the same trend as the accurate ones but are consistently lower (except at $x = 1$ where they are forced to be equal). The major source of inaccuracy arises from forcing the difference between the estimated values of $\phi(4) = \phi_3$ and $\phi(3) = \phi_2$ to be 2; the accurate solution has an average slope in this range of $8 \ln(4/3) = 2.30$, i.e. significantly higher than 2.

27.22 *Use the isocline approach to sketch the family of curves that satisfies the non-linear first-order differential equation*

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 + y^2}}.$$

At each point in the xy-plane, the equation determines the slope of the solution. A solution curve must therefore pass through a (continuous) series of points, at each of which its tangent has the relevant slope. A computer-generated plot, together with typical solution curves, is shown in figure 27.2.

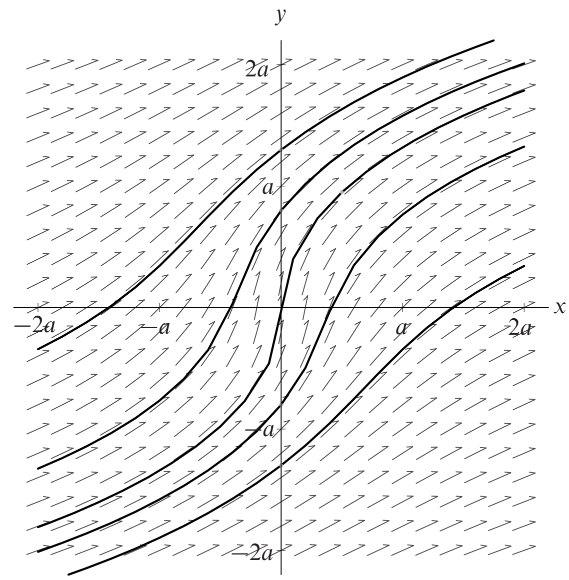

Figure 27.2 Typical solutions $y = y(x)$, shown by solid lines, of $dy/dx = a(x^2 + y^2)^{-1/2}$. The short arrows give the direction that the tangent to any solution must have at that point.

27.24 *In the previous exercise (27.23) the difference scheme for solving*

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0,$$

in which A has been set equal to unity, was one-sided in both space (x) and time (t). A more accurate procedure (known as the Lax–Wendroff scheme) is

$$\frac{u(p, n+1) - u(p, n)}{\Delta t} + \frac{u(p+1, n) - u(p-1, n)}{2\Delta x} = \frac{\Delta t}{2}\left[\frac{u(p+1, n) - 2u(p, n) + u(p-1, n)}{(\Delta x)^2}\right].$$

(a) *Establish the orders of accuracy of the two finite difference approximations on the LHS of the equation.*
(b) *Establish the accuracy with which the expression in the brackets approximates $\partial^2 u/\partial x^2$.*
(c) *Show that the RHS of the equation is such as to make the whole difference scheme accurate to second order in both space and time.*

(a) and (b) We start with the (accurate) Taylor expansion in space for $u(p \pm 1, n)$;

$$u(p \pm 1, n) = u(p, n) \pm \Delta x \frac{\partial u(p, n)}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u(p, n)}{\partial x^2} \pm \frac{(\Delta x)^3}{3!} \frac{\partial^3 u(p, n)}{\partial x^3} +$$

The second term on the LHS of the Lax–Wendroff scheme is thus

$$\frac{u(p+1, n) - u(p-1, n)}{2\Delta x} = \frac{\partial u(p, n)}{\partial x} + \mathrm{O}\left((\Delta x)^2\right),$$

whilst its RHS is

$$\frac{\Delta t}{2} \left[\frac{\partial^2 u(p, n)}{\partial x^2} + \mathrm{O}\left((\Delta x)^2\right) \right].$$

Both are accurate to second order in Δx. We note at this point that the second spatial derivative does not actually appear on the RHS of the original equation; in the original equation the RHS is zero.

For the first term on the LHS we need a Taylor expansion in time:

$$u(p, n+1) = u(p, n) + \Delta t \frac{\partial u(p, n)}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u(p, n)}{\partial t^2} + \cdots.$$

Thus this term in the calculational scheme is

$$\frac{\partial u(p, n)}{\partial t} + \frac{\Delta t}{2!} \frac{\partial^2 u(p, n)}{\partial t^2} + \cdots.$$

So far as a representation of $\partial u/\partial t$ in the original equation is concerned, this is only accurate to first order in Δt and to make it accurate to second order we need to compensate for the term

$$\frac{\Delta t}{2!} \frac{\partial^2 u(p, n)}{\partial t^2}.$$

(c) However, from differentiating the original equation separately with respect to x and t, we have both

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial x} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial x^2} = 0,$$

implying that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

This equality (multiplied through by $\Delta t/2$) allows the unwanted terms on each side of the original equation to be cancelled, leaving the equation accurate to second order in both Δx and Δt.

27.26 *Consider the solution $\phi(x,y)$ of Laplace's equation in two dimensions using a relaxation method on a square grid with common spacing h. As in the main text, denote $\phi(x_0+ih,\ y_0+jh)$ by $\phi_{i,j}$. Further, define $\phi_{i,j}^{m,n}$ by*

$$\phi_{i,j}^{m,n} \equiv \frac{\partial^{m+n}\phi}{\partial x^m\,\partial y^n}$$

evaluated at $(x_0+ih,\ y_0+jh)$.

(a) *Show that*

$$\phi_{i,j}^{4,0} + 2\phi_{i,j}^{2,2} + \phi_{i,j}^{0,4} = 0.$$

(b) *Working up to terms of order h^5, find Taylor series expansions, expressed in terms of the $\phi_{i,j}^{m,n}$, for*

$$S_{\pm,0} = \phi_{i+1,j} + \phi_{i-1,j}$$
$$S_{0,\pm} = \phi_{i,j+1} + \phi_{i,j-1}.$$

(c) *Find a corresponding expansion, to the same order of accuracy, for $\phi_{i\pm1,j+1} + \phi_{i\pm1,j-1}$ and hence show that*

$$S_{\pm,\pm} = \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}$$

has the form

$$4\phi_{i,j}^{0,0} + 2h^2(\phi_{i,j}^{2,0} + \phi_{i,j}^{0,2}) + \frac{h^4}{6}(\phi_{i,j}^{4,0} + 6\phi_{i,j}^{2,2} + \phi_{i,j}^{0,4}).$$

(d) *Evaluate the expression $4(S_{\pm,0}+S_{0,\pm})+S_{\pm,\pm}$ and hence deduce that a possible relaxation scheme, good to the fifth order in h, is to recalculate each $\phi_{i,j}$ as the weighted mean of the current values of its four nearest neighbours (each with weight $\frac{1}{5}$) and its four next-nearest neighbours (each with weight $\frac{1}{20}$).*

(a) In the notation given, Laplace's equation takes the form

$$\phi_{i,j}^{2,0} + \phi_{i,j}^{0,2} = 0.$$

To save space and increase clarity we will omit subscripts that are i, j, but write them explicitly when they are not. Thus $\phi^{m,n} \equiv \phi_{i,j}^{m,n}$.

Differentiating Laplace's equation twice more with respect to x and y (separately) yields

$$\phi^{4,0} + \phi^{2,2} = 0 \text{ and } \phi^{2,2} + \phi^{0,4} = 0.$$

Adding these two equations yields the stated result, but we may, in addition, deduce several equalities to be used later. We start from $\phi^{2,0} = -\phi^{0,2}$ and obtain in an obvious way that

$$\phi^{3,0} = -\phi^{1,2}, \qquad \phi^{0,3} = -\phi^{2,1},$$

and

$$\phi^{3,1} = -\phi^{1,3}, \qquad \phi^{4,0} = \phi^{0,4} = -\phi^{2,2},$$

and

$$\phi^{5,0} = \phi^{1,4} = -\phi^{3,2}, \qquad \phi^{0,5} = \phi^{4,1} = -\phi^{2,3}.$$

(b) The general Taylor series expansion for $\phi_{i\pm1,\,j}$ is

$$\phi_{i\pm1,\,j} = \sum_{m=0}^{\infty} \frac{(\pm h)^m}{m!}\,\phi^{m,0},$$

and, up to order h^5,

$$\begin{aligned} S_{\pm,0} &= \phi_{i+1,\,j} + \phi_{i-1,\,j} \\ &= 2\phi^{0,0} + h^2\phi^{2,0} + \tfrac{1}{12}h^4\phi^{4,0}. \end{aligned}$$

Similarly,

$$S_{0,\pm} = 2\phi^{0,0} + h^2\phi^{0,2} + \tfrac{1}{12}h^4\phi^{0,4}.$$

(c) The expansion of $\phi_{i\pm1,\,j+1}$ requires a 2-variable Taylor series and up to order h^5 takes the form

$$\begin{aligned} \phi_{i\pm1,\,j+1} = \phi^{0,0} &\pm h\phi^{1,0} + h\phi^{0,1} + \frac{h^2}{2!}(\phi^{2,0} \pm 2\phi^{1,1} + \phi^{0,2}) \\ &+ \frac{h^3}{3!}(\pm\phi^{3,0} + 3\phi^{2,1} \pm 3\phi^{1,2} + \phi^{0,3}) \\ &+ \frac{h^4}{4!}(\phi^{4,0} \pm 4\phi^{3,1} + 6\phi^{2,2} \pm 4\phi^{1,3} + \phi^{0,4}) \\ &+ \frac{h^5}{5!}(\pm\phi^{5,0} + 5\phi^{4,1} \pm 10\phi^{3,2} + 10\phi^{2,3} \pm 5\phi^{1,4} + \phi^{0,5}). \end{aligned}$$

Because of the equalities derived in part (a), this can be written more compactly as

$$\begin{aligned} \phi_{i\pm1,\,j+1} = \phi^{0,0} &\pm h\phi^{1,0} + h\phi^{0,1} + \frac{h^2}{2!}(\pm 2\phi^{1,1}) \\ &+ \frac{h^3}{3!}(\pm 2\phi^{1,2} + 2\phi^{2,1}) + \frac{h^4}{4!}(4\phi^{2,2}) + \frac{h^5}{5!}(\pm 4\phi^{3,2} + 4\phi^{2,3}). \end{aligned}$$

For the corresponding expansion of $\phi_{i\pm1,\,j-1}$ those terms for which the n in $\phi^{m,n}$ is odd, will change sign. When the two expansions are added together, such terms will cancel and leave

$$\phi_{i\pm1,\,j+1} + \phi_{i\pm1,\,j-1} = 2\phi^{0,0} \pm 2h\phi^{1,0} \pm \frac{2h^3}{3}\phi^{1,2} + \frac{h^4}{3}\phi^{2,2} \pm \frac{h^5}{15}\phi^{3,2}.$$

Hence,

$$
\begin{aligned}
S_{\pm,\pm} &= \phi_{i+1,\,j+1} + \phi_{i+1,\,j-1} + \phi_{i-1,\,j+1} + \phi_{i-1,\,j-1} \\
&= 4\phi^{0,0} + \tfrac{2}{3}h^4\phi^{2,2} + \mathrm{O}(h^6).
\end{aligned}
$$

This is consistent with the stated expression since $\phi^{2,0} + \phi^{0,2} = 0$ and $\phi^{4,0} + 6\phi^{2,2} + \phi^{0,4} = 4\phi^{2,2}$.

(d) Now consider the sum S given by

$$
\begin{aligned}
S &= 4(S_{\pm,o} + S_{0,\pm}) + S_{\pm,\pm} \\
&= 4\left[4\phi^{0,0} + h^2(\phi^{0,2} + \phi^{2,0}) + \frac{h^4}{12}(\phi^{4,0} + \phi^{0,4})\right] \\
&\qquad\qquad + 4\phi^{0,0} + \tfrac{2}{3}h^4\phi^{2,2} + \mathrm{O}(h^6) \\
&= 20\phi^{0,0} + 4h^2(\phi^{0,2} + \phi^{2,0}) + \frac{h^4}{3}(\phi^{4,0} + \phi^{0,4} + 2\phi^{2,2}) + \mathrm{O}(h^6) \\
&= 20\phi^{0,0} + 0 + 0 + \mathrm{O}(h^6).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\phi^{0,0} &= \frac{4}{20}(S_{\pm,0} + S_{0,\pm}) + \frac{1}{20}S_{\pm,\pm} + \mathrm{O}(h^6) \\
&= \tfrac{1}{5}(\phi_{i+1,\,j} + \phi_{i-1,\,j} + \phi_{i,\,j+1} + \phi_{i,\,j-1}) \\
&\qquad + \tfrac{1}{20}(\phi_{i+1,\,j+1} + \phi_{i+1,\,j-1} + \phi_{i-1,\,j+1}\phi_{i-1,\,j-1}) + \mathrm{O}(h^6) \\
&= \tfrac{1}{5}\sum(\text{nearest neighbours}) \\
&\qquad + \tfrac{1}{20}\sum(\text{next nearest neighbours}) + \mathrm{O}(h^6).
\end{aligned}
$$

This could form the basis of a relaxation scheme, as described in the question.

28

Group theory

> **28.2** *Which of the following relationships between X and Y are equivalence relations? Give a proof of your conclusions in each case:*
>
> (a) *X and Y are integers and $X - Y$ is odd*;
> (b) *X and Y are integers and $X - Y$ is even*;
> (c) *X and Y are people and have the same postcode*;
> (d) *X and Y are people and have a parent in common*;
> (e) *X and Y are people and have the same mother*;
> (f) *X and Y are $n \times n$ matrices satisfying $Y = PXQ$, where P and Q are elements of a group $\mathcal{G}$ of $n \times n$ matrices.*

(a) Defining an odd integer as one that does not divide by 2 exactly to yield another integer, this relationship fails to pass the reflexivity test. The equation $X \sim X$ implies that $X - X = 0$ is odd. However, 0 *does* divide exactly by 2 and so $X \nsim X$.

(b) With $X - Y$ required to be even, this relationship satisfies the reflexivity criterion. Further, since (i) $2n$ and $-2n$ are both even and (ii) $X - Z = (X - Y) + (Y - Z)$ and the sum of two even integers is even, the symmetry and transitivity requirements are also met. Thus this relationship is an equivalence relation. The partition is that of the integers into odd and even integers.

(c) Clearly an equivalence relation. The classes consist of all the people, amongst those considered, who have the same postcode.

(d) Although at first sight this may appear to be an equivalence relation, it is not necessarily a transitive relationship. For three people X, Y and Z for whom

$X \sim Y$ and $Y \sim Z$, if Y's parents both re-marry and X and Z are children of the two second marriages, then $X \not\sim Z$.

(e) Assuming that there is an agreed single definition of mother (so that, for example, a person cannot have two mothers, one natural and one the result of adoption or parental re-marriage) then this is an equivalence relation.

(f) Since the identity I belongs $\mathcal{G}$ and $X = IXI$, $X \sim X$, showing that the relationship is reflexive.

If $Y \sim X$ then $Y = PXQ \quad \Rightarrow \quad X = P^{-1}YQ^{-1}$. But as P and Q belong to $\mathcal{G}$ so do P^{-1} and Q^{-1}. Thus $X \sim Y$ and the relationship is symmetric.

If $Y \sim X$ and $Z \sim Y$ then $Y = PXQ$ and $Z = RYS$, where R and S are also elements of the group. Thus $Z = RPXQS$. However, as P, Q, R and S all belong to $\mathcal{G}$, so do RP and QS. It follows that $Z \sim X$ and that the relationship is transitive.

These three results together show that the relationship is an equivalence relation.

28.4 *Prove that the relationship $X \sim Y$, defined by $X \sim Y$ if Y can be expressed in the form*

$$Y = \frac{aX + b}{cX + d},$$

with a, b, c and d as integers, is an equivalence relation on the set of real numbers $\mathfrak{R}$. Identify the class that contains the real number 1.

(i) Reflexivity is shown by writing $X = \dfrac{1X + 0}{0X + 1}$.

(ii) Symmetry is shown by rewriting the defining equation as

$$cXY + dY = aX + b \quad \Rightarrow \quad X = \frac{-dY + b}{cY - a}.$$

(iii) For transitivity, we have

$$Y = \frac{aX + b}{cX + d} \quad \text{and} \quad Z = \frac{a'Y + b'}{c'Y + d'}$$

giving

$$Z = \frac{a'(aX + b) + b'(cX + d)}{c'(aX + b) + d'(cX + d)} = \frac{(a'a + b'c)X + (a'b + b'd)}{(c'a + d'c)X + (c'b + d'd)}.$$

All of these coefficients are integers and so transitivity is established.

Thus $X \sim Y$ satisfies the three requirements of an equivalence relation on $\mathfrak{R}$, the set of real numbers.

By setting $X = 1$ in the defining relationship, we can see that Y must have the form of the ratio of (any) two integers. Thus the class to which 1 belongs is the set of rational numbers.

28.6 *Prove that the set $\mathcal{M}$ of matrices*

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a, b, c are integers (mod 5) and $a \neq 0 \neq c$, forms a non-Abelian group under matrix multiplication.

Show that the subset containing elements of $\mathcal{M}$ that are of order 1 or 2 do not form a proper subgroup of $\mathcal{M}$,

(a) *using Lagrange's theorem,*
(b) *by direct demonstration that the set is not closed.*

Consider the product of two typical matrices in the set, A and B,

$$\mathsf{AB} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}.$$

Since 5 is prime, the product ad cannot equal 0 unless at least one of a and d is 0; but this is ruled out by the given form of the matrices. Similarly $cf \neq 0$. Thus the set is closed under matrix multiplication.

Matrix multiplication is associative.

The set has an obvious identity element $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The inverse of the typical matrix A is found in the usual way:

$$\mathsf{A}^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix},$$

as is easily verified. We use the fact that $a \neq 0 \neq c$ to deduce that $|\mathsf{A}| = ac \neq 0$ and hence justify the usual inversion calculation. We note that for $x = 1, 2, 3, 4$ the multiplicative inverses (mod 5) are $1, 3, 2, 4$ respectively and are well-defined and unique. In particular, this means that $1/ac$ is a well-defined integer (not the fraction it may appear to be at first sight).

These four results establish the set $\mathcal{M}$ as a group.

There are four choices each for the values of a and c and five choices for b; the group therefore has $4 \times 4 \times 5 = 80$ elements.

The only difference between the products AB and BA is in the $(1, 2)$ element. For an Abelian group the two products need to be equal, i.e. $ae + bf = bd + ec$ (mod 5). But if, for example, $b \neq 0$, $a = c \neq 0$, $d \neq f$ and $e = 0$ then this needed equality is not satisfied. This is sufficent to show that the group is non-Abelian.

(a) The only element of order 1 is the identity. An element of order 2, i.e. $\mathsf{A}^2 = \mathsf{I}$, must have $a^2 = c^2 = 1$ and $ab + bc = 0$. The only possibilities for a and c are therefore 1 and 4.

If $a = c = 1$, then $2b = 0$ (mod 5) $\Rightarrow$ $b = 0$, $\mathsf{A} = \mathsf{I}$, already counted.
If $a = c = 4$, then $8b = 0$ (mod 5) $\Rightarrow$ $b = 0$, one such matrix.
If $a = 1$, $c = 4$, then $5b = 0$ (mod 5) $\Rightarrow$ b arbitrary, five such matrices.
If $a = 4$, $c = 1$, then $5b = 0$ (mod 5) $\Rightarrow$ b arbitrary, five such matrices.
Thus there is a total of 12 matrices of order 1 or 2 in the set.

But, by Lagrange's theorem, the order of any subgroup must divide the order of the group. As 12 does *not* divide 80, the subset cannot be a proper subgroup of $\mathcal{M}$.

(b) To find a counter-example to the closure of the set, consider the product of two matrices of the third type found in part (a):

$$\mathsf{C} = \begin{pmatrix} 1 & b \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & e + 4b \\ 0 & 1 \end{pmatrix}.$$

In view of its diagonal elements, this product can only be a member of the set if it is the identity [as shown in part(a)], thus requiring $e + 4b = 0$. So, to find a counter-example, we are led to consider a specific case, $b = 0$ and $e = 3$, which does not satisfy the requirement.

$$\mathsf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Now,

$$\mathsf{C}^2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus C has order > 2 and does not belong to the set; the set is not closed and cannot form a subgroup of $\mathcal{M}$.

> ***28.8*** *Show that, under matrix multiplication, matrices of the form*
>
> $$\mathsf{M}(a_0, \mathsf{a}) = \begin{pmatrix} a_0 + a_1 i & -a_2 + a_3 i \\ a_2 + a_3 i & a_0 - a_1 i \end{pmatrix},$$
>
> *where a_0 and the components of column matrix $\mathsf{a} = (a_1 \quad a_2 \quad a_3)^{\mathrm{T}}$ are real numbers satisfying $a_0^2 + |\mathsf{a}|^2 = 1$, constitute a group. Deduce that, under the transformation $\mathsf{z} \to \mathsf{M}\mathsf{z}$, where z is any column matrix, $|\mathsf{z}|^2$ is invariant.*

As usual, we take the associativity of matrix multiplication for granted.

An identity element is provided by $\mathsf{M}(1, \mathbf{0})$, i.e. $a_0 = 1,\ a_1 = a_2 = a_3 = 0$, which satisfy $a_0^2 + |\,\mathsf{a}\,|^2 = 1$.

We next note that if we write a typical matrix N in terms of complex numbers n_i as

$$\mathsf{N} = \begin{pmatrix} n_1 & -n_2^* \\ n_2 & n_1^* \end{pmatrix} \quad \text{with} \quad |\mathsf{N}| = (a_0^2 + a_1^2) + (a_2^2 + a_3^2) = |n_1|^2 + |n_2|^2 = 1,$$

then the product $\mathsf{P} = \mathsf{NM}$ of two such matrices is

$$\begin{aligned} \mathsf{P} &= \begin{pmatrix} n_1 & -n_2^* \\ n_2 & n_1^* \end{pmatrix} \begin{pmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{pmatrix} \\ &= \begin{pmatrix} n_1 m_1 - n_2^* m_2 & -n_1 m_2^* - n_2^* m_1^* \\ n_2 m_1 + n_1^* m_2 & -n_2 m_2^* + n_1^* m_1^* \end{pmatrix}. \end{aligned}$$

This is of the form

$$\begin{pmatrix} p_1 & -p_2^* \\ p_2 & p_1^* \end{pmatrix} \quad \text{with} \quad \begin{cases} p_1 = n_1 m_1 - n_2^* m_2 \\ p_2 = n_2 m_1 + n_1^* m_2 \end{cases}.$$

Further $|\mathsf{P}| = |\mathsf{N}|\,|\mathsf{M}| = 1 \times 1 = 1$, i.e. $p_1 p_1^* + p_2 p_2^* = 1$. Thus the set is closed.

It just remains to establish the existence of an inverse N^{-1} for each N within the set. The inverse is constructed in the normal way (recalling that $|\mathsf{N}| = 1$) as

$$\mathsf{N}^{-1} = \begin{pmatrix} n_1^* & n_2^* \\ -n_2 & n_1 \end{pmatrix}.$$

This is of the given form, with $n_1 \to n_1^*$ and $n_2 \to -n_2$. These changes correspond to $a_0 \to a_0$ and $a_i \to -a_i$ for $i = 1, 2, 3$, i.e. $\mathsf{a} \to -\mathsf{a}$.

Clearly, $|\mathsf{N}^{-1}| = |n_1|^2 + |n_2|^2$. This expression was shown to be equal to unity when we considered $|\mathsf{N}|$ earlier Thus, in summary,

$$[\mathsf{M}(a_0, \mathsf{a})]^{-1} = \mathsf{M}(a_0, -\mathsf{a}).$$

The set of matrices have now been shown to satisfy all the conditions for forming a group under matrix multiplication, and so they do so.

Under $\mathsf{z} \to \mathsf{Mz}$,

$$|\,\mathsf{z}\,|^2 = \mathsf{z}^\dagger \mathsf{z} \to \mathsf{z}^\dagger \mathsf{M}^\dagger \mathsf{M} \mathsf{z}.$$

But

$$\mathsf{M}^\dagger\mathsf{M} = \begin{pmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{pmatrix}^\dagger \begin{pmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{pmatrix}$$

$$= \begin{pmatrix} m_1^* & m_2^* \\ -m_2 & m_1 \end{pmatrix} \begin{pmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{pmatrix}$$

$$= \begin{pmatrix} |m_1|^2 + |m_2|^2 & 0 \\ 0 & |m_2|^2 + |m_1|^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathsf{I}.$$

It follows that under the transformation $\mathsf{z} \to \mathsf{Mz}$ for a general matrix z,

$$|\,\mathsf{z}\,|^2 \to \mathsf{z}^\dagger\,\mathsf{I}\,\mathsf{z} = |\,\mathsf{z}\,|^2,$$

i.e. is invariant.

28.10 *The group of rotations (excluding reflections and inversions) in three dimensions that take a cube into itself is known as the group* 432 *(or* O *in the usual chemical notation). Show by each of the following methods that this group has* 24 *elements.*

(a) *Identify the distinct relevant axes and count the number of qualifying rotations about each.*
(b) *The orientation of the cube is determined if the directions of two of its body diagonals are given. Consider the number of distinct ways in which one body diagonal can be chosen to be 'vertical', say, and a second diagonal made to lie along a particular direction.*

(a) As always, the identity (do nothing) operation is *one* of the symmetries of the cube.

About the *three* normals through the centres of opposite faces of the cube, rotations of π take the cube into itself, as do the *six* rotations of $\pm\pi/2$ about the same normals.

Rotations of π about diagonals joining the centre points of opposite edges of the cube are further symmetry operations on the cube; there are *six* of these.

Finally, there are *eight* rotations of $\pm 2\pi/3$ about the cube's body diagonals.

These bring the total up to $1 + 3 + 6 + 6 + 8 = 24$ symmetry operations.

(b) The 'vertical' diagonal can be chosen in 4×2 ways (either end of each diagonal can be 'up'). There are then three equivalent rotational positions (notionally rotations through 0 and $\pm\frac{2}{3}\pi$) about the vertical, each bringing a different body diagonal into any specified position. Thus there are $4 \times 2 \times 3 = 24$ possibilities altogether.

> ***28.12*** *If $\mathcal{A}$ and $\mathcal{B}$ are two groups then their direct product, $\mathcal{A} \times \mathcal{B}$, is defined to be the set of ordered pairs (X, Y), with X an element of $\mathcal{A}$, Y an element of $\mathcal{B}$ and multiplication given by $(X, Y)(X', Y') = (XX', YY')$. Prove that $\mathcal{A} \times \mathcal{B}$ is a group.*
>
> *Denote the cyclic group of order n by $\mathcal{C}_n$ and the symmetry group of a regular n-sided figure (an n-gon) by $\mathcal{D}_n$ – thus $\mathcal{D}_3$ is the symmetry group of an equilateral triangle, as discussed in the text.*
>
> (a) *By considering the orders of each of their elements, show* (i) *that $\mathcal{C}_2 \times \mathcal{C}_3$ is isomorphic to $\mathcal{C}_6$, and* (ii) *that $\mathcal{C}_2 \times \mathcal{D}_3$ is isomorphic to $\mathcal{D}_6$.*
> (b) *Are any of $\mathcal{D}_4$, $\mathcal{C}_8$, $\mathcal{C}_2 \times \mathcal{C}_4$, $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$ isomorphic?*

We consider the four requirements for $\mathcal{A} \times \mathcal{B}$ to be a group.

(1) Closure:

$$X \in \mathcal{A} \text{ and } X' \in \mathcal{A} \quad \Rightarrow \quad XX' \in \mathcal{A}, \text{ since } \mathcal{A} \text{ is a group.}$$

Similarly, $YY' \in \mathcal{B}$. Hence $(XX', YY') \in \mathcal{A} \times \mathcal{B}$, i.e the set is closed.

(2) Associativity holds since it does so in $\mathcal{A}$ and $\mathcal{B}$ separately.

(3) The identity:

$$I_{\mathcal{A}} \in \mathcal{A} \text{ and } I_{\mathcal{B}} \in \mathcal{B} \quad \Rightarrow \quad (I_{\mathcal{A}}, I_{\mathcal{B}}) \in \mathcal{A} \times \mathcal{B},$$

and

$$(I_{\mathcal{A}}, I_{\mathcal{B}})(X, Y) = (I_{\mathcal{A}}X, I_{\mathcal{B}}Y) = (X, Y).$$

Thus $(I_{\mathcal{A}}, I_{\mathcal{B}})$ is in $\mathcal{A} \times \mathcal{B}$ and is its identity.

(4) Inverse: If $X \in \mathcal{A}$ then $X^{-1} \in \mathcal{A}$ (since $\mathcal{A}$ is a group); similarly $Y^{-1} \in \mathcal{B}$. Thus

$$(X^{-1}, Y^{-1}) \in \mathcal{A} \times \mathcal{B}, \text{ and } (X, Y)(X^{-1}, Y^{-1}) = (XX^{-1}, YY^{-1}) = (I_{\mathcal{A}}, I_{\mathcal{B}}).$$

Thus each element of $\mathcal{A} \times \mathcal{B}$ has an inverse in the set.

These four results establish $\mathcal{A} \times \mathcal{B}$ as a group.

(a)(i) In $\mathcal{C}_6$ with generator P ($P^6 = I$), I has order 1, P and P^5 have order 6, P^2 and P^4 have order 3, whilst P^3 has order 2.

How consider $\mathcal{C}_2 \times \mathcal{C}_3$ which has $2 \times 3 = 6$ elements (the same as $\mathcal{C}_6$). With $X \in \mathcal{C}_2$ and $Y \in \mathcal{C}_3$, the elements (X, Y) and their orders are [X and Y are not the identity except where explicitly stated]:

$$\begin{aligned}(I, I) &\text{ has order 1 } \ 1 \text{ element};\\ (I, Y) &\text{ has order 3 } \ 2 \text{ elements};\\ (X, I) &\text{ has order 2 } \ 1 \text{ element};\\ (X, Y) &\text{ has order 6 } \ 1 \times 2 = 2 \text{ elements}.\end{aligned}$$

Thus $\mathcal{C}_6$ and $\mathcal{C}_2 \times \mathcal{C}_3$ both have six elements and have the same numbers of elements of any particular order. Therefore they are isomorphic.

(ii) Consider first the set of symmetry operations on a regular hexagon (the group $\mathcal{D}_6$). In addition to the identity, it includes five rotations of $2\pi k/6$ ($k = 1, 2, \ldots, 5$) about an axis through the hexagon's centre and perpendicular to its plane. The two with $k = 1, 5$ have order 6, the two with $k = 2, 4$ have order 3, whilst that with $k = 3$ has order 2.

Further there are 3 reflection symmetries with respect to diagonals joining opposite corners of the hexagon, and 3 more with respect to diagonals joining the centres of opposite sides. Clearly, all of the reflection symmetries have order 2. In summary, $\mathcal{D}_6$ has twelve elements: one of order 1, seven of order 2, and two each of orders 3 and 6.

As shown in the text, the group $\mathcal{D}_3$ has six elements: one of order 1, two (rotations) of order 3, and three (reflections) of order 2. With the same notation as in part (a), the elements (X, Y) of $\mathcal{C}_2 \times \mathcal{D}_3$ and their orders are:

$$\begin{aligned}(I, I) &\text{ has order 1 } \ 1 \text{ element};\\ (I, Y_{\text{rot}}) &\text{ has order 3 } \ 2 \text{ elements};\\ (I, Y_{\text{ref}}) &\text{ has order 2 } \ 3 \text{ elements};\\ (X, I) &\text{ has order 2 } \ 1 \text{ element};\\ (X, Y_{\text{rot}}) &\text{ has order 6 } \ 1 \times 2 = 2 \text{ elements};\\ (X, Y_{\text{ref}}) &\text{ has order 2 } \ 1 \times 3 = 3 \text{ elements}.\end{aligned}$$

Again, in summary, $\mathcal{C}_2 \times \mathcal{D}_3$ has twelve elements: one of order 1, seven of order 2, and two each of orders 3 and 6. Thus the groups $\mathcal{D}_6$ and $\mathcal{C}_2 \times \mathcal{D}_3$ are isomorphic.

(b) The groups $\mathcal{D}_4$, $\mathcal{C}_8$, $\mathcal{C}_2 \times \mathcal{C}_4$, $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$ each have eight elements [see the previous exercise (28.11) in the case of $\mathcal{D}_4$]. Thus each has the potential to be isomorphic to any other. The same exercise showed that $\mathcal{D}_4$ has five elements of order 2 and two of order 4, as well as the identity of order 1.

However, each cyclic group $\mathcal{C}_n$ must contain at least one element of order n and no element in any product cyclic group can have an order that is greater than the LCM of the orders of the individual groups. Thus $\mathcal{C}_8$ has an element of order 8, $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$ has no element of order greater than 2, and $\mathcal{C}_2 \times \mathcal{C}_4$ has an element of order 4, but no greater. These results, together with the above observations about $\mathcal{D}_4$ mean that only $\mathcal{C}_2 \times \mathcal{C}_4$ and $\mathcal{D}_4$ could *possibly* be isomorphic. These have to be examined further.

$\mathcal{C}_4$ has one element of order 1, one of order 2 and two of order 4. (Elements of order 3 are not possible since 3 does not divide 4 exactly.) Forming the elements of $\mathcal{C}_2 \times \mathcal{C}_4$, we obtain:

$$\begin{aligned}
(I, I) &\text{ has order 1 } \ 1 \text{ element;}\\
(I, Y_2) &\text{ has order 2 } \ 1 \text{ element;}\\
(I, Y_4) &\text{ has order 4 } \ 2 \text{ elements;}\\
(X, I) &\text{ has order 2 } \ 1 \text{ element;}\\
(X, Y_2) &\text{ has order 2 } \ 1 \text{ element;}\\
(X, Y_4) &\text{ has order 4 } \ 2 \text{ elements.}
\end{aligned}$$

In summary, $\mathcal{C}_2 \times \mathcal{C}_4$ has one element of order 1, three of order 2 and four of order 4. This is not the same distribution as for $\mathcal{D}_4$ and so the two groups *cannot* be isomorphic.

28.14 *Show that if p is prime then the set of rational number pairs (a, b), excluding $(0, 0)$, with multiplication defined by*

$$(a, b) \bullet (c, d) = (e, f), \quad \text{where} \quad (a + b\sqrt{p})(c + d\sqrt{p}) = e + f\sqrt{p},$$

forms an Abelian group. Show further that the mapping $(a, b) \rightarrow (a, -b)$ is an automorphism.

From the given combination law

$$e = ac + bdp \qquad \text{and} \qquad f = ad + bc.$$

As a, b, c and d are all rational numbers, so are e and f. The set of rational number pairs is therefore closed. Associativity and commutativity are obvious and the number pair $I = (1, 0)$ clearly has the property that $IX = X$ for any rational number pair X. The existence of inverses is the only requirement remaining to be established in order to complete the proof that the set forms an Abelian group under the given combination law.

Now,

$$(c,d)\bullet(a,b)=(1,0) \quad \Rightarrow \quad c+d\sqrt{p}=\frac{1}{a+b\sqrt{p}}=\frac{a-b\sqrt{p}}{a^2-b^2p}.$$

Further, since a and b are rational (and $\sqrt{p}$ is not), $a^2 \neq b^2p$. Although it is not zero, a^2-b^2p is rational, and so, therefore are c and d. To summarise,

$$(a,b)^{-1}=(c,d)=\left(\frac{a}{a^2-b^2p},\frac{-b}{a^2-b^2p}\right),$$

a rational number pair that is included in the set. The proof is now complete.

For the mapping $(a,b)\to(a,-b)$

$$[\,(a,b)\bullet(c,d)\,]'=(ac+bdp,ad+bc)'=(ac+bdp,-ad-bc),$$

whilst

$$(a,b)'\bullet(c,d)'=(a,-b)\bullet(c,-d)=(ac+bdp,-bc-ad).$$

Hence

$$[\,(a,b)\bullet(c,d)\,]'=(a,b)'\bullet(c,d)',$$

showing that the mapping is a homomorphism. The mapping is clearly one-to-one, making it an isomorphism and, finally, since the object and image sets are the same, it is an automorphism.

28.16 *For the group $\mathcal{G}$ with multiplication table 28.8 and proper subgroup $\mathcal{H}=\{I,A,B\}$, denote the coset $\{I,A,B\}$ by $\mathcal{C}_1$ and the coset $\{C,D,E\}$ by $\mathcal{C}_2$. Form the set of all possible products of a member of $\mathcal{C}_1$ with itself, and denote this by $\mathcal{C}_1\mathcal{C}_1$. Similarly compute $\mathcal{C}_2\mathcal{C}_2$, $\mathcal{C}_1\mathcal{C}_2$ and $\mathcal{C}_2\mathcal{C}_1$. Show that each product coset is equal to $\mathcal{C}_1$ or to $\mathcal{C}_2$ and that a 2×2 multiplication table can be formed, demonstrating that $\mathcal{C}_1$ and $\mathcal{C}_2$ are themselves the elements of a group of order 2. A subgroup like $\mathcal{H}$ whose cosets themselves form a group is a* normal subgroup.

The multiplication table is

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

As can be seen, dividing the six elements into the two cosets based on the subgroup $\{I, A, B\}$ corresponds to dividing the table into four quarters, as shown. The first three rows and the first three columns are labelled by the elements of $\mathcal{C}_1$ whilst the last three rows and columns are labelled by the elements of $\mathcal{C}_2$.

It is clear from inspection that the top LH quarter gives the set of products denoted in the question by $\mathcal{C}_1\mathcal{C}_1$; it is equally clear that it contains only elements that are in $\mathcal{C}_1$, thus $\mathcal{C}_1 \times \mathcal{C}_1 = \mathcal{C}_1$. Similarly, in the top RH quarter, $\mathcal{C}_1 \times \mathcal{C}_2 = \mathcal{C}_2$. In the same way, $\mathcal{C}_2 \times \mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{C}_2 \times \mathcal{C}_2 = \mathcal{C}_1$.

Thus we can also draw up a multiplication table for the cosets themselves

	$\mathcal{C}_1$	$\mathcal{C}_2$
$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_2$
$\mathcal{C}_2$	$\mathcal{C}_2$	$\mathcal{C}_1$

This is the multiplication table of a group of order 2 in which $\mathcal{C}_1$ is the unit element.

28.18 *The group of reflection–rotation symmetries of a square is known as* $\mathcal{D}_4$*; let* X *be one of its elements. Consider a mapping* $\Phi : \mathcal{D}_4 \to S_4$*, the permutation group on four objects, defined by* $\Phi(X) =$ *the permutation induced by* X *on the set* $\{x, y, d, d'\}$*, where* x *and* y *are the two principal axes, and* d *and* d' *the two principal diagonals, of the square. For example, if* R *is a rotation by* $\pi/2$*,* $\Phi(R) =$ (12)(34)*. Show that* $\mathcal{D}_4$ *is mapped onto a subgroup of* S_4 *and, by constructing the multiplication tables for* $\mathcal{D}_4$ *and the subgroup, prove that the mapping is a homomorphism.*

This group of symmetries and its multiplication table were considered and derived in exercise 28.11; we will use the same notation here.

Clearly, the set of principal axes and diagonals is left unchanged by both I and R^2 and so these two symmetries are mapped onto the permutation (1). As explained in the question, R interchanges the two principal axes and also interchanges the two principal diagonals, i.e. it causes the change $\{x, y, d, d'\} \to \{y, x, d', d\}$; it therefore maps onto (12)(34). The same is true of R^3. The reflections m_1 and m_2 about the principal axes leave those axes unchanged but interchange d and d'; they therefore both map onto the permutation (34). Similarly m_3 and m_4 are both mapped onto (12).

In summary, $\mathcal{D}_4$ is mapped by a two-to-one mapping onto the subgroup

$$\{(1), (12)(34), (34), (12)\}$$

of S_4. The entries in the multiplication table for this subgroup,

	(1)	(12)(34)	(34)	(12)
(1)	(1)	(12)(34)	(34)	(12)
(12)(34)	(12)(34)	(1)	(12)	(34)
(34)	(34)	(12)	(1)	(12)(34)
(12)	(12)	(34)	(12)(34)	(1)

are straightforward and can be verified by inspection; for example, the product [(12)(34)][(34)] interchanges the 3rd and 4th objects and then changes them back again whilst at the same time interchanging the 1st and 2nd. The net result is that the 1st and 2nd objects are interchanged and the 3rd and 4th are untouched, i.e. equivalent to (12).

This 4×4 table and the 4×4 table produced by combining the elements of $\mathcal{D}_4$ in appropriate pairs, according to their common effects on the axes and diagonals of the square, clearly have the common structure

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

Here (1), I and R^2 are replaced by I; (12)(34), R and R^3 are replaced by A; (34), m_1 and m_2 are replaced by B; and (12), m_3 and m_4 are replaced by C. This shows that the mapping Φ is a homomorphism. It is not, however, an isomorphism as the mapping is not one-to-one; it is, in fact, an epimorphism.

28.20 *In the quaternion group $\mathcal{Q}$ the elements form the set*

$$\{1, -1, i, -i, j, -j, k, -k\},$$

with $i^2 = j^2 = k^2 = -1$, $ij = k$ and its cyclic permutations, and $ji = -k$ and its cyclic permutations. Find the proper subgroups of $\mathcal{Q}$ and the corresponding cosets. Show that all of the subgroups are normal subgroups. Show further that $\mathcal{Q}$ cannot be isomorphic to the group 4mm (C_{4v}) considered in exercise 28.11.

In order to establish the subgroups of the quaternion group we draw up its multiplication table which reads as follows

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

From this table it can be seen that the proper subgroups are

$$\{1, -1\}, \quad \{1, -1, i, -i\}, \quad \{1, -1, j, -j\}, \quad \{1, -1, k, -k\}.$$

The cosets of the subgroup $\{1, -1\}$, which has order 2, are immediately found to be

$$\mathcal{C}_1 = \{1, -1\}, \quad \mathcal{C}_i = \{i, -i\}, \quad \mathcal{C}_j = \{j, -j\}, \quad \mathcal{C}_k = \{k, -k\}.$$

The multiplication table also shows that the product of two elements drawn one each from $\mathcal{C}_1$ and $\mathcal{C}_n$ always belongs to $\mathcal{C}_n$, i.e. $\mathcal{C}_1 \times \mathcal{C}_n = \mathcal{C}_n$ for $n = 1, i, j, k$. Similarly $\mathcal{C}_n \times \ \mathcal{C}_n = \mathcal{C}_1$. It is only slightly more complicated to see that $\mathcal{C}_i \times \mathcal{C}_j = \mathcal{C}_k$, and corresponding results obtained by interchanging any pair of subscripts, are valid. The cosets themselves obey a group multiplication table of the form

	$\mathcal{C}_1$	$\mathcal{C}_i$	$\mathcal{C}_j$	$\mathcal{C}_k$
$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_i$	$\mathcal{C}_j$	$\mathcal{C}_k$
$\mathcal{C}_i$	$\mathcal{C}_i$	$\mathcal{C}_1$	$\mathcal{C}_k$	$\mathcal{C}_j$
$\mathcal{C}_j$	$\mathcal{C}_j$	$\mathcal{C}_k$	$\mathcal{C}_1$	$\mathcal{C}_i$
$\mathcal{C}_k$	$\mathcal{C}_k$	$\mathcal{C}_j$	$\mathcal{C}_i$	$\mathcal{C}_1$

With $\mathcal{C}_1$ as the identity, this is a group under coset multiplication and establishes $\{1, -1\}$ as a normal subgroup of $\mathcal{Q}$. We note that, unlike $\mathcal{Q}$ itself, the group of cosets is Abelian.

We now consider the three subgroups of order 4 and take $\{1, -1, i, -i\}$ as typical. Its cosets are $\mathcal{D}_i = \{1, -1, i, -i\}$ and $\mathcal{D}_i' = \{j, -j, k, -k\}$; these two cosets exhaust the group.

If we select two elements, one from each coset, and multiply them together we can only obtain one of the four quantities $\pm j$ and $\pm k$ (recall, that, for example, $(-i)(k) = j$). In other words, $\mathcal{D}_i \times \mathcal{D}_i' = \mathcal{D}_i'$. Similar, but even simpler, considerations show that $\mathcal{D}_i \times \mathcal{D}_i = \mathcal{D}_i$ and that $\mathcal{D}_i' \times \mathcal{D}_i' = \mathcal{D}_i$. Thus $\mathcal{D}_i$ and $\mathcal{D}_i'$ form a group of order 2 under coset multiplication, with $\mathcal{D}_i$ as its identity. This shows that the subgroup $\{1, -1, i, -i\}$ is a normal subgroup; corresponding considerations and conclusions apply to the subgroups $\{1, -1, j, -j\}$ and $\{1, -1, k, -k\}$.

Finally, as shown in exercise 28.11, the group $4mm$ has five elements of order 2 (rotation by π and the four reflection symmetries). The quaternion group $\mathcal{Q}$ considered here has only one (-1); this rules out any possibility of isomorphism between the two groups.

28.22 *Show that the matrices*

$$\mathsf{M}(\theta, x, y) = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $0 \le \theta < 2\pi$, $-\infty < x < \infty$, $-\infty < y < \infty$, *form a group under matrix multiplication.*

Show that those $\mathsf{M}(\theta, x, y)$ *for which* $\theta = 0$ *form a subgroup and identify its cosets. Show that the cosets themselves form a group.*

We start by noting that matrix multiplication is associative.

Next consider

$$\begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & x' \\ \sin\phi & \cos\phi & y' \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & X \\ \sin(\theta+\phi) & \cos(\theta+\phi) & Y \\ 0 & 0 & 1 \end{pmatrix},$$

where $X = x'\cos\theta - y'\sin\theta + x$ and $Y = x'\sin\theta + y'\cos\theta + y$. Written in terms of matrices of the given form, this reads

$$\mathsf{M}(\theta, x, y)\,\mathsf{M}(\phi, x', y') = \mathsf{M}(\theta + \phi, X, Y),$$

showing that the set is closed.

Clearly, the unit matrix $\mathsf{M}(0, 0, 0) = \mathsf{I}_3$ acts as an identity element.

To find an inverse for $\mathsf{M}(\theta, x, y)$, set $\phi = -\theta$ and $X = Y = 0$ in the above display equation. This requires that

$$\begin{aligned} & 0 = x' \cos\theta - y' \sin\theta + x, \\ & 0 = x' \sin\theta + y' \cos\theta + y, \\ \Rightarrow \quad & 0 = x'(\cos^2\theta + \sin^2\theta) + x\cos\theta + y\sin\theta, \\ \Rightarrow \quad & x' = -x\cos\theta - y\sin\theta, \\ \text{and} \quad & 0 = -y'(\cos^2\theta + \sin^2\theta) + x\sin\theta - y\cos\theta, \\ \Rightarrow \quad & y' = x\sin\theta - y\cos\theta. \end{aligned}$$

Thus $\mathsf{M}(\theta, x, y)^{-1} = \mathsf{M}(-\theta, -x\cos\theta - y\sin\theta, x\sin\theta - y\cos\theta)$ and establishes the existence of an inverse within the set.
This completes the proof that the set forms a group under matrix multiplication.

For the subset $\mathsf{M}(0, x, y)$, associativity is as before, the identity, $\mathsf{M}(0, 0, 0)$, is included, $\mathsf{M}(0, x, y)^{-1} = \mathsf{M}(0, -x, -y)$ is of the correct form and closure is shown by $\mathsf{M}(0, x, y)\mathsf{M}(0, x', y') = \mathsf{M}(0, x+x', y+y')$. All four group conditions are satisfied and the subset forms a subgroup $\mathcal{N}$ of the group $\{\mathsf{M}(\theta, x, y)\}$.

Since $\mathsf{M}(0, x, y)\mathsf{M}(\theta, x', y') = \mathsf{M}(\theta, x + x', y + y')$, the cosets of $\mathcal{N}$ are

$$\mathcal{C}_\theta = \{\mathsf{M}(\theta, x, y), \text{ for all } -\infty < x, y < \infty\},$$

i.e. all members of any coset have the same value for θ.

If $\mathsf{M}(\theta_1, x_1, y_1)$ is any member of $\mathcal{C}_{\theta_1}$ and $\mathsf{M}(\theta_2, x_2, y_2)$ any member of $\mathcal{C}_{\theta_2}$, then

$$\begin{aligned} & \mathsf{M}(\theta_1, x_1, y_1)\,\mathsf{M}(\theta_2, x_2, y_2) \\ & = \mathsf{M}(\theta_1 + \theta_2, x_1 + x_2\cos\theta_1 - y_2\sin\theta_1, y_1 + y_2\cos\theta_1 + x_2\sin\theta_1) \end{aligned}$$

belongs to $\mathcal{C}_{\theta_1+\theta_2}$. In terms of coset multiplication, $\mathcal{C}_{\theta_1} \times \mathcal{C}_{\theta_2} = \mathcal{C}_{\theta_1+\theta_2}$; the product coset is contained in the set. The identity is $\mathcal{C}_0$ (= $\mathcal{N}$ itself) and $\mathcal{C}_\theta^{-1} = \mathcal{C}_{2\pi-\theta}$ is also in the set of cosets. The cosets therefore form a group.

29

Representation theory

29.2 *Using a square whose corners lie at coordinates* $(\pm 1, \pm 1)$*, form a natural representation of the dihedral group* $\mathcal{D}_4$*. Find the characters of the representation, and, using the information (and class order) in table 29.4 (p. 1102), express the representation in terms of irreps.*

Now form a representation in terms of eight 2×2 *orthogonal matrices, by considering the effect of each of the elements of* $\mathcal{D}_4$ *on a general vector* (x, y)*. Confirm that this representation is one of the irreps found using the natural representation.*

As in figure 29.1, we mark the corners of the square as 1, 2, 3 and 4 and describe the actions of the various symmetry operations by describing to which of the four fixed points $A = (1, 1)$, $B = (1, -1)$, $C = (-1, -1)$ and $D = (-1, 1)$ each of

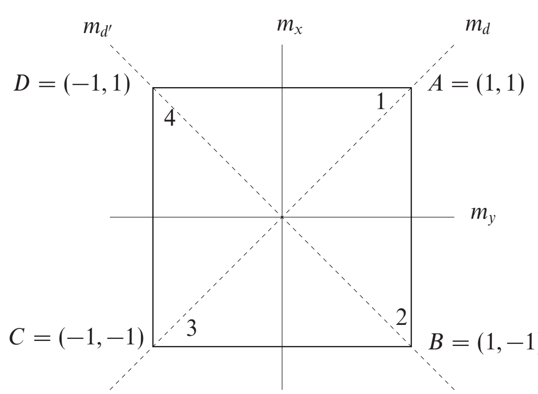

Figure 29.1 The coordinate system and notation used in exercise 29.2.

the corners is carried. For example R, a clockwise rotation by $\pi/2$, carries corner 1 from A to B, corner 2 from B to C, etc. The corresponding matrix is

$$\mathsf{D}(R) = \begin{pmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{pmatrix}$$

This matrix is traceless, as will be those of the matrices corresponding to R^2 and R^3; thus the character for each of these is 0. The identity matrix is I_4 which has a trace, and hence a character, of 4.

The four matrices in the group corresponding to reflections are

$$\mathsf{D}(m_x) = \begin{pmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix} \quad \mathsf{D}(m_y) = \begin{pmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}$$

$$\mathsf{D}(m_d) = \begin{pmatrix} 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \end{pmatrix} \quad \mathsf{D}(m_{d'}) = \begin{pmatrix} . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \end{pmatrix}$$

The first two (in the same class) have character 0, whilst the class consisting of m_d and $m_{d'}$ has character 2.

In summary the representation has characters $(4, 0, 0, 0, 2)$, the five classes being given in the same order as in table 29.4. Application of the standard formula (29.18), or direct inspection, shows that the only combination of irreps that gives the correct character totals for all classes is

$$A_1 \oplus B_2 \oplus E.$$

The relevant character sum is

$$(4, 0, 0, 0, 2) = (1, 1, 1, 1, 1) + (1, 1, -1, -1, 1) + (2, -2, 0, 0, 0).$$

If the combination is calculated from the formula, the expansion coefficients for A_2 and B_1 are found to be zero; for each of the other three irreps they are unity.

As a second representation we consider what happens to a vector (x, y) under each of the symmetries contained in $\mathcal{D}_4$. For example, the rotation R, considered previously, takes (x, y) into $(y, -x)$ and so is represented by

$$\mathsf{D}(R) = \begin{pmatrix} . & 1 \\ -1 & . \end{pmatrix}.$$

The full set of symmetries and corresponding matrices is

$$\begin{array}{cccc} I & R & R^2 & R^3 \\ \begin{pmatrix} 1 & . \\ . & 1 \end{pmatrix}, & \begin{pmatrix} . & 1 \\ -1 & . \end{pmatrix}, & \begin{pmatrix} -1 & . \\ . & -1 \end{pmatrix}, & \begin{pmatrix} . & -1 \\ 1 & . \end{pmatrix}, \\ m_x & m_y & m_d & m_{d'} \\ \begin{pmatrix} -1 & . \\ . & 1 \end{pmatrix}, & \begin{pmatrix} 1 & . \\ . & -1 \end{pmatrix}, & \begin{pmatrix} . & 1 \\ 1 & . \end{pmatrix}, & \begin{pmatrix} . & -1 \\ -1 & . \end{pmatrix}. \end{array}$$

As expected (and required), symmetries in the same class have matrices with equal traces (i.e. have equal characters). The characters [in the same order $(I, R^2, R/R^3, m_x/m_y, m_d/m_{d'})$ as earlier] are $(2, -2, 0, 0, 0)$. This is exactly the character set for irrep E, found earlier in the natural representation. This second representation is two-dimensional but irreducible.

29.4 *Construct the character table for the irreps of the permutation group S_4 as follows.*

(a) *By considering the possible forms of its cycle notation, determine the number of elements in each conjugacy class of the permutation group S_4 and show that S_4 has five irreps. Give the logical reasoning that shows they must consist of two three-dimensional, one two-dimensional, and two one-dimensional irreps.*
(b) *By considering the odd and even permutations in the group S_4 establish the characters for one of the one-dimensional irreps.*
(c) *Form a natural matrix representation of 4×4 matrices based on a set of objects $\{a, b, c, d\}$, which may or may not be equal to each other, and, by selecting one example from each conjugacy class, show that this natural representation has characters* 4, 2, 1, 0, 0. *In the four-dimensional vector space in which each of the four coordinates takes on one of the four values a, b, c or d, the one-dimensional subspace consisting of the four points with coordinates of the form $\{a, a, a, a\}$ is invariant under the permutation group and hence transforms according to the invariant irrep* A_1. *The remaining three-dimensional subspace is irreducible; use this and the characters deduced above to establish the characters for one of the three-dimensional irreps,* T_1.
(d) *Complete the character table using orthogonality properties, and check the summation rule for each irrep. You should obtain table 29.1.*

(a) The group S_4 has $4! = 24$ elements; its possible cyclical forms and their

Irrep	Typical element and class size				
	(1)	(12)	(123)	(1234)	(12)(34)
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	−1	1	−1	1
E	2	0	−1	0	2
T_1	3	1	0	−1	−1
T_2	3	−1	0	1	−1

Table 29.1 The character table for the permutation group S_4.

corresponding orders are

	structure	number	order
(i)	(1)(2)(3)(4)	${}^4C_0 = 1$	1
(ii)	(12)(3)(4)	${}^4C_2 = 6$	2
(iii)	(123)(4)	${}^4C_1\,{}^2C_1 = 8$	3
(iv)	(1234)	${}^3C_1\,{}^2C_1 = 6$	4
(v)	(12)(34)	${}^3C_1 = 3$	2

Now, as shown in the solution to exercise 28.9, elements in the same class must have the same order. This implies that here there are at least four classes. In fact, there are five.

To see this, we note that a permutation structure $(wx)(y)(z)$, involving only one pair interchange, is an odd permutation, whilst one with structure $(pq)(rs)$, which involves two interchanges, is an even one. Thus, since *any* permutation P has the same parity (odd or even) as its inverse P^{-1}, a relationship of the form

$$[\,(pq)(rs)\,] = P^{-1}[\,(wx)(y)(z)\,]P$$

would have even parity on the LHS and odd parity on the RHS; this is not possible and no such relationship can exist. Hence $(wx)(y)(z)$ and $(pq)(rs)$ belong to different classes, implying that there are five classes, and hence five irreps, in all.

Let the five irreps have dimensions n_i, $(i = 1, 2, \ldots, 5)$, with $n_1 = 1$ as the dimension of the identity irrep (which must be present). Then

$$\sum_{i=2}^{5} n_i^2 = 24 - 1 = 23.$$

Since $4 \times 2^2 = 16 < 23$, at least one n_i must be ≥ 3 (and clearly < 5). If one of the n_i were 4, we would require $\sum_{i=2}^{4} n_i^2 = 7$ which has no integral solutions with all $n_i \geq 1$. Thus, one n_i, say n_5, must be equal to 3, leaving $\sum_{i=2}^{4} n_i^2 = 14$. Now, since $3(2^2) < 14 <$ a second n_i, n_4 say, must equal 3. This now leaves $\sum_{i=2}^{3} n_i^2 = 5$, which has only one possible integral solution n_3(say)$= 2$ and $n_2 = 1$.

(b) We have taken the first irrep as A_1. A second one-dimensional irrep, $\mathsf{D}^{(2)}$, will be one in which the odd permutations in S_4 [cycle structures (ii) and (iv)] have -1 as their 1×1 matrix representation whilst the even permutations [structures (i), (iii) and (v)] are represented by $+1$. Then, in the order of classes used above, the character set for $\mathsf{D}^{(2)}$ is $(1, -1, 1, -1, 1)$. This irrep is normally denoted by A_2.

(c) We now consider the action of a typical element of each class (i)-(iv) on the set $\{a, b, c, d\}$ and determine in each case how many of the set are unchanged; this gives the trace of the corresponding (natural) permutation matrix and hence its character. It is clear that this is equal to the number of cycles of length one in the corresponding cycle notation. The character set is therefore $(4, 2, 1, 0, 0)$.

As noted in the question, the vector space on which the permutations act contains one invariant subspace and this must transform as the identity irrep A_1 with character set $(1, 1, 1, 1, 1)$. We are told that the subspace remaining when this invariant subspace is removed is irreducible and must therefore transform according to a three-dimensional irrep T_1. The character set for this space must be $(4-1,\ 2-1,\ 1-1,\ 0-1,\ 0-1) = (3, 1, 0, -1, -1)$.

At this point we have a partial character table

Irrep	Typical element and class size				
	(1)	(12)	(123)	(1234)	(12)(34)
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	-1	1	-1	1
E	2	a	b	c	d
T_1	3	1	0	-1	-1
T_2	3	w	x	y	z

Consider next the characters of E. From the summation rule,

$$4 + 6a^2 + 8b^2 + 6c^2 + 3d^2 = 24.$$

Since all the terms except $3d^2$ are necessarily even, $3d^2$ must be as well. Thus $d = 0$ or $d = \pm 2$.

If $d = 0$, then orthogonality of the character set with those of A_1 and A_2 requires

$$2 + 6a + 8b + 6c = 0,$$
$$2 - 6a + 8b - 6c = 0,$$
$$\Rightarrow \quad 4 + 16b = 0. \quad \text{Impossible, as } b \text{ must be integral.}$$

So $d = \pm 2$, and, from the summation rule,

$$4 + 6a^2 + 8b^2 + 6c^2 + 12 = 24 \quad \Rightarrow \quad 6a^2 + 8b^2 + 6c^2 = 8,$$

for which the only integral solution is $a = c = 0$ with $b = +1$ or -1. From orthogonality with A_1

$$2 + 0 + 8b + 0 + 3d = 0,$$

which has no integral solution if $d = -2$, but gives $b = -1$ if $d = +2$. This is the only acceptable solution and, in summary, the characters for E are $(2, 0, -1, 0, 2)$.

Finally we turn to the character set for T_2 for which we have

$$9 + 6w^2 + 8x^2 + 6y^2 + 3z^2 = 24.$$

Since $24 - 9$ is odd and $6w^2 + 8x^2 + 6y^2$ is even, z must be odd with $|z| < \sqrt{15/3}$, i.e. $z = +1$ or -1. Hence $6w^2 + 8x^2 + 6y^2 = 12$, which can only have an integral solution if $x = 0$ and $|w| = |y| = 1$. Now, orthogonality with the characters of A_1 gives $3 + 6w + 6y + 3z = 0$, and with those of A_2 gives $3 - 6w - 6y + 3z = 0$. Hence $z = -1$ and $w = -y$. The remaining ambiguity is resolved using the orthogonality with $\chi^{(T_1)}$:

$$9 + 6w - 6y + 3 = 0 \quad \Rightarrow \quad 12w + 12 = 0 \quad \Rightarrow \quad w = -1 \text{ and } y = 1.$$

Hence $\chi^{(T_2)} = (3, -1, 0, 1, -1)$ and the table is complete.

The summation rule was used to establish $\chi^{(E)}$ and $\chi^{(T_2)}$ and is easily verified for A_2 and T_1.

29.6 *Consider a regular hexagon orientated so that two of its vertices lie on the x-axis. Find matrix representations of a rotation R through $2\pi/6$ and a reflection m_y in the y-axis by determining their effects on vectors lying in the xy-plane . Show that a reflection m_x in the x-axis can be written as $m_x = m_y R^3$ and that the 12 elements of the symmetry group of the hexagon are given by R^n or $R^n m_y$.*

Using the representations of R and m_y as generators, find a two-dimensional representation of the symmetry group, C_6, of the regular hexagon. Is it a faithful representation?

Under the rotation R,

$$(x, y) \to \left(x \cos\tfrac{\pi}{3} - y \sin\tfrac{\pi}{3}, y \cos\tfrac{\pi}{3} + x \sin\tfrac{\pi}{3}\right) = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Under the reflection m_y

$$(x, y) \to (-x, y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

whilst under the reflection m_x

$$(x, y) \to (x, -y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We now consider $m_y R^3$ whose matrix representation is

$$\begin{aligned} &\tfrac{1}{8} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \\ &= \tfrac{1}{8} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} \\ &= \tfrac{1}{8} \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

i.e. equal to that for m_x, which can also be written $R^3 m_y$ since R^3 and m_y commute.

Denoting the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

by $\mathsf{M}(\theta)$, we see that the representation of R is $\mathsf{M}(\pi/3)$. Now,

$$\begin{aligned} \mathsf{M}(\theta)\mathsf{M}(\phi) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \\ &= \mathsf{M}(\theta+\phi). \end{aligned}$$

It follows that R^n is represented by $\mathsf{M}(n\pi/3)$ and that a rotation by $n\pi/3$ is generated by R^n. This accounts for all the rotational symmetries of the hexagon including, formally, the identity I for which $n = 0$.

The reflection symmetries are of two kinds; three are in axes joining opposite vertices of the hexagon and are exemplified by m_x; three are in axes joining the mid-points of opposite sides as in the case of m_y. In each case, the other two reflections can be obtained by applying either R^2 or R^4 after the reflections.

Figure 29.2 summarises the situation. The label against each of the 12 dots (e.g. $R^5 m_y$) shows the effect on the original ringed point, marked (x, y), of the 12 corresponding symmetry operations. In cases including a reflection, the effective reflection axis for the whole operation is marked with the same symbol. Thus, reflection in the axis (marked $R^5 m_y$) through the vertices in the first and third quadrants carries (x, y) to the point marked $R^5 m_y$. Each of the 12 operations can be expressed either as R^m or as $R^n m_y$.

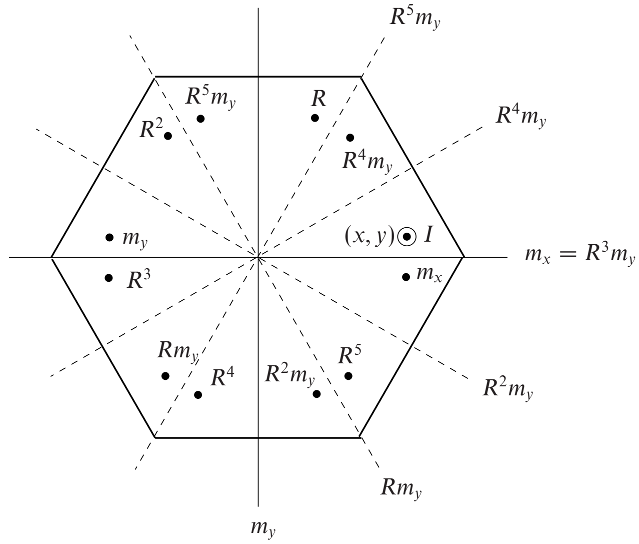

Figure 29.2 The rotation and reflection symmetries of a regular hexagon as discussed in exercise 29.6. A point labelled by the name of any particular symmetry shows the effect of that symmetry on the ringed point (x, y). The axis labelled $R^n m_y$ is the reflection axis corresponding to the point marked $R^n m_y$.

To ease the calculation of the representation, we note that $R^3 = -I$ and obtain

$$R = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = -R^4,$$

$$R^2 = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = -R^5,$$

$$Rm_y = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = -R^4 m_y,$$

$$R^2 m_y = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = -R^5 m_y.$$

To these eight we must add

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$m_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R^3 m_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The twelve matrices are all distinct and so the representation is faithful.

29.8 *Sulphur hexafluoride is a molecule with the same structure as the crystalline compound in exercise 29.7, except that a sulphur atom is now the central atom. The following are the forms of some of the electronic orbitals of the sulphur atom, together with the irreps according to which they transform under the symmetry group* 432 *(or* O*).*

$$\begin{array}{ll} \Psi_s = f(r) & \mathrm{A}_1 \\ \Psi_{p_1} = zf(r) & \mathrm{T}_1 \\ \Psi_{d_1} = (3z^2 - r^2)f(r) & \mathrm{E} \\ \Psi_{d_2} = (x^2 - y^2)f(r) & \mathrm{E} \\ \Psi_{d_3} = xyf(r) & \mathrm{T}_2 \end{array}$$

The function x transforms according to the irrep T_1*. Use the above data to determine whether dipole matrix elements of the form $J = \int \phi_1 x \phi_2 \, d\tau$ can be non-zero for the following pairs of orbitals ϕ_1, ϕ_2 in a sulphur hexafluoride molecule:* (a) Ψ_{d_1}, Ψ_s; (b) Ψ_{d_1}, Ψ_{p_1}; (c) Ψ_{d_2}, Ψ_{d_1}; (d) Ψ_s, Ψ_{d_3}; (e) Ψ_{p_1}, Ψ_s.

For each dipole matrix element we need to determine whether the irrep A_1 is present in the reduction of the representation of the triple product of ϕ_1, ϕ_2 and x; only if it is, can the dipole matrix element be non-zero. To do this we require the character table for the group 432 produced in exercise 29.5, namely

Irrep	Typical element and class size				
	I	2_d	3	4_z	2_z
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	−1	1	−1	1
E	2	0	−1	0	2
T_1	3	1	0	−1	−1
T_2	3	−1	0	1	−1

We also need to use the formula

$$m_\mu = \frac{1}{g} \sum_X \left[\chi^{(\mu)}(X)\right]^* \chi(X) = \frac{1}{g} \sum_i c_i \left[\chi^{(\mu)}(X_i)\right]^* \chi(X_i)$$

with μ set to A_1 [for which all $\chi^{(\mu)}(X) = 1$] in order to calculate whether or not A_1 is present.

In each case we obtain the character set for the dipole matrix element by multiplying together, for each class, the corresponding characters of the two orbitals and x. The resulting character set then has to be tested using the above formula to see whether A_1 is present.

(a) Ψ_{d_1}, Ψ_s:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_{d_1}	E	2	0	−1	0	2
Ψ_s	A_1	1	1	1	1	1
x	T_1	3	1	0	−1	−1
		6	0	0	0	−2

Thus, $m_{A_1} = \dfrac{1(6) + 3(-2)}{24} = 0 \quad \Rightarrow \quad$ No.

(b) Ψ_{d_1}, Ψ_{p_1}:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_{d_1}	E	2	0	−1	0	2
Ψ_{p_1}	T_1	3	1	0	−1	−1
x	T_1	3	1	0	−1	−1
		18	0	0	0	2

Thus, $m_{A_1} = \dfrac{1(18) + 3(2)}{24} = 1 \quad \Rightarrow \quad$ Yes.

(c) Ψ_{d_2}, Ψ_{d_1}:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_{d_1}	E	2	0	−1	0	2
Ψ_{d_2}	E	2	0	−1	0	2
x	T_1	3	1	0	−1	−1
		12	0	0	0	−4

Thus, $m_{A_1} = \dfrac{1(12) + 3(-4)}{24} = 0 \quad \Rightarrow \quad$ No.

(d) Ψ_s, Ψ_{d_3}:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_s	A_1	1	1	1	1	1
Ψ_{d_3}	T_2	3	−1	0	1	−1
x	T_1	3	1	0	−1	−1
		9	−1	0	−1	1

Thus, $m_{A_1} = \dfrac{1(9) + 6(-1) + 6(-1) + 3(1)}{24} = 0 \quad \Rightarrow \quad$ No.

(e) Ψ_{p_1}, Ψ_s:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_{p_1}	T_1	3	1	0	−1	−1
Ψ_s	A_1	1	1	1	1	1
x	T_1	3	1	0	−1	−1
		9	1	0	1	1

Thus, $m_{A_1} = \dfrac{1(9) + 6(1) + 6(1) + 3(1)}{24} = 1 \quad \Rightarrow \quad$ Yes.

In summary, only in cases (b) and (e) is there the possibility of a non-zero dipole matrix element. Further calculation, involving data not provided here, would be needed to determine whether these two matrix elements are in fact non-zero.

29.10 *Investigate the properties of an alternating group and construct its character table as follows.*

(a) *The set of even permutations of four objects (a proper subgroup of S_4) is known as the* alternating group A_4. *List its twelve members using cycle notation.*
(b) *Assume that all permutations with the same cycle structure belong to the same conjugacy class. Show that this leads to a contradiction and hence demonstrates that even if two permutations have the same cycle structure they do not necessarily belong to the same class.*
(c) *By evaluating the products*

$$p_1 = (123)(4) \bullet (12)(34) \bullet (132)(4) \quad \textit{and} \quad p_2 = (132)(4) \bullet (12)(34) \bullet (123)(4)$$

deduce that the three elements of A_4 with structure of the form (12)(34) *belong to the same class.*
(d) *By evaluating products of the form* $(1\alpha)(\beta\gamma) \bullet (123)(4) \bullet (1\alpha)(\beta\gamma)$, *where α, β and γ are various combinations of* 2, 3 *and* 4, *show that the class to which* (123)(4) *belongs contains at least four members. Show the same for* (124)(3).
(e) *By combining results* (b), (c) *and* (d), *deduce that A_4 has exactly four classes, and determine the dimensions of its irreps.*
(f) *Using the orthogonality properties of characters and noting that elements of the form* (124)(3) *have order 3, find the character table for A_4.*

(a) The twelve members of A_4 are those permutations that involve an even number (0 or 2) of pair interchanges. For future identification, and as a shorthand, we list them, each with a label, as:

$I = (1)(2)(3)(4)$,
$2A = (12)(34)$, $2B = (13)(24)$, $2C = (14)(23)$,
$3A = (123)(4)$, $3B = (124)(3)$, $3C = (134)(2)$, $3D = (234)(1)$,
$3E = (132)(4)$, $3F = (142)(3)$, $3G = (143)(2)$, $3H = (243)(1)$.

(b) If permutations with the same cycle structure all belonged to the same conjugacy class there would be 3 classes in A_4. This would imply that it has 3 irreps, one of which would have to be the identity irrep A_1. The dimensions of the other two would then have to satisfy

$$1 + n_2^2 + n_3^2 = 12.$$

This equation has no integral solutions for n_2 and n_3 and we conclude that the assumption that permutations with the same cycle structure all belong to the same conjugacy class leads to a contradiction and is therefore wrong.

(c) With the meaning of (pqr) as in the text, i.e. p is replaced by q, q is replaced by r and r is replaced by p, we evaluate the following products of the form $P^{-1}XP$:

$$\begin{aligned} p_1 &= (123)(4) \bullet (12)(34) \bullet (132)(4)\ abcd = (123)(4) \bullet (12)(34)\ cabd \\ &= (123)(4)\ acdb = cdab = (13)(24)\ abcd \\ p_2 &= (132)(4) \bullet (12)(34) \bullet (123)(4)\ abcd = (132)(4) \bullet (12)(34)\ bcad \\ &= (132)(4)\ cbda = dcba = (14)(23)\ abcd \end{aligned}$$

Thus (13)(24) and (14)(23) both belong to the same conjugacy class as (12)(34) and therefore to the same class as each other, i.e. all three permutations with structure $(pq)(rs)$ belong to the same class.

(d) These evaluations follow the same lines as in (c) and we summarise the results using the labels allocated in (a).

$(2A)^{-1}3A\ 2A = 3F$, $(2B)^{-1}3A\ 2B = 3C$, $(2C)^{-1}3A\ 2C = 3H$.

Thus the class to which 3A=(123)(4) belongs also contains 3F, 3C and 3H, i.e. it has at least four members. Further,

$(2A)^{-1}3B\ 2A = 3E$, $(2B)^{-1}3B\ 2B = 3D$, $(2C)^{-1}3B\ 2C = 3G$,

showing that the class containing 3B also contains at least three other members.

(e) As always, I is in a class by itself and, as we have shown, the class of permutations with structure $(pq)(rs)$ has 3 members. Permutations with structure $(pqr)(s)$ are contained in a maximum of two classes since we have already shown that there exist two classes with at least four members each — and this exhausts the group. These two sets of four cannot form one class of eight as this would reduce the total number of classes to three, and we have shown in (b) that this is not possible. We conclude that there are exactly four classes containing 1, 4, 4 and 3 members. It also follows that there are four irreps, whose dimensions must satisfy

$$1 + n_2^2 + n_3^2 + n_4^2 = 12 \quad \Rightarrow \quad n_4 = 3,\ n_2 = n_3 = 1,$$

as the only integer solution.

(f) The character set for A_1 is, of course, $(1, 1, 1, 1)$. Suppose that for the three-dimensional irrep T it is $(3, x, y, z)$. Then, firstly,

$$1|3|^2 + 4|x|^2 + 4|y|^2 + 3|z|^2 = 12 \quad \Rightarrow \quad x = y = 0,\ |z| = 1,$$

for an integer solution. Secondly, the orthogonality of the two irreps gives

$$1(1)(3) + 4(1)(0) + 4(1)(0) + 3(1)z = 0 \quad \Rightarrow \quad z = -1.$$

The character table thus takes the form

Irrep	Typical element and class size			
	(1)	(123)	(132)	(12)(34)
	1	4	4	3
A_1	1	1	1	1
A_2	1	a	b	c
A_3	1	d	e	f
T	3	0	0	-1

The orthogonality of T to each of A_2 and A_3 implies that $c = f = 1$.

This leaves only a, b, d and e to be determined. Since the elements in both the classes of which they are the characters have order 3, each character can only be the sum of a number of cube roots of unity (see the text). As they are further restricted by summation rules of the form $1 + 4|a|^2 + 4|b|^2 + 3 = 12$, $|a| \leq \sqrt{2}$ and similarly for the other three characters.

The cube roots of unity are 1, $\omega = \exp(2\pi i/3)$ and ω^2, with $1 + \omega + \omega^2 = 0$. There must be at least one character in each of the sets for A_2 and A_3 that is not unity – otherwise they become A_1. Let $a = \omega$; then, from the orthogonality with A_1, $1 + 4\omega + 4b + 3 = 0$ implying that $b = \omega^2$. Since all the character sets must be different, we must have for A_3 that $d = \omega^2$ and $e = \omega$, thus completing the character table for the group A_4 as

Irrep	Typical element and class size			
	(1)	(123)	(132)	(12)(34)
	1	4	4	3
A_1	1	1	1	1
A_2	1	ω	ω^2	1
A_3	1	ω^2	ω	1
T	3	0	0	-1

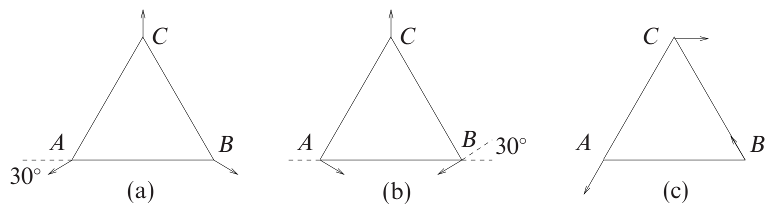

Figure 29.3 The three normal vibration modes of the equilateral array. Mode (a) is known as the 'breathing mode'. Modes (b) and (c) transform according to irrep E and have equal vibrational frequencies.

The orthogonality of the character sets for A_2 and A_3 may verified as follows:

$$\begin{aligned}\sum_X \left[\chi^{(A_2)}(X)\right]^* \chi^{(A_3)}(X) &= 1(1)(1) + 4\omega^*\omega^2 + 4(\omega^2)^*\omega + 3(1)(1)\\ &= 1 + 4e^{-i2\pi/3}e^{i4\pi/3} + 4e^{-i4\pi/3}e^{i2\pi/3} + 3\\ &= 4 + 4[\,2\cos(2\pi/3)\,] = 4 + 4(-1) = 0.\end{aligned}$$

29.12 *Demonstrate that equation (29.24) does indeed generate a set of vectors transforming according to an irrep λ, by sketching and superposing drawings of an equilateral triangle of springs and masses, based on that shown in figure 29.4.*

(a) *Make an initial sketch showing an arbitrary small mass displacement from, say, vertex C. Draw the results of operating on this initial sketch with each of the symmetry elements of the group* 3*m* *(C_{3v}).*

(b) *Superimpose the results, weighting them according to the characters of irrep* A_1 *(table 29.1 in section 29.6) and verify that the resultant is a symmetrical arrangement in which all three masses move symmetrically towards (or away from) the centroid of the triangle. The mode is illustrated in figure 29.4*(a).

(c) *Start again, now considering a displacement δ of C parallel to the x-axis. Form a similar superposition of sketches weighted according to the characters of irrep* E *(note that the reflections are not needed). The resultant contains some bodily displacement of the triangle, since this also transforms according to* E. *Show that the displacement of the centre of mass is $\bar{x} = \delta$, $\bar{y} = 0$. Subtract this out and verify that the remainder is of the form shown in figure 29.4*(c).

(d) *Using an initial displacement parallel to the y-axis, and an analogous procedure, generate the remaining normal mode, degenerate with that in* (c) *and shown in figure 29.4*(b).

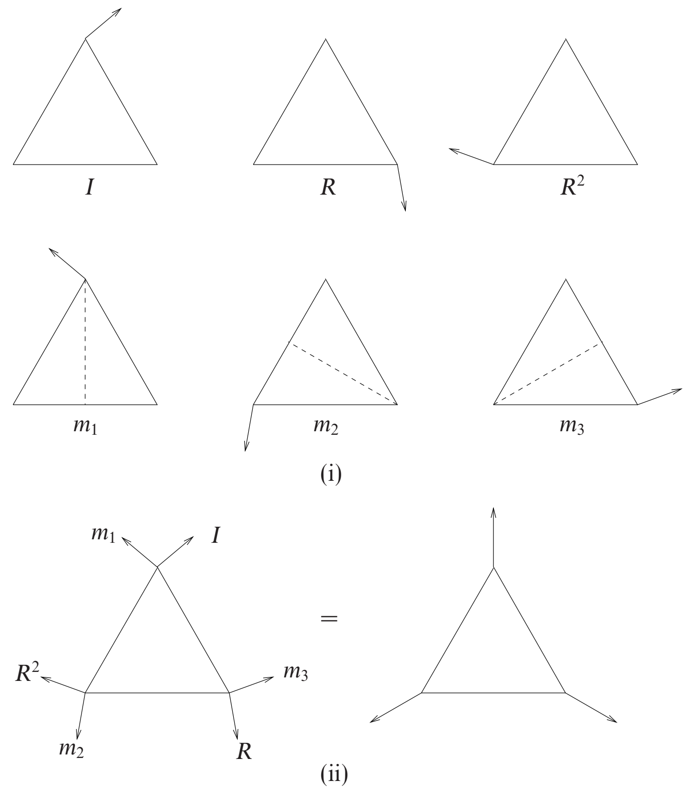

Figure 29.4 The construction of the 'breathing mode' of an equilateral array of equal springs and masses, as discussed in exercise 29.12.

(a) Part (i) of figure 29.4 shows the triangle with an initial displacement at vertex C and the results of operating on this with rotations R and R^2 and reflections m_1, m_2 and m_3. Of course, the original is also the result of operating with the identity I.

(b) To use equation (29.24),

$$\Psi_i^{(\lambda)} = \sum_X \chi^{(\lambda)^*}(X) X \Psi_i$$

with $\lambda = \mathrm{A}_1$ we need the character set for A_1 which is $(1, 1, 1)$. Thus the six triangles shown in part (i) of the figure have to be superimposed, all with equal weights. This has been done in part (ii) of the figure and, after the two displacements at each vertex have been added vectorially, the result is shown to be that of the 'breathing mode'. In this mode all movements are directed away from (or towards) the centroid of the triangle.

(c) For the two-dimensional irrep E the character set is $(2, -1, 0)$, showing that the projection operator will contain no contribution from the third class, namely the reflections.

The original displacement δ parallel to the x-axis and the results of operating on this with R and R^2 are shown in part (i) of figure 29.5. These have to be superimposed with the original (in the role of the result of I) having weight 2 and the results of the rotations having weight -1, i.e. the directions of the displacements are reversed. This is done in part (ii) of the figure. The superposition can be broken down into an overall bodily displacement of the triangle and displacements about its centroid as follows:

$$\begin{aligned}
(\bar{x}, \bar{y}) &= \tfrac{1}{3}(2\delta + \delta \cos\frac{\pi}{3} + \delta \cos\frac{\pi}{3}, 0 + \delta \sin\frac{\pi}{3} - \delta \sin\frac{\pi}{3}) = (\delta, 0),\\
(x, y)_C &= (\bar{x}, \bar{y}) + (\delta, 0),\\
(x, y)_A &= (\bar{x}, \bar{y}) + (-\frac{1}{2}\,\delta, -\frac{\sqrt{3}}{2}\,\delta),\\
(x, y)_B &= (\bar{x}, \bar{y}) + (-\frac{1}{2}\,\delta, \frac{\sqrt{3}}{2}\,\delta).
\end{aligned}$$

This breakdown is also shown in part (ii). Note that all the vibrational displacements are of magnitude δ.

(d) The final normal mode, degenerate with that in (c), is shown in part (iii) of figure 29.5. The construction parallels that in (c) and so only the calculational details are given. They are:

$$\begin{aligned}
(\bar{x}, \bar{y}) &= \tfrac{1}{3}(\delta \cos\frac{\pi}{3} - \delta \cos\frac{\pi}{3}, 2\delta + \delta \sin\frac{\pi}{3} + \delta \sin\frac{\pi}{3}) = (0, \delta),\\
(x, y)_C &= (\bar{x}, \bar{y}) + (0, \delta),\\
(x, y)_A &= (\bar{x}, \bar{y}) + (\frac{\sqrt{3}}{2}\,\delta, -\frac{1}{2}\,\delta),\\
(x, y)_B &= (\bar{x}, \bar{y}) + (-\frac{\sqrt{3}}{2}\,\delta, -\frac{1}{2}\,\delta).
\end{aligned}$$

This is mode (b) as given in the question.

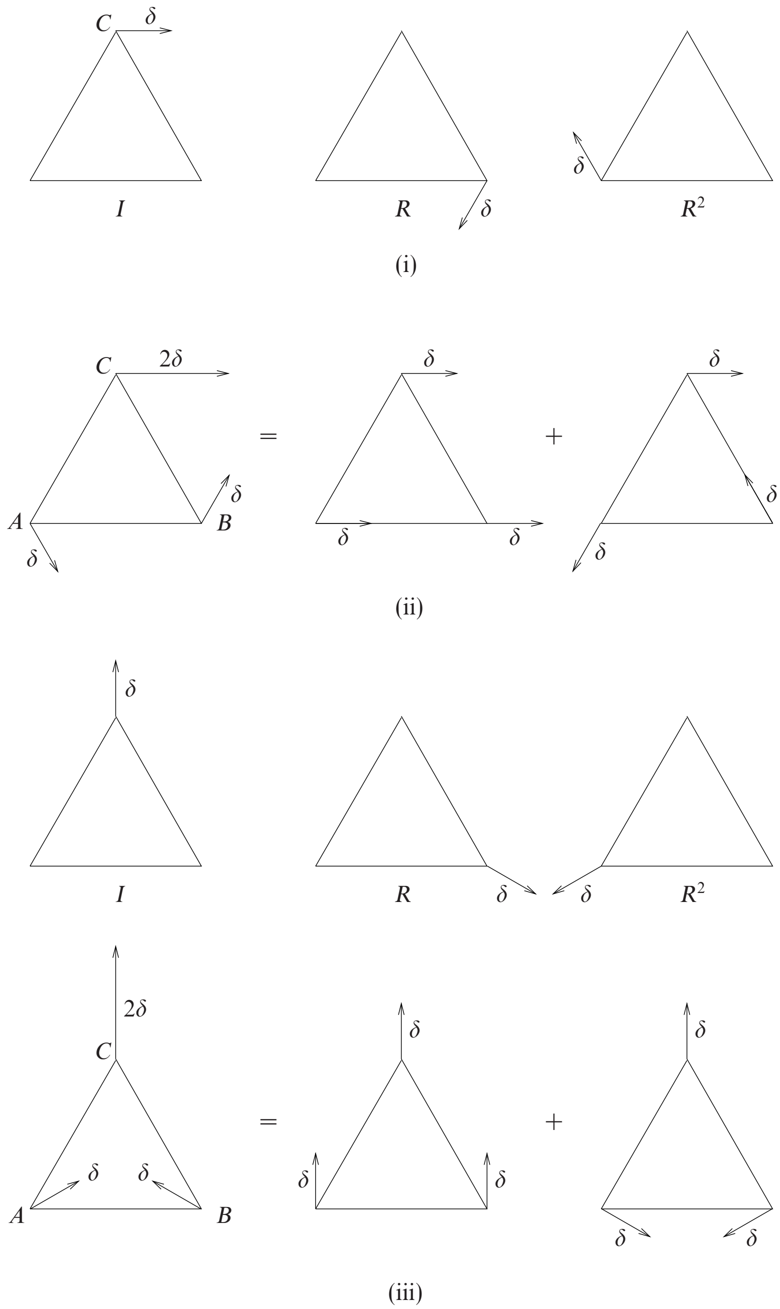

Figure 29.5 The construction of each of the two degenerate normal modes of an equilateral array of equal springs and masses, as discussed in exercise 29.12. Construction of the first mode is shown in (i) and (ii); that of the second mode is shown in (iii).

30

Probability

> **30.2** *Given that events X, Y and Z satisfy*
> $$(X \cap Y) \cup (Z \cap X) \cup \overline{(\bar{X} \cup \bar{Y})} = \overline{(Z \cup \bar{Y})} \cup \{[\overline{(\bar{Z} \cup \bar{X})} \cup (\bar{X} \cap Z)] \cap Y\},$$
> *prove that $X \supset Y$ and either $X \cap Z = \emptyset$ or $Y \supset Z$.*

We start by simplifying both sides of the equation separately using the commutativity and distributional properties of $\cup$ and $\cap$ and de Morgan's laws. For the LHS:

$$\begin{aligned}(X \cap Y) \cup (Z \cap X) \cup \overline{(\bar{X} \cup \bar{Y})} &= [(Y \cup Z) \cap X] \cup (X \cap Y) \\ &= X \cap (Y \cup Z \cup Y) \\ &= X \cap (Y \cup Z).\end{aligned}$$

For the RHS we have

$$\begin{aligned}&\overline{(Z \cup \bar{Y})} \cup \{[\overline{(\bar{Z} \cup \bar{X})} \cup (\bar{X} \cap Z)] \cap Y\} \\ &= (\bar{Z} \cap Y) \cup \{[(Z \cap X) \cup (\bar{X} \cap Z)] \cap Y\} \\ &= (\bar{Z} \cap Y) \cup [Z \cap (X \cup \bar{X}) \cap Y] \\ &= (\bar{Z} \cap Y) \cup (Z \cap Y) \\ &= Y \cap (\bar{Z} \cup Z) = Y.\end{aligned}$$

Thus the equation reduces to $X \cap (Y \cup Z) = Y$. This implies that X contains everything that is in Y, i.e. $X \supset Y$, and that X contains no part of Z that is not also in Y. This latter requirement means that *either* Z is wholly contained in Y, i.e. $Y \supset Z$ *or* X and Z have no events in common, i.e. $X \cap Z = \emptyset$.

30.4 *Use the method of induction to prove equation (30.16), the probability addition law for the union of n general events.*

We are required to prove that

$$\begin{aligned}\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) &= \sum_i \Pr(A_i) - \sum_{i,j} \Pr(A_i \cap A_j) \\ &\quad + \sum_{i,j,k} \Pr(A_i \cap A_j \cap A_k) - \cdots \\ &\quad \cdots + (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n). \quad (*)\end{aligned}$$

We do so by first assuming that $(*)$ is true for some particular value of n and use this to prove that this implies that it is true for $n \to n+1$. The relationship is obvious for $n = 1$.

Let event B be the union of events $A_1, A_2, \ldots, A_n$ and apply

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B), \qquad (**)$$

with event A as A_{n+1}. The probability $\Pr(A)$ is simply $\Pr(A_{n+1})$ and $\Pr(B)$ is the assumed result for the probability of the union of n events. This leaves only the calculation of the final term $\Pr(A \cap B)$. This is given by

$$\begin{aligned}\Pr(B \cap A_{n+1}) &= \Pr[\,(A_1 \cup A_2 \cup \cdots \cup A_n) \cap A_{n+1}\,] \\ &= \Pr[\,(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \cdots \cup (A_n \cap A_{n+1})\,] \\ &\equiv \Pr(C_1 \cup C_2 \cup \cdots \cup C_n),\end{aligned}$$

where we have defined the set of n events C_i as $C_i = A_i \cap A_{n+1}$.

We now apply the result assumed valid for n events to the C_i and obtain

$$\begin{aligned}&\Pr(B \cap A_{n+1}) \\ &= \sum_i^n \Pr(C_i) - \sum_{i,j}^n \Pr(C_i \cap C_j) + \cdots + (-1)^{n+1} \Pr(C_1 \cap C_2 \cap \cdots \cap C_n) \\ &= \sum_i^n \Pr(A_i \cap A_{n+1}) - \sum_{i,j}^n \Pr[\,(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})\,] + \\ &\quad \cdots + (-1)^{n+1} \Pr[\,(A_1 \cap A_{n+1}) \cap (A_2 \cap A_{n+1}) \cap \cdots \cap (A_n \cap A_{n+1})\,] \\ &= \sum_i^n \Pr(A_i \cap A_{n+1}) - \sum_{i,j}^n \Pr[\,(A_i \cap A_j) \cap A_{n+1})\,] + \\ &\quad \cdots + (-1)^{n+1} \Pr[\,(A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1})\,].\end{aligned}$$

We now substitute for the various terms in (**) and obtain

$$\begin{aligned}
&\Pr(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) \\
&= \Pr(A_{n+1}) + \sum_i^n \Pr(A_i) - \sum_{i,j}^n \Pr(A_i \cap A_j) + \\
&\quad \cdots + (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n) \\
&\quad -\{ \sum_i^n \Pr(A_i \cap A_{n+1}) - \sum_{i,j}^n \Pr[(A_i \cap A_j) \cap A_{n+1})] + \\
&\quad \cdots + (-1)^{n+1} \Pr[(A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1})] \}.
\end{aligned}$$

Finally, collecting together similar terms and noting that, for example, $(P \cap Q) \cap R = P \cap Q \cap R$, we obtain

$$\begin{aligned}
&\Pr(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) \\
&= \sum_i^{n+1} \Pr(A_i) - \sum_{i,j}^{n+1} \Pr(A_i \cap A_j) + \sum_{i,j,k}^{n+1} \Pr(A_i \cap A_j \cap A_k) - \\
&\quad \cdots + (-1)^{n+2} \Pr(A_1 \cap A_2 \cap \cdots \cap A_{n+1}).
\end{aligned}$$

All summations now run over $i, j, \ldots = 1, 2, \ldots, n+1$ and this expression is the same as (*) but with n increased to $n+1$. This, together with our earlier observation that the formula is valid for $n = 1$ completes the proof by induction.

30.6 *$X_1, X_2, \ldots, X_n$ are independent identically distributed random variables drawn from a uniform distribution on* $[0, 1]$. *The random variables A and B are defined by*

$$A = \min(X_1, X_2, \ldots, X_n), \qquad B = \max(X_1, X_2, \ldots, X_n).$$

For any fixed k such that $0 \le k \le \frac{1}{2}$, find the probability p_n that both

$$A \le k \qquad \text{and} \qquad B \ge 1 - k.$$

Check your general formula by considering directly the cases (a) $k = 0$, (b) $k = \frac{1}{2}$, (c) $n = 1$ *and* (d) $n = 2$.

There are four possible situations, according as to whether A is less than or greater than k and as to whether B is less than or greater than $1 - k$. We need to calculate the probability for just one of these combinations and we do so by finding the probabilities for the other three and subtracting them from unity.

If $A \ge k$ then all X_i must lie in $k \le X_i \le 1$. This has probability $(1-k)^n$. Similarly,

$$\Pr(B \le 1 - k) = (1 - k)^n \quad \text{and} \quad \Pr(A \ge k \text{ and } B \le 1 - k) = (1 - 2k)^n.$$

Now,

$$\Pr(A \geq k) = \Pr(A \geq k \text{ and } B \leq 1-k) + \Pr(A \geq k \text{ and } B \geq 1-k),$$
$$(1-k)^n = (1-2k)^n + \Pr(A \geq k \text{ and } B \geq 1-k),$$

and, substituting from this result into

$$\Pr(B \geq 1-k) = \Pr(B \geq 1-k \text{ and } A \geq k) + \Pr(B \geq 1-k \text{ and } A \leq k),$$

gives

$$1-(1-k)^n = [\,(1-k)^n - (1-2k)^n\,] + \Pr(B \geq 1-k \text{ and } A \leq k).$$

Thus

$$p_n = \Pr(B \geq 1-k \text{ and } A \leq k) = 1 - 2(1-k)^n + (1-2k)^n.$$

In retrospect, the structure of this answer is straightforward to understand. Its RHS is

$$\Pr(A \text{ and } B \text{ have some values}) - \Pr(\text{all the } X_i \text{ lie in } k < X_i < 1)$$
$$- \Pr(\text{all the } X_i \text{ lie in } 0 < X_i < 1-k)$$
$$+ \Pr(\text{all the } X_i \text{ lie in } k < X_i < 1-k),$$

the final term being added back in to account for the fact that the range $k < X_i < 1-k$ has been subtracted out twice (instead of once) by the previous two terms.

For the special cases given we have:

(a) $k = 0$, i.e $A \leq 0$ and $B \geq 1$. This clearly has zero probability, in agreement with $1 - 2(1-0)^n + (1-0)^n = 0$ for all n.

(b) $k = \frac{1}{2}$, i.e $A \leq \frac{1}{2}$ and $B \geq \frac{1}{2}$. This outcome requires the avoidance of a situation in which all the X_i are in one half of the range. For all of them to be in one half has probability $(\frac{1}{2})^n$; this has to be doubled as there are two possible half ranges. The probability of $A \leq \frac{1}{2}$ and $B \geq \frac{1}{2}$ is therefore $1 - (\frac{1}{2})^{n-1}$. The formula derived earlier gives $1 - 2(1-\frac{1}{2})^n + (1-1)^n = 1 - (\frac{1}{2})^{n-1}$, which is in agreement with this direct calculation.

(c) $n = 1$. Clearly a single random variable, which has to act as both the minimum and the maximum of its set, cannot satisfy both inequalities. The probability in this case must be zero, as given by the general formula $1-2(1-k)+(1-2k) = 0$ for any k.

(d) $n = 2$. In this case, the first of the two X_i has to be less than k or greater than $1-k$. This has probability $2k$. The second then has to fall in a particular one of these two ranges; this has probability k. The overall probability is therefore $2k^2$. The derived formula gives $1 - 2(1 - 2k + k^2) + (1 - 4k + 4k^2) = 2k^2$, as expected.

30.8 *This exercise shows that the odds are hardly ever 'evens' when it comes to dice rolling.*

(a) *Gamblers A and B each roll a fair six-faced die, and B wins if his score is strictly greater than A's. Show that the odds are 7 to 5 in A's favour.*
(b) *Calculate the probabilities of scoring a total T from two rolls of a fair die for $T = 2, 3, \ldots, 12$. Gamblers C and D each roll a fair die twice and score respective totals T_C and T_D, D winning if $T_D > T_C$. Realising that the odds are not equal, D insists that C should increase her stake for each game. C agrees to stake £1.10 per game, as compared to D's £1.00 stake. Who will show a profit?*

(a) Out of the 36 equally likely outcomes for a single roll of each die, there are 6 in which the scores are equal and result in a win for A. The other 30 cases, in which the scores are unequal, provide 15 wins for A and 15 wins for B. Thus A wins in $6 + 15 = 21$ of the 36 cases and B in only 15 cases, giving A favourable odds of 7 to 5.

(b) The probability distribution $p(T)$ for the total T from two rolls of a die is

T	2	3	4	5	6	7	8	9	10	11	12
$36p(T)$	1	2	3	4	5	6	5	4	3	2	1

The probability that $T_D > T_C$ could be calculated for each T_D by adding all the probabilities for T_C up to $T_D - 1$ and the overall probability found by then weighting each sum by the probability of T_D. However, it is simpler to add up the probabilities for the two totals being equal, subtract this from unity and then take $\Pr(T_D > T_C)$ as one-half of what is left, i.e.

$$\Pr(T_D > T_C) = \frac{1}{2}\left\{1 - \frac{1}{(36)^2}\left[\,2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 6^2\,\right]\right\}$$
$$= \frac{1}{2}\left[1 - \frac{146}{(36)^2}\right] = 0.4437.$$

This is the chance that D wins and so C's expected return for a $£1.10$ stake is $£2.10(1 - 0.4437) = £1.168$, i.e. a profit of about 7 pence per game.

A straightforward calculation shows that C should stake just over $£1.25$ to make the game 'fair'.

30.10 *As every student of probability theory will know, Bayesylvania is awash with natives, not all of whom can be trusted to tell the truth, and lost and apparently somewhat deaf travellers who ask the same question several times in an attempt to get directions to the nearest village.*

One such traveller finds himself at a T-junction in an area populated by the Asciis and Bisciis in the ratio 11 *to* 5. *As is well known, the Biscii always lie but the Ascii tell the truth three quarters of the time, giving independent answers to all questions, even to immediately repeated ones.*

(a) *The traveller asks one particular native twice whether he should go to the left or to the right to reach the local village. Each time he is told 'left'. Should he take this advice, and, if he does, what are his chances of reaching the village?*
(b) *The traveller then asks the same native the same question a third time and for a third time receives the answer 'left'. What should the traveller do now? Have his chances of finding the village been altered by asking the third question?*

If the native is an Ascii then the chance of getting the same answer twice is

$$\frac{3}{4}\frac{3}{4} + \frac{1}{4}\frac{1}{4} = \frac{5}{8}.$$

The same calculation shows that a repeated answer by an Ascii is 9 times more likely to be the truth than a lie. If the native is a Biscii, then a repeated answer is guaranteed; it is also guaranteed to be a lie.

With A denoting the event that the native is an Ascii and B that he/she is a Biscii, let E be the event that the same answer is given twice by the native. Then, by Bayes' theorem,

$$\Pr(A|E) = \frac{\Pr(E|A)\ \Pr(A)}{\Pr(E|A)\ \Pr(A) + \Pr(E|B)\ \Pr(B)}$$
$$= \frac{\frac{5}{8}\,\frac{11}{16}}{\frac{5}{8}\,\frac{11}{16} + 1\,\frac{5}{16}} = \frac{55}{95}.$$

The traveller can only be told the truth if the native is an Ascii and the two identical responses are the truth. The combined probability for this is $(55/95) \times (9/10) = 99/190 = 52.11\%$. As this is more than 50%, the traveller should go *left*.

(b) The corresponding calculations for the event F of the same answer of 'left' being given three times are:

$$\Pr(F|A) = \left(\frac{3}{4}\right)^3 + \left(\frac{1}{4}\right)^3 = \frac{28}{64},$$

with the truth 27 times more likely than a lie;

$$\Pr(A|F) = \frac{\frac{28}{64}\,\frac{11}{16}}{\frac{28}{64}\,\frac{11}{16} + 1\,\frac{5}{16}} = \frac{308}{628};$$

the overall probability of being told the truth is $(308/628) \times (27/28) = 297/628$. As this is less than half he should go *right* with a probability of $331/628 = 52.71\%$ of being correct — a very marginal improvement in his chances!

> ***30.12*** *Villages A, B, C and D are connected by overhead telephone lines joining AB, AC, BC, BD and CD. As a result of severe gales, there is a probability p (the same for each link) that any particular link is broken.*
>
> (a) *Show that the probability that a call can be made from A to B is*
>
> $$1 - p^2 - 2p^3 + 3p^4 - p^5.$$
>
> (b) *Show that the probability that a call can be made from D to A is*
>
> $$1 - 2p^2 - 2p^3 + 5p^4 - 2p^5.$$

We denote the probability that a link is intact by q, equal to $1 - p$.

(a) Situations with a total n of breaks have a probability of occuring equal to $p^n q^{5-n}$. For each value of n we need to identify the number m out of the ${}^5C_n$ possible combinations of breaks that will still allow a call to be made from A to B. Denote an intact link AB by its name and a break in AB by $\overline{AB}$.

n	probability	${}^5C_n$	acceptable break patterns	m
0	q^5	1	any (with 0 breaks)	1
1	pq^4	5	any with 1 break	5
2	p^2q^3	10	all except $\overline{AB} + \overline{AC}$	9
3	p^3q^2	10	AB with any of ${}^4C_3$ break patterns	4
			$AC + BC$	1
4	p^4q	5	only AB	1
5	p^5	1	none	0

The total probability of being able to make a call from A to B is

$$\begin{aligned}
P_{AB} &= q^5 + 5pq^4 + 9p^2q^3 + 5p^3q^2 + p^4q \\
&= (p+q)^5 - p^2q^3 - 5p^3q^2 - 4p^4q - p^5 \\
&= 1 - p^2(q^3 + 5pq^2 + 4p^2q + p^3) \\
&= 1 - p^2[\,(q+p)^3 + 2pq^2 + p^2q\,] \\
&= 1 - p^2(1 + 2p - 4p^2 + 2p^3 + p^2 - p^3) \\
&= 1 - p^2 - 2p^3 + 3p^4 - p^5.
\end{aligned}$$

(b) A similar count for a call from A to D gives

n	probability	${}^5C_n$	acceptable break patterns	m
0	q^5	1	any (with 0 breaks)	1
1	pq^4	5	any with 1 break	5
2	p^2q^3	10	all except $\overline{AB}+\overline{AC}$ and $\overline{BD}+\overline{CD}$	8
3	p^3q^2	10	only $AB+BD$ and $AC+CD$	2
4	p^4q	5	none	0
5	p^5	1	none	0

The total probability of being able to make a call from A to D is

$$\begin{aligned}
P_{AD} &= q^5 + 5pq^4 + 8p^2q^3 + 2p^3q^2 \\
&= (p+q)^5 - 2p^2q^3 - 8p^3q^2 - 5p^4q - p^5 \\
&= 1 - p^2(2q^3 + 8pq^2 + 5p^2q + p^3) \\
&= 1 - p^2(2 - 6p + 6p^2 - 2p^3 + 8p - 16p^2 + 8p^3 + 5p^2 - 5p^3 + p^3) \\
&= 1 - 2p^2 - 2p^3 + 5p^4 - 2p^5.
\end{aligned}$$

As they should, both calculations give unit probability if $p = 0$ and zero probability if $p = 1$.

30.14 *A certain marksman never misses his target, which consists of a disc of unit radius with centre O. The probability that any given shot will hit the target within a distance t of O is t^2 for $0 \le t \le 1$. The marksman fires n independendent shots at the target, and the random variable Y is the radius of the smallest circle with centre O that encloses all the shots. Determine the PDF for Y and hence find the expected area of the circle.*

The shot that is furthest from O is now rejected and the corresponding circle determined for the remaining $n-1$ shots. Show that its expected area is

$$\frac{n-1}{n+1}\,\pi.$$

Let the n shots be at distances r_i ($i = 1, 2, \dots, n$) from O; Y is equal to the largest of these. The cumulative distribution function for each shot is $F(y) = y^2$ and the probability that all n shots lie within y of the centre is y^{2n}. But this is also the CDF for Y, whose PDF must therefore be its derivative, i.e. $2ny^{2n-1}$. The expected area for covering the n shots is thus

$$A_n = \int_0^1 \pi y^2 \, 2ny^{2n-1} \, dy = 2\pi n \int_0^1 y^{2n+1} \, dy = \frac{2\pi n}{2n+2} = \frac{n\pi}{n+1}.$$

As already found, the probability that the worst shot lies in the range y to $y + dy$ is $2ny^{2n-1} \, dy$. The other $n - 1$ shots are individually distributed as given in the question, but only over the region $0 < z < y$. Their common CDF is therefore z^2/y^2 (reaching unity at $z = y$). The CDF for all $n - 1$ shots is thus $(z^2/y^2)^{n-1}$ and the corresponding PDF is its derivative, $(2n-2)z^{2n-3}/y^{2n-2}$.

We now need to average the area of the circle covering the best $n - 1$ shots over all values of the radius of the worst shot. This gives

$$\begin{aligned} A_{n-1} &= \int_0^1 2ny^{2n-1} \, dy \int_0^y \pi z^2 \, \frac{(2n-2)z^{2n-3}}{y^{2n-2}} \, dz \\ &= 2n(2n-2)\pi \int_0^1 y \, dy \int_0^y z^{2n-1} \, dz \\ &= 2n(2n-2)\pi \int_0^1 \frac{y^{2n+1}}{2n} \, dy \\ &= (2n-2)\pi \, \frac{1}{2n+2} = \frac{n-1}{n+1} \, \pi. \end{aligned}$$

30.16 *Kittens from different litters do not get on with each other and fighting breaks out whenever two kittens from different litters are present together. A cage initially contains x kittens from one litter and y from another. To quell the fighting, kittens are removed at random, one at a time, until peace is restored. Show, by induction, that the expected number of kittens finally remaining is*

$$N(x, y) = \frac{x}{y+1} + \frac{y}{x+1}.$$

This result is trivially true if either of x or y is zero, as no kittens need to be removed. We therefore consider $x, y \geq 1$. It is also clear that $N(a, b)$ must be equal to $N(b, a)$, as this is only a matter of labelling the litters.

Let the cage contain x kittens from the first litter and $y + 1$ from the second and consider the removal of one randomly chosen kitten. With probability $(y +$

$1)/(x+y+1)$ it will be from the second litter and the expected final number of kittens will be $N(x,y)$. Correspondingly, with probability $x/(x+y+1)$ it will be from the first litter and the expected final number of kittens will be $N(x-1,y+1)$. Thus we have the recurrence relation

$$N(x,y+1) = \frac{y+1}{x+y+1}\,N(x,y) + \frac{x}{x+y+1}\,N(x-1,y+1).$$

Now *suppose* that

$$N(x,y) = \frac{x}{y+1} + \frac{y}{x+1} \qquad (*)$$

for all x, y such that $x+y=n$ and apply the assumption to the RHS of the above equation.

$$\begin{aligned}
N(x,y+1) &= \frac{y+1}{x+y+1}\left(\frac{x}{y+1}+\frac{y}{x+1}\right) \\
&\qquad + \frac{x}{x+y+1}\left(\frac{x-1}{y+2}+\frac{y+1}{x}\right) \\
&= \frac{1}{x+y+1}\left(x+\frac{y^2+y}{x+1}+\frac{x^2-x}{y+2}+y+1\right) \\
&= \frac{1}{x+y+1}\left[\left(x+\frac{x^2-x}{y+2}\right)+\left(y+1+\frac{y^2+y}{x+1}\right)\right] \\
&= \frac{1}{x+y+1}\left(\frac{xy+x+x^2}{y+2}+\frac{xy+x+2y+1+y^2}{x+1}\right) \\
&= \frac{x}{y+2}+\frac{yx+y^2+y+x+y+1}{(x+y+1)(x+1)} \\
&= \frac{x}{y+2}+\frac{y+1}{x+1} \qquad\qquad (**) \\
&= \frac{x}{(y+1)+1}+\frac{(y+1)}{x+1}.
\end{aligned}$$

Either by interchanging x and y in $(**)$ to obtain $N(y,x+1)$ and then using $N(a,b)=N(b,a)$, or by a similar calculation to the above, we can also show that

$$N(x+1,y) = \frac{(x+1)}{y+1} + \frac{y}{(x+1)+1}.$$

Combining the two results then shows that the assumption that $(*)$ is valid for $x+y=n$ implies that it is valid for $x+y=n+1$.

However, it is valid by direct inspection for $n=2$, since $N(2,0)=2=N(0,2)$ (no kitten need be removed) and $N(1,1)=1$ (one kitten must be removed). Thus $(*)$ is valid for all n and hence for all x and y. It is formally valid, by inspection, for $n=1$, but the proof given then involves negative arguments of N, albeit with N multiplied by zero.

> ***30.18*** *A particle is confined to the one-dimensional space $0 \le x \le a$ and classically it can be in any small interval dx with equal probability. However, quantum mechanics gives the result that the probability distribution is proportional to $\sin^2(n\pi x/a)$, where n is an integer. Find the variance in the particle's position in both the classical and quantum mechanical pictures and show that, although they differ, the latter tends to the former in the limit of large n, in agreement with the correspondence principle of physics.*

Classical Mechanics

Here, since the probability is uniformly distributed throughout the interval $0 \le x \le a$, we have

$$p(x)\,dx = \frac{1}{a}\,dx \quad \Rightarrow \quad \bar{x} = \frac{a}{2}.$$

The corresponding variance in the position of the particle is

$$V[X] = \int_0^a \left(x - \frac{a}{2}\right)^2 \frac{1}{a}\,dx = \frac{1}{a}\left[\frac{(x-\frac{1}{2}a)^3}{3}\right]_0^a = \frac{a^2}{12}.$$

Quantum Mechanics

The probability density is

$$p(x)\,dx = A\sin^2\frac{n\pi x}{a}\,dx, \text{ where } \int_0^a A\sin^2\frac{n\pi x}{a}\,dx = 1 \quad \Rightarrow \quad A = \frac{2}{a}.$$

The mean value of x is

$$\bar{x} = \frac{2}{a}\int_0^a x\sin^2\frac{n\pi x}{a}\,dx = \frac{a}{2}, \text{ by symmetry.}$$

We compute the variance as $\overline{x^2} - \bar{x}^2$.

$$\begin{aligned}
V(x) &= \frac{2}{a}\int_0^a x^2\sin^2\frac{n\pi x}{a}\,dx - \frac{a^2}{4} \\
&= \frac{1}{a}\int_0^a x^2\left(1 - \cos\frac{2n\pi x}{a}\right)dx - \frac{a^2}{4} \\
&= \frac{1}{a}\frac{a^3}{3} - \frac{1}{a}\left[\frac{ax^2}{2n\pi}\sin\frac{2n\pi x}{a}\right]_0^a + \frac{1}{a}\int_0^a \frac{a2x}{2n\pi}\sin\frac{2n\pi x}{a}\,dx - \frac{a^2}{4} \\
&= \frac{a^2}{12} - 0 + \frac{1}{n\pi}\left[-\frac{ax}{2n\pi}\cos\frac{2n\pi x}{a}\right]_0^a + \frac{1}{n\pi}\int_0^a \frac{a}{2n\pi}\cos\frac{2n\pi x}{a}\,dx \\
&= \frac{a^2}{12} - \frac{a^2}{2n^2\pi^2} + 0 + 0.
\end{aligned}$$

The classical and quantum results differ by an amount that depends upon n, but the latter tends to the former as $n \to \infty$.

30.20 *For a non-negative integer random variable X, in addition to the probability generating function $\Phi_X(t)$ defined in equation (30.71) it is possible to define the probability generating function*

$$\Psi_X(t) = \sum_{n=0}^{\infty} g_n t^n,$$

where g_n is the probability that $X > n$.

(a) *Prove that Φ_X and Ψ_X are related by*

$$\Psi_X(t) = \frac{1 - \Phi_X(t)}{1 - t}.$$

(b) *Show that $E[X]$ is given by $\Psi_X(1)$ and that the variance of X can be expressed as $2\Psi'_X(1) + \Psi_X(1) - [\Psi_X(1)]^2$.*

(c) *For a particular random variable X, the probability that $X > n$ is equal to α^{n+1} with $0 < \alpha < 1$. Use the results in* (b) *to show that $V[X] = \alpha(1-\alpha)^{-2}$.*

(a) We first note, from the definition of Ψ_X, that $g_0 = 1 - f_0$ and, for general n, that $g_n = f_{n+1} + f_{n+2} + \cdots$. Now consider

$$\begin{aligned}
(1-t)\Psi_X(t) &= (1-t)\sum_{n=0}^{\infty} g_n t^n \\
&= \sum_{n=1}^{\infty}(g_n - g_{n-1})t^n + g_0 \\
&= -\sum_{n=1}^{\infty} f_n t^n + 1 - f_0 \\
&= 1 - \sum_{n=0}^{\infty} f_n t^n \\
&= 1 - \Phi_X(t),
\end{aligned}$$

thus establishing the given result.

(b) We wish to express the standard result that $E[X] = \Phi'_X(1)$ in terms of $\Psi_X(t)$ and to do so differentiate the equation derived in (a) with respect to t and then set $t = 1$:

$$(1-t)\Psi'_X(t) - \Psi_X(t) = 0 - \Phi'_X(t) \quad \Rightarrow \quad \Psi_X(1) = \Phi'_X(1) = E[X].$$

For the variance, we need to obtain alternative expressions for the terms that appear in the general result

$$V[X] = \Phi''_X(1) + \Phi'_X(1) - [\Phi'_X(1)]^2.$$

The final two terms are already dealt with; for the first we differentiate the earlier result a second time and obtain

$$(1-t)\Psi_X''(t) - \Psi_X'(t) - \Psi_X'(t) = -\Phi_X''(t).$$

Setting $t = 1$ shows that $-\Phi_X''(1) = -2\Psi_X'(1)$. Substitution in the expression for the variance then shows that $V[X] = 2\Psi_X'(1) + \Psi_X(1) - [\Psi_X(1)]^2$.

(c) As the probability that $X > n$ is equal to α^{n+1} with $0 < \alpha < 1$, $g_n = \alpha^{n+1}$ and so

$$\Psi_X(t) = \sum_{n=0}^{\infty} \alpha^{n+1} t^n = \frac{\alpha}{1-\alpha t},$$
$$\Psi_X'(t) = \frac{\alpha^2}{(1-\alpha t)^2}.$$

The mean of the distribution is $\Psi_X(1) = \dfrac{\alpha}{1-\alpha}$ and the variance is given by

$$\begin{aligned} V[X] &= 2\Psi_X'(1) + \Psi_X(1) - [\Psi_X(1)]^2 \\ &= \frac{2\alpha^2}{(1-\alpha)^2} + \frac{\alpha}{1-\alpha} - \frac{\alpha^2}{(1-\alpha)^2} \\ &= \frac{2\alpha^2 + \alpha - \alpha^2 - \alpha^2}{(1-\alpha)^2} \\ &= \frac{\alpha}{(1-\alpha)^2}, \end{aligned}$$

as stated in the question.

30.22 *Use the formula obtained in subsection 30.8.2 for the moment generating function of the geometric distribution to determine the CGF $K_n(t)$ for the number of trials needed to record n successes. Evaluate the first four cumulants and use them to confirm the stated results for the mean and variance and to show that the distribution has skewness and kurtosis given, respectively, by*

$$\frac{2-p}{\sqrt{n(1-p)}} \qquad \text{and} \qquad 3 + \frac{6-6p+p^2}{n(1-p)}.$$

The MGF obtained in the text for the number of trials required to obtain the first success is

$$M(t) = \frac{pe^t}{1-qe^t},$$

and so it follows that the MGF for the number of trials needed to record n successes is

$$M_n(t) = \left(\frac{pe^t}{1 - qe^t}\right)^n.$$

The CGF of the distribution is therefore given by

$$\begin{aligned} K_n(t) &= \ln M_n(t) \\ &= n\ln(pe^t) - n\ln(1 - qe^t) \\ &= n\ln p + nt + n\sum_{r=1}^{\infty}\frac{(qe^t)^r}{r} \\ &= n\ln p + nt + n\sum_{r=1}^{\infty}\frac{q^r}{r}\sum_{s=0}^{\infty}\frac{(tr)^s}{s!}. \end{aligned}$$

This must be the same as

$$\kappa_1 t + \kappa_2\frac{t^2}{2!} + \kappa_3\frac{t^3}{3!} + \cdots,$$

where κ_i is the ith cumulant.

The coefficient of t^0 is

$$n\ln p + n\sum_{r=1}^{\infty}\frac{q^r}{r}\frac{1}{0!} = n\ln p - n\ln(1 - q) = 0,$$

as expected, since no CGF contains a constant term.

The coefficient of t^1 is $\kappa_1 = \mu_1 = \mu$ and given by

$$n + n\sum_{r=1}^{\infty}\frac{q^r}{r}\frac{r}{1!} = n + \frac{nq}{1 - q} = \frac{n}{1 - q} = \frac{n}{p},$$

in agreement with the stated result for $n = 1$.

The coefficient of t^2 is $\kappa_2/(2!)$ with $\kappa_2 = \nu_2 = \sigma^2$ and given by

$$\begin{aligned} \frac{\kappa_2}{2!} &= n\sum_{r=1}^{\infty}\frac{q^r}{r}\frac{r^2}{2!} = \frac{n}{2!}\sum_{r=1}^{\infty} rq^r = \frac{nq}{2!}\frac{d}{dq}\left(\sum_{r=1}^{\infty} q^r\right) \\ &= \frac{nq}{2!}\frac{d}{dq}\left(\frac{q}{1 - q}\right) = \frac{nq}{2!}\frac{1 - q + q}{(1 - q)^2}, \\ \kappa_2 &= \frac{nq}{p^2}, \end{aligned}$$

again in agreement with the stated result for $n = 1$.

The coefficient of t^3 is $\kappa_3/(3!)$ with $\kappa_3 = \nu_3$ and given by

$$\frac{\kappa_3}{3!} = n\sum_{r=1}^{\infty}\frac{q^r}{r}\frac{r^3}{3!} = \frac{n}{3!}\sum_{r=1}^{\infty} r^2 q^r.$$

To evaluate this sum we make further use of the result that we have just derived,

$$\sum_{r=1}^{\infty} rq^r = \frac{q}{(1-q)^2},$$

by differentiating both sides with respect to q. This gives

$$\sum_{r=1}^{\infty} r^2 q^{r-1} = \frac{(1-q)^2 + 2q(1-q)}{(1-q)^4} = \frac{1+q}{(1-q)^3}.$$

The sum on the LHS is closely related to the one appearing in the expression for κ_3 and substituting for it gives

$$\kappa_3 = \frac{nq(1+q)}{p^3}.$$

The skewness, equal to $\nu_3/(\nu_2)^{3/2}$, therefore has the value

$$\gamma_3 = \frac{\kappa_3}{(\kappa_2)^{3/2}} = \frac{nq(1+q)}{p^3}\,\frac{p^3}{n^{3/2}q^{3/2}} = \frac{1+q}{\sqrt{nq}} = \frac{2-p}{\sqrt{n(1-p)}}.$$

The kurtosis γ_4 of the distribution is given by $\gamma_4 = \nu_4/(\nu_2)^2$ with $\nu_4 = \kappa_4 + 3(\nu_2)^2$. And so, to determine it, we need an explicit expression for κ_4. This is obtained from the coefficient of t^4, which is $\kappa_4/(4!)$ and given by

$$\frac{\kappa_4}{4!} = n\sum_{r=1}^{\infty} \frac{q^r}{r}\,\frac{r^4}{4!} = \frac{n}{4!}\sum_{r=1}^{\infty} r^3 q^r.$$

Differentiating the result obtained for $\sum_{r=1}^{\infty} r^2 q^r$ when finding κ_3, we deduce that

$$\begin{aligned}\sum_{r=1}^{\infty} r^3 q^{r-1} &= \frac{(1-q)^3(1+2q) + 3q(1+q)(1-q)^2}{(1-q)^6}\\ &= \frac{1+q-2q^2+3q+3q^2}{(1-q)^4}\\ &= \frac{1+4(1-p)+(1-p)^2}{p^4}.\end{aligned}$$

We conclude that

$$\begin{aligned}\kappa_4 &= \frac{n(1-p)(6-6p+p^2)}{p^4},\\ \Rightarrow\quad \gamma_4 &= 3 + \frac{n(1-p)(6-6p+p^2)}{n^2(1-p)^2} = 3 + \frac{6-6p+p^2}{n(1-p)}.\end{aligned}$$

30.24 *As assistant to a celebrated and imperious newspaper proprietor, you are given the job of running a lottery in which each of his five million readers will have an equal independent chance p of winning a million pounds; you have the job of choosing p. However, if nobody wins it will be bad for publicity, whilst, if more than two readers do so, the prize cost will more than offset the profit from extra circulation – in either case you will be sacked! Show that, however you choose p, there is more than a* 40% *chance you will soon be clearing your desk.*

The number of winners x will follow a Poisson distribution; let its mean be μ. I will keep my job provided the number of winners is 1 or 2. The probability $y(\mu)$ of this is

$$y(\mu) = \frac{\mu}{1!}\,e^{-\mu} + \frac{\mu^2}{2!}\,e^{-\mu}.$$

This is maximal when μ is chosen to satisfy

$$0 = \frac{dy}{d\mu} = -(\tfrac{1}{2}\mu^2 + \mu)e^{-\mu} + (1+\mu)e^{-\mu} \quad \Rightarrow \quad \mu = \sqrt{2}.$$

The corresponding value of p is $\sqrt{2}/(5 \times 10^6)$ and the chance that I keep my job is

$$y(\sqrt{2}) = (\sqrt{2} + 1)e^{-\sqrt{2}} = 0.587,$$

i.e. a 41% chance that I will be clearing my desk.

30.26 *In the game of Blackball, at each turn Muggins draws a ball at random from a bag containing five white balls, three red balls and two black balls; after being recorded, the ball is replaced in the bag. A white ball earns him* \$1 *whilst a red ball gets him* \$2*; in either case he also has the option of leaving with his current winnings or of taking a further turn on the same basis. If he draws a black ball the game ends and he loses all he may have gained previously. Find an expression for Muggins' expected return if he adopts the strategy of drawing up to n balls if he has not been eliminated by then.*

Show that, as the entry fee to play is \$3*, Muggins should be dissuaded from playing Blackball, but if that cannot be done what value of n would you advise him to adopt?*

Suppose that Muggins draws all of the n balls dictated by his strategy and let the respective numbers of the different colours be w, r and b, where $w + r + b = n$. If b is non-zero, his winnings s, given otherwise by $S = w + 2r$, will be zero. Now,

$$\Pr(S = 0) = 1 - \left(\frac{8}{10}\right)^n$$

$$\text{and} \quad \Pr(S = w + 2r = n + r) = \left(\frac{8}{10}\right)^n \left[{}^nC_r \left(\frac{3}{8}\right)^r \left(\frac{5}{8}\right)^{n-r} \right].$$

Thus his expected return is (noting that for non-zero contributions $w+2r = n+r$)

$$\begin{aligned}
S(n) &= \left(\frac{8}{10}\right)^n \sum_{r=0}^{n} {}^nC_r \left(\frac{3}{8}\right)^r \left(\frac{5}{8}\right)^{n-r} (n+r) \\
&= \left(\frac{8}{10}\right)^n \left[n\left(\frac{3}{8}+\frac{5}{8}\right)^n + \sum_{r=0}^{n} {}^nC_r\, r \left(\frac{3}{8}\right)^r \left(\frac{5}{8}\right)^{n-r} \right] \\
&= \left(\frac{8}{10}\right)^n \left[n + \sum_{r=1}^{n} \frac{n!}{(r-1)!\,(n-r)!} \left(\frac{3}{8}\right)^r \left(\frac{5}{8}\right)^{n-r} \right] \\
&= \left(\frac{8}{10}\right)^n \left[n + \frac{3n}{8} \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\,[\,n-1-(r-1)\,]!} \right. \\
&\qquad\qquad \left. \times \left(\frac{3}{8}\right)^{r-1} \left(\frac{5}{8}\right)^{[\,n-1-(r-1)\,]} \right] \\
&= \left(\frac{8}{10}\right)^n \left[n + \frac{3n}{8} \sum_{s=0}^{n-1} \frac{(n-1)!}{s!\,[\,n-1-s\,]!} \left(\frac{3}{8}\right)^s \left(\frac{5}{8}\right)^{[\,n-1-s\,]} \right] \\
&= \left(\frac{4}{5}\right)^n \frac{11n}{8}.
\end{aligned}$$

In hindsight, this should have been expected, since the average gain for a non-zero return is $n \times [\,(5 \times 1) + (3 \times 2)\,]/8 = 11n/8$. The bag could have been more easily treated as one containing 2 black balls and 8 non-black balls, each of the latter offering a return of \$11/8 if drawn.

To optimise this return n should be chosen so that $\ln y$, where $y = n(\frac{4}{5})^n$, is optimised, i.e.

$$\begin{aligned}
\ln y &= \ln n + n \ln \tfrac{4}{5}, \\
0 = \frac{dy}{dn} &= \frac{1}{n} + \ln \frac{4}{5}, \\
\Rightarrow \quad n &= \frac{1}{\ln \frac{5}{4}} = 4.48.
\end{aligned}$$

Since n must be integral, we calculate $S(4) = 2.2528$ and $S(5) = 2.2528$. These are equal as the calculated formula shows they must be. However, they are both less than 3 and Muggins would be well advised to keep his \$3 in his pocket; if he will not take this advice then he should probably choose $n = 5$ and lose his money marginally more slowly.

> ***30.28*** *A husband and wife decide that their family will be complete when it includes two boys and two girls – but that this would then be enough! The probability that a new baby will be a girl is p. Ignoring the possibility of identical twins, show that the expected size of their family is*
>
> $$2\left(\frac{1}{pq} - 1 - pq\right),$$
>
> *where $q = 1 - p$.*

The 'experiment' will end after n 'trials' if the previous $n-1$ trials have produced either $n-2$ boys and 1 girl or $n-2$ girls and one boy, and the nth trial produces the girl or boy (respectively) needed to complete the desired family. These two situations have respective probabilities

$$({}^{n-1}C_1\, pq^{n-2})p \quad \text{and} \quad ({}^{n-1}C_1\, qp^{n-2})q.$$

Thus the probability that the size of the family is n is

$${}^{n-1}C_1\, p^2q^{n-2} + {}^{n-1}C_1\, q^2p^{n-2} = (n-1)(p^2q^{n-2} + q^2p^{n-2}).$$

Averaging this over all possible values of n (≥ 4) gives the expected size as

$$\langle n \rangle = \sum_{n=4}^{\infty} n(n-1)(p^2q^{n-2} + q^2p^{n-2}).$$

Now, from the formula for the sum of a geometric series, we have

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Differentiating this (twice) with respect to r gives

$$\sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2},$$

$$\sum_{n=2}^{\infty} n(n-1)r^{n-2} = \frac{2}{(1-r)^3}.$$

So, as the minimum size of their family is 4,

$$\begin{aligned}
\langle n \rangle &= \sum_{n=4}^{\infty} n(n-1)(p^2q^{n-2} + q^2p^{n-2}) \\
&= p^2\left[\frac{2}{(1-q)^3} - (2)(1)q^0 - (3)(2)q^1\right] \\
&\qquad\qquad + q^2\left[\frac{2}{(1-p)^3} - (2)(1)p^0 - (3)(2)p^1\right] \\
&= p^2\left[\frac{2}{p^3} - 2 - 6q\right] + q^2\left[\frac{2}{q^3} - 2 - 6p\right] \\
&= 2\left[\frac{1}{p} + \frac{1}{q} - p^2 - q^2 - 3pq(p+q)\right] \\
&= 2\left[\frac{p+q}{pq} - (1^2 - 2pq) - 3pq(1)\right] \\
&= 2\left[\frac{1}{pq} - 1 - pq\right],
\end{aligned}$$

as given in the question.

30.30 *A shopper buys* 36 *items at random in a supermarket where, because of the sales tax imposed, the final digit (the number of pence) in the price is uniformly and randomly distributed from* 0 *to* 9. *Instead of adding up the bill exactly she rounds each item to the nearest* 10 *pence, rounding up or down with equal probability if the price ends in a '5'. Should she suspect a mistake if the cashier asks her for* 23 *pence more than she estimated?*

The probability distribution for the rounding (in pence) is

$$\Pr(i) = p_i = \begin{cases} \frac{1}{10} & -4 \le i \le 4 \\ \frac{1}{20} & i = 5,\ i = -5 \end{cases}$$

This clearly has mean $\mu = 0$ and the variance is given by

$$\sigma^2 = \sum_{i=-5}^{5} i^2 p_i - (\mu)^2 = 0 + \frac{2}{10}(1^2 + 2^2 + 3^2 + 4^2) + \frac{2}{20}\,5^2 - 0 = \frac{17}{2}.$$

The standard deviation for 36 items is therefore $\sqrt{36}\sigma = 6\sqrt{17/2} = 17.49$. The extra 23 pence asked for is therefore only $23/17.49 = 1.31$ s.d. and, for such a bill of items, this discrepancy can be expected to be exceeded (either way) about 20% of the time.

> ***30.32*** *In a certain parliament the government consists of* 75 *New Socialites and the opposition consists of* 25 *Preservatives. Preservatives never change their mind, always voting against government policy without a second thought; New Socialites vote randomly, but with probability p that they will vote for their party leader's policies.*
>
> *Following a decision by the New Socialites' leader to drop certain manifesto commitments, N of his party decide to vote consistently with the opposition. The leader's advisors reluctantly admit that an election must be called if N is such that, at any vote on government policy, the chance of a simple majority in favour would be less than* 80%. *Given that* $p = 0.8$, *estimate the lowest value of N that would precipitate an election.*

From interpolation in the tabulation of the CPF for the Gaussian distribution the value of z for which $\Phi(z) = 0.8$ is 0.841. It follows that the chance of a defeat becomes more than 20% when the difference between 50 and the predictable number of anti-government votes is reduced to 0.841 times the standard deviation in the number of pro-government votes.

The number of assured anti-government votes is $25 + N$ whilst the remaining number of unpredictable New Socialites is $75 - N$. For these members of parliament, voting is a series of Bernoulli trials with a probability that they will vote with the opposition of $1 - p$. The mean number of votes they will cast with the opposition is therefore $(1 - p)(75 - N)$. The standard deviations of the number of their votes cast either with or against the government are the same and equal to $\sqrt{(75 - N)p(1 - p)}$.

An election will be precipitated if

$$25 + N + (1 - p)(75 - N) + 0.841\sqrt{(75 - N)p(1 - p)} = 50$$

Setting $p = 0.8$ and rearranging gives

$$\begin{aligned}
0.841\sqrt{0.16(75 - N)} &= 10 - 0.8N, \\
(0.7073)(0.16)(75 - N) &= 100 - 16N + 0.64N^2, \\
0.64N^2 - 15.89N + 91.51 &= 0, \\
\Rightarrow \quad N &= 9.08 \text{ or } 15.75.
\end{aligned}$$

The second value corresponds to taking -0.841 standard deviations and is not relevant here. The conclusion is that 10 rebel New Socialites would be enough to precipitate an election.

> ***30.34*** *The random variables X and Y take integer values ≥ 1 such that $2x+y \leq 2a$, where a is an integer greater than 1. The joint probability within this region is given by*
>
> $$\Pr(X = x, Y = y) = c(2x + y),$$
>
> *where c is a constant, and it is zero elsewhere.*
>
> *Show that the marginal probability $\Pr(X = x)$ is*
>
> $$\Pr(X = x) = \frac{6(a-x)(2x+2a+1)}{a(a-1)(8a+5)},$$
>
> *and obtain expressions for $\Pr(Y = y)$, (a) when y is even and (b) when y is odd. Show further that*
>
> $$E[Y] = \frac{6a^2+4a+1}{8a+5}.$$
>
> [*You will need the results about series involving the natural numbers given in subsection 4.2.5.*]

Since the boundary of the region is $2x + y \leq 2a$, the maximal value of y for any fixed x will be even. The marginal probability $\Pr(X = x)$ is obtained by summing over the probabilities for all the allowed values of y in the range $1 \leq y \leq 2a-2x$, i.e.

$$\begin{aligned}
\Pr(X = x) &= \sum_{y=1}^{2a-2x} c(2x + y) \\
&= 2cx(2a - 2x) + c\,\tfrac{1}{2}(2a - 2x)(2a - 2x + 1) \\
&= 2c(a - x)(2x + a - x + \tfrac{1}{2}) \\
&= c(a - x)(2x + 2a + 1).
\end{aligned}$$

However, the overall normalisation requires that $\sum_{x=1}^{a-1} \Pr(X = x)$ must be equal to unity:

$$\begin{aligned}
1 &= \sum_{x=1}^{a-1} \Pr(X = x) \\
&= \sum_{x=1}^{a-1} c(a - x)(2x + 2a + 1) = c \sum_{x=1}^{a-1} 2a^2 - 2x^2 + a - x \\
&= c\,[\,2a^2(a-1) - \tfrac{2}{6}(a-1)(a)(2a-1) + a(a-1) - \tfrac{1}{2}(a-1)a\,] \\
&= ca(a-1)[\,2a - \tfrac{1}{3}(2a-1) + 1 - \tfrac{1}{2}\,] \\
&= \tfrac{1}{6}ca(a-1)(8a+5).
\end{aligned}$$

The normalisation condition therefore requires that

$$c = \frac{6}{a(a-1)(8a+5)},$$
$$\Rightarrow \quad \Pr(X = x) = \frac{6(a-x)(2x+2a+1)}{a(a-1)(8a+5)}.$$

(a) If y is even then, for the calculation of the marginal probability $\Pr(Y = y)$, the largest value of x to be included in the sum over x lies on the boundary of the region at $x = \frac{1}{2}(2a - y)$. The sum is therefore

$$\begin{aligned}\Pr(y) &= \sum_{x=1}^{(2a-y)/2} c(2x+y)\\ &= 2c\,\frac{1}{2}\,\frac{2a-y}{2}\,\frac{2a-y+2}{2} + cy\,\frac{2a-y}{2}\\ &= \frac{1}{4}c(2a-y)(2a-y+2+2y)\\ &= \frac{3(2a-y)(2a+y+2)}{2a(a-1)(8a+5)}.\end{aligned}$$

(b) When y is odd the largest value of x does *not* lie on the boundary but is given by $\frac{1}{2}(2a - y - 1)$. Hence

$$\begin{aligned}\Pr(y) &= \sum_{x=1}^{(2a-y-1)/2} c(2x+y)\\ &= 2c\,\frac{1}{2}\,\frac{2a-y-1}{2}\,\frac{2a-y+1}{2} + cy\,\frac{2a-y-1}{2}\\ &= \frac{1}{4}c(2a-y-1)(2a-y+1+2y)\\ &= \frac{3(2a-y-1)(2a+y+1)}{2a(a-1)(8a+5)}.\end{aligned}$$

The mean value $E[\,Y\,]$ is equal to the sum $\sum_y y\Pr(Y = y)$, the minimum value of y being 1 and the maximum value $2a-2$, i.e. there are an even number of terms. We group the values in pairs, $y = 2m - 1$ and $y = 2m$, for $m = 1, 2, \ldots, a-1$. Denoting by k the constant $3/[\,2a(a-1)(8a+5)\,]$, we have that

$$\begin{aligned}E[\,Y\,] &= \sum_{m=1}^{a-1}[\,k(2a-2m)(2a+2m)(2m-1) + k(2a-2m)(2a+2m+2)2m\,]\\ &= 4k\sum_{m=1}^{a-1}[\,(a^2-m^2)(2m-1) + (a^2-m^2+a-m)2m\,]\\ &= 4k\sum_{m=1}^{a-1}[\,-a^2 + (4a^2+2a)m - m^2 - 4m^3\,].\end{aligned}$$

This sum may be evaluated by using the formulae for the sums of the powers of the natural numbers (see chapter 4), and reads

$$\begin{aligned}E[\,Y\,] &= 4k\left[-a^2\,(a-1) + 2a(2a+1)\,\frac{1}{2}(a-1)a \right.\\ &\qquad\left. -\ \frac{1}{6}(a-1)a(2a-1) - 4\,\frac{1}{4}(a-1)^2a^2\right]\\ &= 4k(a-1)(-a^2 + 2a^3 + a^2 - \tfrac{1}{3}a^2 + \tfrac{1}{6}a - a^3 + a^2)\\ &= \frac{4k(a-1)a}{6}(6a^2+4a+1)\\ &= \frac{3}{2a(a-1)(8a+5)}\,\frac{4(a-1)a}{6}(6a^2+4a+1)\\ &= \frac{6a^2+4a+1}{8a+5}.\end{aligned}$$

30.36 *A discrete random variable X takes integer values $n = 0, 1, \ldots, N$ with probabilities p_n. A second random variable Y is defined as $Y = (X-\mu)^2$, where μ is the expectation value of X. Prove that the covariance of X and Y is given by*

$$Cov[X, Y] = \sum_{n=0}^{N} n^3 p_n - 3\mu \sum_{n=0}^{N} n^2 p_n + 2\mu^3.$$

Now suppose that X takes all its possible values with equal probability and hence demonstrate that two random variables can be uncorrelated even though one is defined in terms of the other.

The covariance of X and Y is given by

$$\begin{aligned}\mathrm{Cov}[X, Y] &= E[XY] - E[X]E[Y]\\ &= \sum_{n=0}^{N}([\,n(n-\mu)^2 p_n\,] - \mu \sum_{n=0}^{N}(n-\mu)^2 p_n\\ &= \sum_{n=0}^{N} n^3 p_n - 2\mu \sum_{n=0}^{N} n^2 p_n + \mu^2 \sum_{n=0}^{N} n p_n\\ &\qquad - \mu \sum_{n=0}^{N} n^2 p_n + 2\mu^2 \sum_{n=0}^{N} n p_n - \mu^3 \sum_{n=0}^{N} p_n.\end{aligned}$$

But $\sum_{n=0}^{N} np_n = \mu$ and $\sum_{n=0}^{N} p_n = 1$, and so

$$\mathrm{Cov}[X, Y] = \sum_{n=0}^{N} n^3 p_n - 3\mu \sum_{n=0}^{N} n^2 p_n + 2\mu^3.$$

Now suppose that $p_n = (N+1)^{-1}$ for all values of n. In this case, the mean $\mu = N/2$ and, using the sums of the first, second and third powers of the natural numbers derived in subsection 4.2.5, we have

$$\begin{aligned}\text{Cov}[X,Y] &= \frac{N^2(N+1)^2}{4(N+1)} - \frac{3N}{2}\,\frac{N(N+1)(2N+1)}{6(N+1)} + \frac{2N^3}{8}\\ &= \frac{N^2}{4}[\,N+1-2N-1+N\,] = 0.\end{aligned}$$

Thus, as their covariance is zero, the random variables X and Y are uncorrelated – even though Y is defined in terms of X.

> ***30.38*** *A continuous random variable X is uniformly distributed over the interval $[-c,c]$. A sample of $2n+1$ values of X is selected at random and the random variable Z is defined as the* median *of that sample. Show that Z is distributed over $[-c,c]$ with probability density function*
>
> $$f_n(z) = \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}(c^2-z^2)^n.$$
>
> *Find the variance of Z.*

For the median of the sample of $2n+1$ values of X to lie in the interval $z \to z+dz$ we require that n values lie in the range $-c \le X < z$, n lie in the range $z+dz < X \le c$ and one is in the interval $z \le X \le z+dz$. We are thus considering a multinomial distribution and, as all the sample values in any one interval are equivalent, the probability density function is

$$\begin{aligned}f_n(z)\,dz &= \frac{(2n+1)!}{n!\,n!\,1!}\left(\frac{z+c}{2c}\right)^n\left(\frac{c-z}{2c}\right)^n\left(\frac{dz}{2c}\right)^1\\ &= \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}(c^2-z^2)^n\,dz\\ &\equiv A_n(c^2-z^2)^n\,dz,\end{aligned}$$

where A_n is defined (for any $n \ge 1$) by

$$A_n\int_{-c}^{c}(c^2-z^2)^n\,dz = 1.$$

Now, from symmetry, it is clear that $E[Z]=0$ and so the variance of Z is given by

$$\begin{aligned}
V[Z] &= E[Z^2] - (E[Z])^2 \\
&= A_n \int_{-c}^{c} z^2 (c^2 - z^2)^n \, dz - 0 \\
&= A_n \left[-\frac{z}{2} \frac{(c^2 - z^2)^{n+1}}{n+1} \right]_{-c}^{c} + \frac{A_n}{2} \int_{-c}^{c} \frac{(c^2 - z^2)^{n+1}}{n+1} \, dz \\
&= 0 + \frac{A_n}{2(n+1)} \frac{1}{A_{n+1}} \\
&= \frac{(2n+1)!}{2(n+1)\,(n!)^2\,(2c)^{2n+1}} \frac{[\,(n+1)!\,]^2\,(2c)^{2n+3}}{(2n+3)!} \\
&= \frac{(n+1)^2\,4c^2}{2(n+1)(2n+2)(2n+3)} \\
&= \frac{c^2}{2n+3}.
\end{aligned}$$

We note that this result has been obtained without having to explicitly evaluate the integrals involved.

30.40 *The variables X_i, $i = 1, 2, \ldots, n$, are distributed as a multivariate Gaussian, with means μ_i and a covariance matrix V. If the X_i are required to satisfy the linear constraint $\sum_{i=1}^{n} c_i X_i = 0$, where the c_i are constants (and not all equal to zero), show that the variable*

$$\chi_n^2 = (\mathsf{x} - \mu)^{\mathrm{T}} \mathsf{V}^{-1} (\mathsf{x} - \mu)$$

follows a chi-squared distribution of order $n - 1$.

As shown in the text, the PDF of the multivariate Gaussian can be written

$$f(\mathsf{x}) = \frac{1}{(2\pi)^{n/2}\,|V|^{1/2}} \exp\left[-\frac{1}{2} (\mathsf{x} - \mu)^{\mathrm{T}} \mathsf{V}^{-1} (\mathsf{x} - \mu) \right].$$

Now let S be the orthogonal matrix whose normalised columns are the eigenvectors of V with corresponding eigenvalues λ_i and define new variables y_i by $\mathsf{y} = \mathsf{S}^{\mathrm{T}}(\mathsf{x} - \mu)$. Using the fact that $\mathsf{S}\mathsf{S}^{\mathrm{T}} = \mathsf{I}$, the argument of the exponential function becomes

$$\begin{aligned}
-\frac{1}{2} (\mathsf{x} - \mu)^{\mathrm{T}} \mathsf{V}^{-1} (\mathsf{x} - \mu) &= -\frac{1}{2} (\mathsf{x} - \mu)^{\mathrm{T}} \mathsf{S}\,\mathsf{S}^{\mathrm{T}} \mathsf{V}^{-1} \mathsf{S}\,\mathsf{S}^{\mathrm{T}} (\mathsf{x} - \mu) \\
&= -\frac{1}{2} [\,\mathsf{S}^{\mathrm{T}}(\mathsf{x} - \mu)\,]^{\mathrm{T}}\, \mathsf{S}^{\mathrm{T}} \mathsf{V}^{-1} \mathsf{S}\, [\,\mathsf{S}^{\mathrm{T}}(\mathsf{x} - \mu)\,] \\
&= -\frac{1}{2} \mathsf{y}^{\mathrm{T}} \operatorname{diag}(\lambda_i^{-1})\, \mathsf{y}.
\end{aligned}$$

A further scaling of the variables, $z_i = y_i/\sqrt{\lambda_i}$, reduces the argument to $-\frac{1}{2}\chi_n^2 = -\frac{1}{2}\sum_{i=1}^{n} z_i^2$ and

$$\begin{aligned} f(\mathsf{z})\, d^n\mathsf{z} &= f(z_1, z_2, \ldots, z_n)\, dz_1\, dz_2 \cdots dz_n \\ &= \frac{1}{(2\pi)^{n/2}\, |V|^{1/2}} \exp(-\frac{1}{2}\chi_n^2)\, dV_\chi, \end{aligned}$$

where $dz_1\, dz_2 \cdots dz_n$ is the infinitesimal volume enclosed by the intersection of the n-dimensional spherical shell of radius χ_n^2 and thickness $d\chi_n^2$ with the $(n-1)$-dimensional hyperplane $\sum_{i=1}^{n} c_i x_i = 0$.

From the way each y_i was constructed, both it and z_i have zero means; the scaling from y_i to z_i ensures that z_i has unit variance.

Under the successive transformations the linear constraint $\sum_{i=1}^{n} c_i x_i = 0$, with not all $c_i = 0$, will become another linear constraint $\sum_{i=1}^{n} c_i' Z_i = 0$, again with not all $c_i' = 0$ (since the λ_i are neither zero nor infinite).

The constraint can be incorporated by writing one z_k for which $c_k' \neq 0$ in terms of the others:

$$\chi_n^2 = z_1^2 + z_2^2 + \ldots + \left(-\sum_{j \neq k}^{n} \frac{c_j}{c_k} z_j \right)^2 + \ldots + z_n^2.$$

A further transformation can then be made to carry this into the form

$$\chi_n^2 = v_1^2 + v_2^2 + \cdots + v_{n-1}^2,$$

which can be considered as the square of the distance from the origin in the $(n-1)$-dimensional **V**-space. In this space the element of volume is

$$dV_\chi = A\chi_n^{n-2}\, d\chi_n = A\chi_n^{n-2} \frac{d(\chi_n^2)}{2\chi_n} = \frac{A}{2} (\chi_n^2)^{(n-3)/2}\, d\chi_n^2.$$

Collecting these results together gives

$$h(\chi_n^2)\, d\chi_n^2 \propto (\chi_n^2)^{(n-3)/2} \exp(-\frac{1}{2}\chi_n^2)\, d\chi_n^2,$$

i.e. χ_n^2 follows a chi-squared distribution of order $n-1$.

31

Statistics

> ***31.2*** *Measurements of a certain quantity gave the following values:* 296, 316, 307, 278, 312, 317, 314, 307, 313, 306, 320, 309. *Within what limits would you say there is a* 50% *chance that the correct value lies*?

Since all the other readings are within ± 12 of 308 and the reading of 278 is 30 away from this value, it should probably be rejected, as erroneous rather than a statistical fluctuation.

The other readings do not look as though they are Gaussian distributed and the best estimate is probably obtained by considering the distribution as approximating to a uniform distribution and using the inter-quartile range of the remaining 11 readings. Arranged in order, they are

296, 306, 307, 307, 309, 312, 313, 314, 316, 317, 320,

and their mean is 310.6.

This number of readings does not divide into four equal-sized groups and the perhaps over-cautious approach is to discard only two readings from each end of the range i.e. give the range in which the correct value lies with 50% probability as 307–316. An additional reading would probably have justified discarding three reading from each end.

31.4 *Two physical quantities x and y are connected by the equation*

$$y^{1/2} = \frac{x}{ax^{1/2} + b},$$

and measured pairs of values for x and y are as follows:

x:	10	12	16	20
y:	409	196	114	94.

Determine the best values for a and b by graphical means and (either by hand or by using a built-in calculator routine) by a least-squares fit to an appropriate straight line.

We aim to put this equation into a 'straight-line' form. One way to do this is to re-arrange it as

$$\frac{x}{y^{1/2}} = ax^{1/2} + b$$

and plot $(x/y^{1/2})$ against $x^{1/2}$. The slope of the graph will give a and its intercept on the $(x/y^{1/2})$-axis will give b. We therefore tabulate the required quantities:

x	10	12	16	20
y	409	196	114	94
$x^{1/2}$	3.16	3.46	4.00	4.47
$x/y^{1/2}$	0.494	0.857	1.499	2.063

Plotting the graph over the range $3.0 \leq x^{1/2} \leq 4.5$ gives a good straight line of slope $(2.09 - 0.31)/(4.50 - 3.00) = 1.19$. Thus $a = 1.19$. The fit to the line is sufficiently good that it is hard to estimate the uncertainty in a and a least-squares fit would result in a small but virtually meaningless value.

However, the measured values of $x^{1/2}$ are bunched in a range that is small compared to their distance from the $(x/y^{1/2})$-axis, where the intercept is b. Such a long graphical extrapolation could result in a serious error in the value of b. It is better to calculate b using the straight-line values at one point (say $x/y^{1/2} = 0.31$ at $x^{1/2} = 3.00$) and the slope just found:

$$b = 0.31 - (1.19 \times 3.00) = -3.26.$$

An alternative is to re-arrange the original equation as

$$\frac{x^{1/2}}{y^{1/2}} = a + \frac{b}{x^{1/2}}$$

and then plot values from the following table,

x	10	12	16	20
y	409	196	114	94
$x^{-1/2}$	0.316	0.288	0.250	0.223
$(x/y)^{1/2}$	0.156	0.247	1.375	2.461

over the range $0.200 \leq x^{-1/2} \leq 0.330$. An equally good straight-line fit is obtained with a slope, this time being equal to b (rather than a), of $(0.110-0.534)/(0.330-0.200) = -3.26$. A similar calculation to that used earlier now determines a as $0.534 + (3.26 \times 0.200) = 1.19$.

31.6 *Prove that the sample mean is the best* linear unbiased estimator *of the population mean* μ *as follows.*

(a) *If the real numbers* $a_1, a_2, \ldots, a_n$ *satisfy the constraint* $\sum_{i=1}^n a_i = C$*, where* C *is a given constant, show that* $\sum_{i=1}^n a_i^2$ *is minimised by* $a_i = C/n$ *for all* i*.*
(b) *Consider the linear estimator* $\hat{\mu} = \sum_{i=1}^n a_i x_i$*. Impose the conditions* (i) *that it is* unbiased*, and* (ii) *that it is as* efficient *as possible.*

(a) To minimise $S = \sum_{i=1}^n a_i^2$ subject to the constraint $\sum_{i=1}^n a_i = C$, we introduce a Lagrange multiplier and consider

$$\begin{aligned} T &= \sum_{i=1}^{n} a_i^2 - \lambda \sum_{i=1}^{n} a_i, \\ 0 = \frac{\partial T}{\partial a_i} &= 2a_i - \lambda \\ &\Rightarrow \quad a_i = \tfrac{1}{2}\lambda, \text{ for all } i. \end{aligned}$$

Re-substitution in the constraint gives $C = \frac{1}{2}n\lambda$, leading to $a_i = C/n$ for all i. The corresponding minimum value of S is C^2/n.

(b) If the sample values x_i are drawn from a population with mean μ and variance σ^2, consider the linear estimator $\hat{\mu} = \sum_{i=1}^n a_i x_i$. For the estimator to be *unbiased* we require that

$$\begin{aligned} 0 = \langle \hat{\mu} - \mu \rangle = \left\langle \sum_{i=1}^{n} a_i x_i - \mu \right\rangle &= \sum_{i=1}^{n} a_i \langle x_i \rangle - \mu \\ = \sum_{i=1}^{n} a_i \mu - \mu &= \mu \left(\sum_{i=1}^{n} a_i - 1 \right). \end{aligned}$$

Thus the first requirement is that $\sum_{i=1}^n a_i = 1$.

Now we add the further requirement of *efficiency* by minimising the variance of $\hat{\mu}$. The expression for the variance is

$$\begin{aligned}
\langle(\hat{\mu}-\mu)^2\rangle &= \left\langle \left(\sum_{i=1}^{n} a_i(\mu+z_i) - \mu \right)^2 \right\rangle, \text{ with } \langle z_i\rangle = 0 \text{ and } \langle z_i^2\rangle = \sigma^2, \\
&= \left\langle \left(\sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} a_i\mu - \mu \right)^2 \right\rangle, \\
&= \left\langle \left(\sum_{i=1}^{n} a_i z_i + \mu - \mu \right)^2 \right\rangle, \text{ since } \sum_{i=1}^{n} a_i = 1, \\
&= \left\langle \left(\sum_{i=1}^{n} a_i z_i \right)^2 \right\rangle \\
&= \sum_{i=1}^{n} a_i^2\,\sigma^2, \text{ since } \langle z_i^2\rangle = \sigma^2 \text{ and the } z_i \text{ are independent.}
\end{aligned}$$

Now, from part (a), this expression is minimised subject to $\sum_{i=1}^{n} a_i = 1$ when $a_i = 1/n$ for all i, i.e. when $\hat{\mu}$ is taken as the mean of the sample. The minimum value for the variance is σ^2/n. This completes the proof that the sample mean is the best *linear unbiased estimator* of the population mean μ.

> ***31.8*** *Carry through the following proofs of statements made in subsections 31.5.2 and 31.5.3 about the ML estimators $\hat{\tau}$ and $\hat{\lambda}$.*
>
> (a) *Find the expectation values of the ML estimators $\hat{\tau}$ and $\hat{\lambda}$ given respectively in (31.71) and (31.75). Hence verify equations (31.76), which show that, even though an ML estimator is unbiased, it does not follow that functions of it are also unbiased.*
> (b) *Show that $E[\hat{\tau}^2] = (N+1)\tau^2/N$ and hence prove that $\hat{\tau}$ is a minimum-variance estimator of τ.*

(a) As shown in the text [equation (27.67)] the likelihood of the measured intervals x_k is

$$\prod_{k=1}^{N} \frac{1}{\tau} \exp\left(-\frac{x_k}{\tau}\right) = \frac{1}{\tau^N} \exp\left[-\frac{1}{\tau}(x_1 + x_2 + \cdots + x_N)\right].$$

The expectation value $E[\hat{\tau}]$ of the estimator $\hat{\tau} = N^{-1}\sum_{i=1}^{N} x_i$ is therefore

$$\frac{1}{N}\sum_{i=1}^{N} \int \cdots \int x_i \frac{1}{\tau^N} \exp\left[-\frac{1}{\tau}(x_1 + x_2 + \cdots + x_N)\right] dx_1\, dx_2 \cdots dx_N.$$

In each term of the sum we can carry out the integrations over all the x_k variables except the one with $k = i$ (each gives τ) thereby reducing the sum to

$$\begin{aligned} E[\hat{\tau}] &= \frac{1}{N}\sum_{i=1}^{N} x_i \frac{1}{\tau}\, e^{-x_i/\tau}\, dx_i \\ &= \frac{1}{N}\sum_{i=1}^{N}\left(\left[-\frac{x_i}{\tau}\,\tau\, e^{-x_i/\tau}\right]_0^{\infty} + \int_0^{\infty} e^{-x_i/\tau}\, dx_i\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\tau = \frac{1}{N}\, N\tau = \tau, \text{ as expected.} \end{aligned}$$

We note that this estimator is unbiased and now turn to the expectation value of the estimator

$$\hat{\lambda} = \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^{-1} = \bar{x}^{-1}.$$

For typographical clarity we will omit explicit limits from the sum $\sum_{i=1}^{N} x_i$ where it appears in the equations that follow.

$$\begin{aligned} E[\hat{\lambda}] &= \int \cdots \int \left(\frac{N}{\sum x_i}\right) \lambda e^{-\lambda x_1}\, \lambda e^{-\lambda x_2} \cdots \lambda e^{-\lambda x_N}\, dx_1\, dx_2 \cdots dx_N \\ &= \int \frac{N}{\sum x_i}\, \lambda^N\, e^{-\lambda \sum x_i}\, d^N x_i. \end{aligned}$$

To evaluate this integral we differentiate both sides of its definition with respect to λ. The RHS is a product of two functions of λ; differentiating it produces one term in which $\lambda^N \to N\lambda^{N-1}$ and the other produces a factor that cancels the $\sum x_i$ in the denominator. The result is

$$\begin{aligned} \frac{dE[\hat{\lambda}]}{d\lambda} &= \frac{N}{\lambda}\, E[\hat{\lambda}] - N\int \lambda^N\, e^{-\lambda\sum x_i}\, d^N x_i \\ &= \frac{N}{\lambda}\, E[\hat{\lambda}] - N, \end{aligned}$$

since the distribution function for each x_i is normalised (they are all the same). The integrating factor for this first-order equation is λ^{-N} giving

$$\frac{d}{d\lambda}\left(\frac{E}{\lambda^N}\right) = -\frac{N}{\lambda^N} \quad \Rightarrow \quad \frac{E}{\lambda^N} = \frac{N}{(N-1)\lambda^{N-1}} + c.$$

We must have $E[\hat{\lambda}] \to \lambda$ as $N \to \infty$ and so $c = 0$, yielding

$$E[\hat{\lambda}] = \frac{N}{N-1}\,\lambda.$$

Thus, although the bias tends to zero as $N \to \infty$, $\hat{\lambda}$ is a biased estimator of λ. Since it is directly given as the reciprocal of $\hat{\tau}$, the two results obtained, taken together, show that even though an ML estimator is unbiased, it does not follow that functions of it are also unbiased.

(b) We start by using the Fisher inequality to determine the minimum variance that any estimator of τ *could* have; for this we need $\ln P(\mathbf{x}|\tau)$. This is given by

$$\ln P(\mathbf{x}|\tau) = \ln\left[\prod_{i=1}^{N}\left(\frac{e^{-x_i/\tau}}{\tau}\right)\right] = -\sum_{i=1}^{N}\left(\ln\tau + \frac{1}{\tau}x_i\right).$$

Hence,

$$\begin{aligned} E\left[-\frac{\partial^2}{\partial\tau^2}\ln P\right] &= E\left[\sum_{i=1}^{N}\left(-\frac{1}{\tau^2}+\frac{2}{\tau^3}\,x_i\right)\right] \\ &= -\frac{N}{\tau^2}+\frac{2N\tau}{\tau^3} = \frac{N}{\tau^2}. \end{aligned}$$

We have already shown that the estimator is unbiased; thus $\partial b/\partial\tau = 0$ and Fisher's inequality reads

$$V[\,\hat{\tau}\,] \geq \frac{1}{N/\tau^2} = \frac{\tau^2}{N}.$$

Next we compute

$$E[\,\hat{\tau}^2\,] = \int\cdots\int \frac{1}{N^2}\left(\sum x_i\right)^2 \frac{1}{\tau^N}\,e^{-(\sum x_i)/\tau}\,dx_1\,dx_2,\cdots dx_N.$$

We now separate off the N terms in the square of the sum that contain factors typified by x_i^2 from the $N(N-1)$ terms containing factors typified by $x_i x_j$ with $i \neq j$. All integrals over sample values not involving i, or i and j, (as the case may be) integrate to τ. Within each group all integrals have the same value and so we can write

$$\begin{aligned} E[\,\hat{\tau}^2\,] &= \frac{1}{N^2}\left[N\int_0^\infty \frac{x^2}{\tau}\,e^{-x/\tau}\,dx \right.\\ &\qquad \left. +\; N(N-1)\int_0^\infty\int_0^\infty \frac{x_1}{\tau}\,\frac{x_2}{\tau}\,e^{-x_1/\tau}\,e^{-x_2/\tau}\,dx_1\,dx_2\right] \\ &= \frac{1}{N^2}\,[\,2N\tau^2 + N(N-1)\tau^2\,] \\ &= \frac{N+1}{N}\,\tau^2. \end{aligned}$$

Finally, the variance of $\hat{\tau}$ is calculated as

$$V[\,\hat{\tau}\,] = E[\,\hat{\tau}^2\,] - (E[\,\hat{\tau}\,])^2 = \frac{N+1}{N}\,\tau^2 - \tau^2 = \frac{\tau^2}{N}.$$

This is equal to the minimum allowed by the Fisher inequality; thus $\hat{\tau}$ is a minimum-variance estimator of τ.

31.10 *This exercise is intended to illustrate the dangers of applying formalised estimator techniques to distributions that are not well behaved in a statistical sense.*

The following are five sets of 10 values, all drawn from the same Cauchy distribution with parameter a.

(i)	4.81	−1.24	1.30	−0.23	2.98
	−1.13	−8.32	2.62	−0.79	−2.85
(ii)	0.07	1.54	0.38	−2.76	−8.82
	1.86	−4.75	4.81	1.14	−0.66
(iii)	0.72	4.57	0.86	−3.86	0.30
	−2.00	2.65	−17.44	−2.26	−8.83
(iv)	−0.15	202.76	−0.21	−0.58	−0.14
	0.36	0.44	3.36	−2.96	5.51
(v)	0.24	−3.33	−1.30	3.05	3.99
	1.59	−7.76	0.91	2.80	−6.46

Ignoring the fact that the Cauchy distribution does not have a finite variance (or even a formal mean), show that $\hat{a}$, the ML estimator of a, has to satisfy

$$s(\hat{a}) = \sum_{i=1}^{10} \frac{1}{1 + x_i^2/\hat{a}^2} = 5. \quad (*)$$

Using a programmable calculator, spreadsheet or computer, find the value of $\hat{a}$ that satisfies $(*)$ for each of the data sets and compare it with the value $a = 1.6$ used to generate the data. Form an opinion regarding the variance of the estimator.

Show further that if it is assumed that $(E[\hat{a}])^2 = E[\hat{a}^2]$ then $E[\hat{a}] = \nu_2^{1/2}$, where ν_2 is the second (central) moment of the distribution, which for the Cauchy distribution is infinite!

The Cauchy distribution with parameter a has the form

$$f(x) = \frac{a}{\pi} \frac{1}{a^2 + x^2}.$$

It follows that the likelihood function for 10 sample values is

$$L(\mathbf{x}|a) = \left(\frac{a}{\pi}\right)^{10} \prod_{i=1}^{10} \frac{1}{a^2 + x_i^2},$$

and that the log-likelihood function

$$\ln L = -10 \ln \pi + 10 \ln a - \sum_{i=1}^{10} \ln(a^2 + x_i^2).$$

The equation satisfied by the ML estimator $\hat{a}$ is therefore

$$0 = \frac{\partial(\ln L)}{\partial a} = \frac{10}{a} - \sum_{i=1}^{10} \frac{2a}{a^2 + x_i^2} \quad \Rightarrow \quad s(\hat{a}) = \sum_{i=1}^{10} \frac{1}{1 + x_i^2/\hat{a}^2} = 5.$$

Using a simple spread sheet to calculate the sum on the LHS for various assumed values of a and then manual or automated interpolation to make the sum equal to 5, the following values for $\hat{a}$ are obtained for the five sets of data:

(i) 1.85, (ii) 1.66, (iii) 2.46, (iv) 0.68, (v) 2.44.

Although the estimates have the correct order of magnitude, there is clearly a very large (perhaps infinite) sampling variance. Even if all 50 samples are combined, the resulting estimated value for a of 1.84 is 0.24 away from that used to generate the data.

It is clear that for sets of N sample values (∗) reads

$$\sum_{i=0}^{N} \frac{1}{\hat{a}^2 + x_i^2} = \frac{N}{2\hat{a}^2},$$

and we take this as the *definition* of $\hat{a}$. Multiplying both sides of this equation by $\prod_{k=1}^{N}(\hat{a}^2 + x_k^2)$, we obtain

$$\frac{N}{2\hat{a}^2} \prod_{k=1}^{N} (\hat{a}^2 + x_k^2) = \sum_{i=0}^{N} \prod_{k \neq i}^{N} (\hat{a}^2 + x_k^2).$$

Now we take expectation values over all the x_i, writing $E[\,x_i^r\,] = v_r$,

$$\frac{N}{2E[\,\hat{a}^2\,]} \left(E[\,\hat{a}^2\,] + v_2\right)^N = N \left(E[\,\hat{a}^2\,] + v_2\right)^{N-1}$$

$$E[\,\hat{a}^2\,] + v_2 = 2E[\,\hat{a}^2\,] \quad \Rightarrow \quad E[\,\hat{a}\,] = v_2^{1/2},$$

assuming that $E[\,\hat{a}\,]^2 = E[\hat{a}^2\,]$. As shown in exercise 31.8, this is not necessarily so, but any possible fractional bias is typically $\mathrm{O}(N^{-1})$. However, for the Cauchy distribution,

$$v_2 = \int_{-\infty}^{\infty} \frac{a}{\pi} \frac{x^2}{a^2 + x^2} \, dx = \infty.$$

This is rather more serious than an $\mathrm{O}(N^{-1})$ error and the statistically unsound procedure used leads to the false conclusion that the expected value of the estimator is infinite, when it ought to have a value equal to the finite parameter a of the sample distribution.

31.12 *On a certain (testing) steeplechase course there are* 12 *fences to be jumped and any horse that falls is not allowed to continue in the race. In a season of racing a total of* 500 *horses started the course and the following numbers fell at each fence:*

Fence:	1	2	3	4	5	6	7	8	9	10	11	12
Falls:	62	75	49	29	33	25	30	17	19	11	15	12

Use this data to determine the overall probability of a horse's falling at a fence, and test the hypothesis that it is the same for all horses and fences as follows.

(a) *Draw up a table of the expected number of falls at each fence on the basis of the hypothesis.*
(b) *Consider for each fence i the standardised variable*

$$z_i = \frac{\text{estimated falls} - \text{actual falls}}{\text{standard deviation of estimated falls}}$$

and use it in an appropriate χ^2 *test.*
(c) *Show that the data indicates that the odds against all fences being equally testing are about* 40 *to* 1. *Identify the fences that are significantly easier or harder than the average.*

(a) The information as presented does not give statistically independent data for each fence, as a horse that falls at an early fence cannot attempt a later one. To extract the necessary data we extend the table by adding rows for the number of attempts at each fence and the number of successful jumps there.

Fence:	1	2	3	4	5	6	
Falls:	62	75	49	29	33	25	
Clearances:	438	363	314	285	252	227	
Attempts:	500	438	363	314	285	252	
Fence:	7	8	9	10	11	12	Total
Falls:	30	17	19	11	15	12	377
Clearances:	197	180	161	150	135	123	2825
Attempts:	227	197	180	161	150	135	3202

On the hypothesis that all fences are equally difficult the best estimator of the probability p of a fall at any particular fence i is $377/3202 = 0.1177$, independent of i. If the number of attempts at fence i is n_i then the expected number of falls at that fence is $x_i = pn_i$. Since each attempt is a Bernoulli trial the s.d. of x_i is given by $\sqrt{n_i p(1-p)} = 0.3223\sqrt{n_i}$.

(b) We may now draw up a further table of the expected number of falls and of the standardised variable

$$z_i = \frac{\text{estimated falls} - \text{actual falls}}{\text{standard deviation of estimated falls}}$$

for each fence. The corresponding contribution to the overall χ^2 statistic is $\chi_i^2 = z_i^2$.

Fence:	1	2	3	4	5	6	
Falls:	62	75	49	29	33	25	
Estimated Falls:	58.9	51.6	42.7	37.0	33.6	29.7	
z_i:	-0.43	-3.47	-1.02	1.40	0.11	0.92	
χ_i^2:	0.2	12.0	1.0	2.0	0.0	0.8	
Fence:	7	8	9	10	11	12	Total
Falls:	30	17	19	11	15	12	377
Estimated Falls:	26.7	23.2	21.2	19.0	17.7	15.9	377
z_i:	-0.68	1.37	0.51	1.96	0.68	1.04	
χ_i:	0.5	1.9	0.3	3.8	0.5	1.1	24.1

Thus $\chi^2 = 24.1$ for $12 - 1 = 11$ degrees of freedom. This is close to the 99% limit and therefore it is *exceedingly unlikely* (odds of almost 100 to 1 against) that all fences are equally difficult and that the variations in the success rate are due to statistical fluctuations. Fence 2 is especially difficult, whilst fences 4, 8 and (particularly) 10 are easier than average.

A similar (slightly erroneous) calculation treating the number of falls as governed by a Poisson distribution (rather than each jump being a Bernoulli trial) gives a χ^2 value of 21.2 for 11 degrees of freedom and leads to the odds against uniform difficulty of the jumps of about 40 to 1.

31.14 *Three candidates X, Y and Z were standing for election to a vacant seat on their college's Student Committee. The members of the electorate (current first-year students, consisting of* 150 *men and* 105 *women) were each allowed to cross out the name of the candidate they least wished to be elected, the other two candidates then being credited with one vote each. The following data are known.*

(a) *X received* 100 *votes from men, whilst Y received* 65 *votes from women.*
(b) *Z received five more votes from men than X received from women.*
(c) *The total votes cast for X and Y were equal.*

Analyse this data in such a way that a χ^2 test can be used to determine whether voting was other than random (i) *amongst men, and* (ii) *amongst women.*

The numbers of votes cast for each candidate are not independent quantities since for each vote a candidate receives another candidate also receives a vote. The independent quantities are the numbers of times each name has been crossed out. We must first determine the latter quantities. Suppose that the correlation table for crossings out is

	Not X	Not Y	Not Z	Total
Men	a	c	e	150
Women	b	d	f	105
Total	?	?	?	255

As the questions to be answered deal with men and women's voting patterns separately, we do not need to estimate overall percentages; the theoretical expectation of the result of random voting is $\frac{1}{3} \times 150 = 50$ crossings out by men and $\frac{1}{3} \times 105 = 35$ by women for each candidate. The corresponding variances, for what are essentially Bernoulii trials, are $\frac{1}{3} \times \frac{2}{3} \times 150$ and $\frac{1}{3} \times \frac{2}{3} \times 105$.

To determine the values in the table we know that $a+c+e = 150$ and $b+d+f = 105$. Further, from the information (a) - (c) provided:

(a) $c + e = 100$ and $b + f = 65$,
(b) $a + c = d + f + 5$,
(c) $c + d + e + f = a + b + e + f$.

From these it follows (in approximately deducible order) that $a = 50$, $d = 40$, $5 + c = f$, $c = b + 10$, $(c - 10) + c + 5 = 65 \Rightarrow c = 35$, $b = 25$, $f = 40$ and $e = 65$.

To test for random voting amongst the men we calculate

$$\chi^2 = \frac{(50 - a)^2}{33.3} + \frac{(50 - c)^2}{33.3} + \frac{(50 - e)^2}{33.3} = 13.5$$

for $3 - 1 = 2$ d.o.f. Similarly for the women

$$\chi^2 = \frac{(35 - b)^2}{23.3} + \frac{(35 - d)^2}{23.3} + \frac{(35 - f)^2}{23.3} = 6.4$$

for 2 d.o.f.
The χ^2 value for the men is significantly greater, at almost the 0.1% level, than would be expected for random voting, making the latter extremely unlikely. The corresponding value for women voters is only significant at about the 5% level and random voting cannot be ruled out.

Incidentally, X and Y, who each received 180 votes, tied for first place and a (more conventional) run-off was needed!

31.16 *The function $y(x)$ is known to be a quadratic function of x. The following table gives the measured values and uncorrelated standard errors of y measured at various values of x (in which there is negligible error):*

x	1	2	3	4	5
$y(x)$	3.5 ± 0.5	2.0 ± 0.5	3.0 ± 0.5	6.5 ± 1.0	10.5 ± 1.0

Construct the response matrix R *using as basis functions* 1, x, x^2. *Calculate the matrix* $\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R}$ *and show that its inverse, the covariance matrix* V, *has the form*

$$\mathsf{V} = \frac{1}{9184}\begin{pmatrix} 12\,592 & -9708 & 1580 \\ -9708 & 8413 & -1461 \\ 1580 & -1461 & 269 \end{pmatrix}.$$

Use this matrix to find the best values, and their uncertainties, for the coefficients of the quadratic form for $y(x)$.

As the measured data has uncorrelated, but unequal, errors, the covariance matrix N, whilst being diagonal, will not be a multiple of the unit matrix; it will be

$$\mathsf{N} = \text{diag}(0.25, 0.25, 0.25, 1.0, 1.0).$$

Using as base functions the three functions $h_1(x) = 1$, $h_2(x) = x$ and $h_3(x) = x^2$, we calculate the elements of the 5×3 response matrix $\mathsf{R}_{ij} = h_j(x_i)$. To save space we display its 3×5 transpose:

$$\mathsf{R}^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}$$

Then

$$\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R} = \begin{pmatrix} 4 & 4 & 4 & 1 & 1 \\ 4 & 8 & 12 & 4 & 5 \\ 4 & 16 & 36 & 16 & 25 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 33 & 97 \\ 33 & 97 & 333 \\ 97 & 333 & 1273 \end{pmatrix}.$$

The determinant of the square matrix $\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R}$ is

$$\begin{aligned} &14[(97 \times 1273) - (333 \times 333)] + 33[(333 \times 97) - (33 \times 1273)] \\ &\quad + 97[(33 \times 333) - (97 \times 97)] \\ &= 14 \times 12592 - 33 \times 9708 + 97 \times 1580 = 9184. \end{aligned}$$

This is non-zero and so the matrix has an inverse. It is tedious to calculate the inverse V by the standard methods and it is just as good for practical purposes to verify the given form for V, knowing that it is unique. The following matrix equation, $\mathsf{V}\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R} = \mathsf{I}_3$, can be verified numerically

$$\frac{1}{9184}\begin{pmatrix} 12\,592 & -9708 & 1580 \\ -9708 & 8413 & -1461 \\ 1580 & -1461 & 269 \end{pmatrix}\begin{pmatrix} 14 & 33 & 97 \\ 33 & 97 & 333 \\ 97 & 333 & 1273 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The best estimators $\hat{a}_1$, $\hat{a}_2$ and $\hat{a}_3$ for the coefficients in the quadratic form are now given by $\hat{\mathsf{a}} = \mathsf{V}\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{y}$, where y is the data column vector $(3.5,\ 2.0,\ 3.0,\ 6.5,\ 10.5)^{\mathrm{T}}$. The column vector $\hat{\mathsf{a}}$ is calculated as

$$\begin{pmatrix} 1.371 & -1.057 & 0.1720 \\ -1.057 & 0.9160 & -0.1591 \\ 0.1720 & -0.1591 & 0.0293 \end{pmatrix}\begin{pmatrix} 4 & 4 & 4 & 1 & 1 \\ 4 & 8 & 12 & 4 & 5 \\ 4 & 16 & 36 & 16 & 25 \end{pmatrix}\begin{pmatrix} 3.5 \\ 2.0 \\ 3.0 \\ 6.5 \\ 10.5 \end{pmatrix},$$

yielding the three components as 6.73, −4.34 and 1.03. The corresponding standard errors in these coefficients are given by the square roots of the diagonal elements of V, namely 1.17, 0.96 and 0.17.

Thus the best quadratic fit to the measured data, giving weight to the standard errors in them, is

$$y(x) = (6.73 \pm 1.17) - (4.34 \pm 0.96)x + (1.03 \pm 0.17)x^2.$$

The off-diagonal elements of V are not used here, but are closely related to the correlations between the fitted parameters.

31.18 *Prove that the expression given for the Student's t-distribution in equation (31.118) is correctly normalised.*

The given expression is

$$P(t|H_0) = \frac{1}{\sqrt{(N-1)\pi}}\frac{\Gamma\left(\frac{1}{2}N\right)}{\Gamma\left(\frac{1}{2}N - \frac{1}{2}\right)}\left(1 + \frac{t^2}{N-1}\right)^{-N/2},$$

Denoting the product of constants multiplying the t-dependent parentheses by $A(N)$, we require that

$$\int_{-\infty}^{\infty} P(t|H_0)\,dt = A(N)\int_{-\infty}^{\infty}\left(1 + \frac{t^2}{N-1}\right)^{-N/2} dt = 1.$$

Set $t = \sqrt{N-1}\tan\theta$ for $-\pi/2 \le \theta \le \pi/2$ giving

$$\int_{-\infty}^{\infty} P(t|H_0)\,dt = A(N)\int_{-\pi/2}^{\pi/2}(1+\tan^2\theta)^{-N/2}\sqrt{N-1}\sec^2\theta\,d\theta$$
$$= 2\sqrt{N-1}\,A(N)\int_0^{\pi/2}\sec^{-N+2}\theta\,d\theta$$
$$= 2\sqrt{N-1}\,A(N)\int_0^{\pi/2}\cos^{N-2}\theta\,d\theta.$$

Now, integrals of this form can be expressed in term of beta and gamma functions by

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = B(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta.$$

It follows that

$$\int_0^{\pi/2}\cos^{N-2}\theta\,d\theta = \frac{1}{2}B(\tfrac{1}{2},\tfrac{1}{2}N-\tfrac{1}{2})$$
$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}N-\frac{1}{2})}{2\Gamma(\frac{1}{2}N)}$$
$$= \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}N-\frac{1}{2})}{2\Gamma(\frac{1}{2}N)}.$$

Hence

$$\int_0^{\infty} P(t|H_0)\,dt = 2\sqrt{N-1}\,\frac{\Gamma(\frac{1}{2}N)}{\sqrt{(N-1)\pi}\,\Gamma(\frac{1}{2}N-\frac{1}{2})}\,\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}N-\frac{1}{2})}{2\Gamma(\frac{1}{2}N)} = 1,$$

as expected.

31.20 *It is claimed that the two following sets of values were obtained* (a) *by randomly drawing from a normal distribution that is* $N(0,1)$ *and then* (b) *randomly assigning each reading to one of two sets A and B:*

Set A:	−0.314	0.603	−0.551	−0.537	−0.160	−1.635	0.719
	0.610	0.482	−1.757	0.058			
Set B:	−0.691	1.515	−1.642	−1.736	1.224	1.423	1.165

Make tests, including t- and F-tests, to establish whether there is any evidence that either claims is, or both claims are, false.

(a) The mean and variance of the whole sample are −0.068 and 1.180, leading to an estimated standard deviation, including the Bessel correction for 18 readings, of 1.12. These are obviously compatible with samples drawn from a $N(0,1)$ distribution, without the need for statistical tests.

(b) The means and sample variances of the two sets are: A, −0.226 and 0.741; B, 0.180 and 2.189, with estimated standard deviations of the populations from which they are drawn of 0.861 and 1.480 respectively.

The best estimator of $\hat{\sigma}$ for calculating t is

$$\hat{\sigma} = \left[\frac{(11 \times 0.741) + (7 \times 2.189)}{11 + 7 - 2}\right]^{1/2} = 1.21.$$

On the null hypothesis that the two samples are drawn from the same distribution, t is given by

$$t = \frac{0.180 - (-0.226)}{1.21} \left(\frac{11 \times 7}{11 + 7}\right)^{1/2} = 0.694.$$

This is for $11 + 7 - 2 = 16$ degrees of freedom. From the table $C_{16}(0.694) = 0.74$. Thus, this or a greater value of t (in magnitude) can be expected in marginally more than half of all cases (recall that here a two-tailed distribution is needed) and there is no evidence for a significant difference between the means of the two samples.

The value of the estimated variance ratio of the parent populations is

$$F = \frac{u^2}{v^2} = \frac{7 \times 2.189}{6} \frac{10}{11 \times 0.741} = 3.13.$$

For $n_1 = 6$ and $n_2 = 10$, this value is very close to the 95% confidence limit of 3.22. Thus it is rather unlikely that the allocation between the two groups was made at random – set B has significantly more readings that are more than one standard deviation from the mean for a $N(0,1)$ distribution than it should have.